A GENERIC MULTIPLICATION IN QUANTISED SCHUR ALGEBRAS

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Abstract. We define a generic multiplication in quantised Schur algebras and thus obtain a new algebra structure in the Schur algebras. We prove that via a modified version of the map from quantum groups to quantised Schur algebras, defined in [1], a subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied in [10]. We also prove that the subalgebra of the new algebra gives a geometric realisation of a positive part of 0-Schur algebras, defined in [4]. Consequently, we obtain a multiplicative basis for the positive part of 0-Schur algebras.

Introduction

Schur algebras $S(n,r)$ were invented by I. Schur to classify the polynomial representations of the complex general linear group $\mathrm{Gl}_n(\mathbb{C})$. Quantised Schur algebras $S_q(n,r)$ are quantum analogues of Schur algebras. Both quantised Schur algebras $S_q(n,r)$ and classical Schur algebras $S(n,r)$ have applications to the representation theory of Gl_n over fields of undescribing characteristics.

In [1] A. A. Beilinson, G. Lusztig and R. MacPherson gave a geometric construction of quantised enveloping algebras of type $\mathbb{A}$. Among other important results they defined surjective algebra homomorphisms θ, from the integral form of the quantised enveloping algebras to certain finite dimensional associative algebras. They first defined a multiplication of pairs of n-step partial flags in a vector space k^r over a finite field k and thus obtained a finite dimensional associative algebra. They also studied how the structure constants behave when r increases by a multiple of n. Then, by taking a certain limit they obtained the quantised enveloping algebras of type $\mathbb{A}$. J. Du remarked in [5] that the finite dimensional associative algebras studied in [1] are canonicallly isomorphic to the quantised Schur algebras studied by R. Dipper and G. James in [2].

The aim of this paper is to study a generic version of the multiplication of pairs of partial flags defined in [1]. By this generic multiplication we get an algebra structure in the quantised Schur algebras. We prove that a certain subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied by M. Reineke in [10]. Via a modified version of the surjective algebra homomorphism θ, defined in [1], we prove that the subalgebra is isomorphic to a positive part of 0-Schur algebras, studied by S. Donkin in [4]. Thus we achieve a geometric construction of the positive parts of the 0-Schur algebras.

This paper is organized as follows. In Section 1 we recall definitions and results in [1] on the multiplication of pairs of partial flags (see also [5, 6]). In Section 2 we recall definitions and results on the monoid given by generic extensions studied in [10]. In Section 3 we study a generic version of the multiplication of pairs of partial flags in [1], and prove that this generic multiplication gives us a new algebra structure in quantised Schur algebras. In Section 4 we prove results on connection between our new algebras and the monoid algebras given by generic extensions in [10] and to 0-Schur algebras. We also provide a multiplicative basis for a positive part of 0-Schur algebras. As a remark, we would like to mention that this multiplicative basis is related to Lusztig's canonical basis.

1. q-Schur algebras as quotients of quantised enveloping algebras

In this section we recall some definitions and results from [1] on q-Schur algebras as quotients of quantised enveloping algebras (see also [5, 6]).

1.1. **q-Schur algebras.** Denote by Θ_r the set of $n \times n$ matrices whose entries are non-negative integers and sum to r. Let V be an r-dimensional vector space over a field k. Let $\mathcal{F}$ be the set of all n-steps flags in V:

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V.$$

The group $\mathrm{GL}(V)$ acts naturally by change of basis on $\mathcal{F}$. We let $\mathrm{GL}(V)$ act diagonally on $\mathcal{F} \times \mathcal{F}$. Let $(f, f') \in \mathcal{F} \times \mathcal{F}$, we write

$$f = V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V \text{ and } f' = V_1' \subseteq V_2' \subseteq \cdots \subseteq V_n' = V.$$

Let $V_0 = V_0' = 0$ and define

$$a_{ij} = \dim(V_{i-1} + V_i \cap V_j') - \dim(V_{i-1} + V_i \cap V_{j-1}').$$

Then the map $(f, f') \mapsto (a_{ij})_{ij}$ induces a bijection between the set of $\mathrm{GL}(V)$-orbits in $\mathcal{F} \times \mathcal{F}$ and the set Θ_r. We denote by $\mathcal{O}_A$ the $\mathrm{GL}(V)$-orbit in $\mathcal{F} \times \mathcal{F}$ corresponding to the matrix $A \in \Theta_r$.

Now suppose that k is a finite field with q elements. Let $A,\ A',\ A'' \in \Theta_r$ and let $(f_1, f_2) \in \mathcal{O}_{A''}$. Following Proposition 1.1 in [1], there exists a polynomial $g_{A,A',A''} = c_0 + c_1 q \cdots + c_m q^m$, given by

$$g_{A,A',A''} = |\{f \in \mathcal{F} | (f_1, f) \in \mathcal{O}_A, (f, f_2) \in \mathcal{O}_{A'}\}|,$$

where c_i are integers that do not depend on q, the cardinality of the field k, and $(f_1, f_2) \in \mathcal{O}_{A''}$.

