MA10207: Exercise sheet 14

Please hand in solutions to homework problems by Monday, 11th March, 2:15pm.

(*Power series & uniform convergence*)

Warmup problems

Problem T 14.1. Let $\sum_{n=0}^{\infty} a_n$ be a real series; let $f(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n$ denote a power series.

a) Recall some criteria for convergence of infinite series.
b) What is the radius of convergence of a power series?
c) How (and why) can the radius of convergence be determined using convergence criteria?

Problem T 14.2. Let $(f_n)_{n\in N} \subset C^0(I)$ be a sequence of continuous functions on a *compact* interval I.

(i) Prove that $f_n \stackrel{\text{unif.}}{\longrightarrow} f$ if and only if $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : \|f_n - f\|_\infty < \varepsilon$.
(ii) Prove that $(f_n)_{n\in N}$ is a uniform Cauchy sequence if and only if $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N$: $\|f_n - f_m\|_\infty < \varepsilon$, that is, $(f_n)_{n\in N}$ is a "Cauchy sequence in $(C^0(I), \|.\|_\infty)$".

Homework problems

Problem H 14.1. Prove that a uniformly convergent sequence $(f_n)_{n\in N}$ of functions is a uniform Cauchy sequence. Discuss the converse.

Problem H 14.2. Decide whether the sequence $(f_n)_{n\in N}$ converges uniformly on $[0, 1]$ in the following cases:

$$\text{(a)} \;\; f_n(x) = \tfrac{nx}{1+nx}, \quad \text{(b)} \;\; f_n(x) = x^n(1 - x^n).$$

[Hint for (b): check $f_n(x_n)$ for $x_n = \frac{1}{\sqrt[n]{2}}$.]

Problem H 14.3. Use Theorems 4.10 and 4.12 to prove Theorem 3.25 (continuity of power series).

Quiz questions

Problem Q 14.1.

(i) There are nowhere convergent power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$. [Hint]
(ii) If a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R = 1$ then it always converges for either $x = 1$ or $x = -1$ (depending on whether $(a_k)_{k\in N}$ is alternating or not). [Hint]
(iii) If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 1$, then there is a number M so that $|a_k| \leq M$ for all k. [Hint]
(iv) The sum of the series $\sum_{k=1}^{\infty} \frac{1}{2^k}(x - 3)^k$ and $\sum_{k=1}^{\infty} \frac{1}{3^k}(x - 2)^k$ is convergent at $x = 4$. [Hint]
(v) Suppose $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ have radii of convergence $R_1 > 0$ and $R_2 > 0$, respectively. Then $\sum_{k=0}^{\infty}(a_k + b_k)\, x^k$ has radius of convergence $R \geq \min\{R_1, R_2\} > 0$. [Hint]

<u>*Evaluate*</u>

Problem Q 14.2. Let I be an interval and $f_n : I \to \mathbb{R}$ for $n \in \mathbb{N}$.

(i) For $(f_n)_{n\in N}$ to be a uniform Cauchy sequence, it is necessary that $(f_n(x))_{n\in N}$ converges for every $x \in I$. [Hint]
(ii) For $(f_n)_{n\in N}$ to be uniformly convergent on I, it is sufficient that $(f_n(x))_{n\in N}$ be a Cauchy sequence for every $x \in I$. [Hint]
(iii) If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is defined by a power series with radius of convergence $R > 0$, then the sequence of partial sums converges uniformly on $(-R, R)$. [Hint]

<u>*Evaluate*</u>