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AM 125c

Engineering Mathematical Principles

Midterm Exam

120 minutes. Open notes and textbooks. There are 4 problems (40 points) on the exam. Please show all your work: we cannot give credits for unsupported results.

1. *(10 points).* Consider Hill's equation

$$\ddot{x}(t) + p(t)x(t) = 0,$$

where p is continuous and periodic with period T. Let $\underline{\mathrm{U}}(t)$ be the main fundamental matrix (of the corresponding first order system) and C the transition matrix.

a) Show that the eigenvalues of C are given by

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}(C) \pm \frac{1}{2}\sqrt{(\mathrm{tr}(C))^2 - 4}.$$

b) What can you say about existence of normal solutions and their behavior for $t \to \infty$ (Exponential growth, exponential convergence, boundedness)?

c) What can you say about the existence of periodic solutions? In case periodic solutions exist, what is their period?

2. *(10 points).* Find the Green's function for

a) the quadrant $Q := \{(x, y) \mid x > 0, y > 0\}$,

b) the square $S := \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$.

3. *(10 points).* The mapping $(x, y) \mapsto (x^2 - y^2, 2xy)$ maps the quadrant $Q := \{(x, y) \mid x > 0, y > 0\}$ onto the upper half plane.

a) Check that this mapping is conformal.

b) Solve the problem

$$\begin{aligned} \Delta u &= 0 \text{ in } Q, \\ u(x, 0) &= A, \\ u(0, y) &= B, \end{aligned}$$

where $A, B \in \mathbb{R}$.
Hint: You may want to use

$$\int \frac{1}{(x-a)^2 + b^2}\mathrm{d}x = \frac{1}{b}\arctan\left(\frac{x-a}{b}\right).$$

4. *(10 points).* Let U be a bounded region in $\mathbb{R}^3$. Consider the Neumann problem for Poisson's equation:

$$
\begin{aligned}
-\Delta u &= f \text{ in } U & (1)\\
\partial_n u &= g \text{ on } \partial U & (2)
\end{aligned}
$$

a) Show that a necessary condition for solvability of the Neumann problem is

$$\int_{\partial U} g(x) \mathrm{d}S(x) = -\int_U f(x) \mathrm{d}x.$$

Hint: Use the Divergence Theorem or one of its variants.

b) Assume the Neumann problem has a smooth solution. Let x_0 be an arbitrary point in U. Show that there exists a solution of (1)–(2) with $u(x_0) = 0$.

c) Show that the solution in b) is unique.