Johannes Zimmer

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Problem Set #1

MA50175 Topics in Differential Equations

Mathematical Materials Science

H1. *Scaling regimes*
Take an A4 sized sheet of paper and fold a paper ship. Repeat the exercise with the smallest sheet of paper you manage to fold a ship with (who can fold the smallest microsized ship?). What do you observe? If you try to describe folding paper ships by energy minimisation, which energetic contribution gets large as the ship size goes to zero?

H2. *Square root of a symmetric, positive definite matrix*
Let A be a symmetric ($A^T = A$) and positive definite ($x^T Ax > 0$ for every vector $x \neq 0$) matrix. Show that there exists a unique symmetric positive definite matrix B such that $B^2 = A$. We write $B = \sqrt{A}$ and say that B is the *square root* of A.
Remark. This is also true for linear operators. Rephrased in the language commonly used in operator theory, the above Theorem reads: A self-adjoint, positive semi-definite operator A has a square root B, i.e., there exists a unique self-adjoint and positive semi-definite operator B such that $B^2 = A$.

H3. Express C and E in terms of Du.

H4. Let $\Omega \subset \mathbb{R}^2$, and let $F = QU$ be the polar decomposition of an invertible matrix F.

a) Consider the characteristic polynomial χ for U, $\chi(\lambda) = \lambda^2 - I_U\lambda + II_U$. Determine the scalars I_U and II_U.
b) By the Cayley-Hamilton Theorem from linear algebra, $\chi(U) = 0$, i.e., $U^2 - I_U U + II_U \mathrm{Id} = 0$, where Id is the identity matrix in $\mathbb{R}^2$. Express this equation in terms of C and II_C as far as possbile. Take the trace to show that

$$U = \frac{1}{\sqrt{\left(I_C + 2\sqrt{II_C}\right)}} \left(C + \sqrt{II_C}\mathrm{Id}\right).$$

H5. *Shear*
Let $\Omega \subset \mathbb{R}^2$ be the unit disc $\Omega := \{(x^1, x^2) \mid (x^1)^2 + (x^2)^2 \leq 1\}$. Suppose this body undergoes the deformation $y^1 := x^1 + \kappa x^2$, $y^2 := x^2$, with $\kappa > 0$. Compute F, C, B, and Q and U of the polar decomposition $F = QU$. Is this deformation orientation-preserving and volume-preserving?

H6. *Singular value decomposition and polar decomposition*
Suppose F is an invertible matrix linear operator on a finite-dimensional Hilbert space H. Let $F = QU$ be the polar decomposition of F in a orthogonal matrix Q and a symmetric positive definite matrix U. Let $F = W_1^T \mathrm{diag}(\sigma_1, \ldots, \sigma_p) W_2$ be the singular value decomposition of F. Express Q and U in terms of the singular values and the columns of W_1 and W_2.