Now recall that the q-Schur algebra $S_q(n, r)$ is the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{e_A | A \in \Theta_r\}$, and with an associative multiplication given by

$$e_A e_{A'} = \sum_{A'' \in \Theta_r} g_{A,A',A''} e_{A''}.$$

For a matrix $A \in \Theta_r$, denote by $\mathrm{ro}(A)$ the vector $(\sum_j a_{1j}, \sum_j a_{2j}, \cdots, \sum_j a_{nj})$ and by $\mathrm{co}(A)$ the vector $(\sum_j a_{j1}, \sum_j a_{j2}, \cdots, \sum_j a_{jn})$. By the definition of the multiplication it is easy to see that

$$e_A e_{A'} = 0 \text{ if } \mathrm{co}(A) \neq \mathrm{ro}(A').$$

Denote by E_{ij} the elementary $n \times n$ matrix with 1 at the entry (i, j) and 0 elsewhere. We recall a lemma, which we will use later, on the multiplication defined above.

Lemma 1.1 ([1])**.** *Assume that* $1 \leq h < n$. *Let* $A = (a_{ij}) \in \Theta_r$. *Assume that* $B = (b_{ij}) \in \Theta_r$ *such that* $B - E_{h,h+1}$ *is a diagonal matrix and* $\mathrm{co}(B) = \mathrm{ro}(A)$. *Then*

$$e_B e_A = \sum_{p: a_{h+1,p} > 0} v^{2\sum_{j>p} a_{hj}} \frac{v^{2(a_{hp}+1)} - 1}{v^2 - 1} e_{A + E_{h,p} - E_{h+1,p}}.$$

1.2. **The map $\theta : U_{\mathcal{A}}(\mathrm{gl}_n) \to S_v(n, r)$.** Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $v^2 = q$. Let $U_{\mathcal{A}}(\mathrm{gl}_n)$ be the integral form of the quantised enveloping algebra of the Lie algebra gl_n. Denote by $S_v(n, r)$ the algebra $\mathcal{A} \otimes S_q(n, r)$. Let

$$\theta : U_{\mathcal{A}}(\mathrm{gl}_n) \to S_v(n, r)$$

be the surjective algebra homomorphism defined by A. A. Beilinson, G. Lusztig and R. MacPherson in [1]. Through the map θ we can view the Schur algebra $S_v(n, r)$ as a quotient of the quantised enveloping algebra $U_{\mathcal{A}}(\mathrm{gl}_n)$. We are interested in the restriction of θ to the positive part U^+ of $U_{\mathcal{A}}(\mathrm{gl}_n)$.

Unless stated otherwise, we let Q be the linearly oriented quiver of type $\mathbb{A}_{n-1}$:

$$Q : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 .$$

By a well-known result of C. M. Ringel (see [11, 12]), the algebra U^+ is isomorphic to the twisted Ringel-Hall algebra $H_q(Q)$, which is generated by isomorphism classes of simple kQ-modules via Hall multiplication.

Denote by S_i the simple module of the path algebra kQ associated to vertex i of Q. By abuse of notation we also denote by M the isomorphism class of a kQ-module M. For any $s \in \mathbb{N}$, denote by D_s the set of diagonal matrices satisfying that the entries are non-negative integers and that the sum of the entries is s. For a matrix $A \in \Theta_r$, denote by $[A] = v^{-\dim \mathcal{O}_A + \dim pr_1(\mathcal{O}_A)} e_A$, where pr_1 is the natural projection to the first component of $\mathcal{F} \times \mathcal{F}$. Now the map θ can be defined on the twisted Ringel-Hall algebra as follows:

$$\theta : H_q(Q) \to S_v(n,r), \quad S_i \mapsto \sum_{D \in D_{r-1}} [E_{i,i+1} + D].$$

2. A monoid given by generic extensions

In this section we briefly recall definitions and results on the monoid of generic extensions in [10], and we let k be an algebraically closed field. Results in this section work for any Dynkin quiver $Q = (Q_0,\ Q_1)$, where $Q_0 = \{1, \cdots, n\}$ is the set of vertices of Q and Q_1 is the set of arrows of Q. We denote by modkQ and Rep(Q), respectively, the category of finitely generated left kQ-modules and the category of finite dimensional representations of Q. We don't distinguish a representation of Q from the corresponding kQ-module.

Let $\mathbf{b} \in \mathbb{N}^{n-1}$. Denote by

$$\mathrm{Rep}(\mathbf{b}) = \Pi_{i \to j \in Q_1} \mathrm{Hom}_k(k^{b_i}, k^{b_j})$$

the representation variety of Q, which is an affine space consisting of representations with dimension vector $\mathbf{b}$. The group $\mathrm{GL}(\mathbf{b}) = \Pi_i \mathrm{GL}(b_i)$ acts on Rep$(\mathbf{b})$ by conjugation and there is a one-to-one correspondence between GL$(\mathbf{b})$-orbits in Rep$(\mathbf{b})$ and isomorphism classes of representations in Rep$(\mathbf{b})$. Denote by $\mathcal{E}(M,N)$ the subset of Rep$(\mathbf{b})$, containing points which are extensions of M by N.

Lemma 2.1 ([10])**.** *The set $\mathcal{E}(M,N)$ is an irreducible subset of* Rep$(\mathbf{b})$.

Thus there exists a unique open GL$(\mathbf{b})$-orbit in $\mathcal{E}(M,N)$. We say a point in $\mathcal{E}(M,N)$ is generic if it is contained in the open orbit. Generic points in $\mathcal{E}(M,N)$ are also called generic extensions of M by N.

Definition 2.2 ([10])**.** *Let M and N be two isomorphism classes in* modkQ*. Define a multiplication*

$$M * N = G,$$

where G is the isomorphism class of the generic points in $\mathcal{E}(M,N)$.

Denote by $\mathbf{H}_q(Q)$ the Ringel-Hall algebra over $\mathbb{Q}[q]$ and by $\mathbf{H}_0(Q)$ the specialisation of $\mathbf{H}_q(Q)$ at $q = 0$.

Theorem 2.3 ([10])**.** *(1) $\mathcal{M} = (\{M | M$ is an isomorphism class in $\mathrm{mod}\ kQ\},\ *)$ is a monoid.*

(2) $\mathbb{Q}\mathcal{M} \cong \mathbf{H}_0(Q)$ as algebras.

3. A monoid given by a generic multiplication in q-Schur algebras

In this section we define a generic multiplication in the q-Schur algebra $S_q(n,r)$. Via this multiplication we obtain a new algebra in $S_q(n,r)$. We let k be an algebraically closed field in this section.

Denote by

$$\Theta_r^u = \{A \in \Theta_r | A \text{ is an upper triangular matrix}\}.$$

Let $A, A' \in \Theta_r^u$, define

$$\mathcal{E}(A, A') = \{(f_1, f_2) \in \mathcal{F} \times \mathcal{F} | \exists f \text{ such that } (f_1, f) \in \mathcal{O}_A \text{ and } (f, f_2) \in \mathcal{O}_{A'}\}.$$

We denote by $M(ij)$ the indecomposable representation of Q with dimension vector $\sum_{l=i}^{j-1} \mathbf{e}_l$, where $\mathbf{e}_l$ is the simple root of Q associated to vertex i, that is, $\mathbf{e}_i$ is the dimension vector of the simple module S_i. By the module determined by a matrix $A \in \Theta_r^u$ we mean the module

$\bigoplus_{i<j} M(ij)^{a_{ij}}$. Note that for any $(f_1, f_2) \in \mathcal{E}(A, A')$, we have f_2 is a subflag of f_1. Note also that by omitting the last step of a partial flag f in $\mathcal{F}$, we can view f as a projective kQ-module and by abuse of notation we still denote the projective module by f. Now for $A \in \Theta_r^u$, suppose that $(f, h) \in \mathcal{O}_A$ and that M is the module determined by A. We have a short exact sequence

$$0 \longrightarrow h \longrightarrow f \longrightarrow M \longrightarrow 0 ,$$

that is, $h \subseteq f$ is a projective resolution of M.

Let $\mathbf{b} \in \mathbb{N}^{n-1}$ and denote by $k^{\mathbf{b}}$ the Q_0-graded vector space with k^{b_i} as its i-th homogeneous component, where i is a vertex of Q. Denote by $\mathrm{Hom}_{\mathrm{gr}}(f, k^{\mathbf{b}})$ the set of graded linear maps between f and $k^{\mathbf{b}}$, where f is a partial flag in $\mathcal{F}$ viewed as a Q_0-graded vector space by omitting its last step.

3.1. **Relation between generic points in $\mathcal{E}(A, A')$ and in $\mathcal{E}(M, N)$.** For $A, A' \in \Theta_r$, we write $A \leq A'$ if $\mathcal{O}_{A'}$ is contained in the Zariski closure of $\mathcal{O}_A$. In this case we say that $(f_1, f_2) \in \mathcal{O}_A$ degenerates to $(f_1', f_2') \in \mathcal{O}_{A'}$. Lemma 3.7 in [1] implies the existence of generic points in $\mathcal{E}(A, A')$, in the sense that the closure of their orbit contains orbits of all the other points in $\mathcal{E}(A, A')$. That is, there is a unique open orbit in $\mathcal{E}(A, A')$. In this subsection we will show that there is a nice correspondence between generic points $\mathcal{E}(A, A')$ and generic points in subset $\mathcal{E}(M, N)$, where M and N are the modules determined by A and A', respectively.

Let $(f_1, f_2) \in \mathcal{F} \times \mathcal{F}$ with f_2 a subflag of f_1. Denote by $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, the dimension vectors of the projective modules f_1 and f_2. Let $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2$. We define some sets as follows.

$$\mathrm{Inj}(f_2, f_1) = \{\sigma \in \mathrm{Hom}_{kQ}(f_2, f_1) | \sigma \text{ is injective}\};$$
$$\mathcal{S}_1 = \{(\sigma, \eta) \in \mathrm{Inj}(f_2, f_1) \times \mathrm{Hom}_{\mathrm{gr}}(f_1, k^{\mathbf{b}}) | \eta \text{ is surjective and } \eta\sigma = 0\};$$
$$\mathcal{S}_1' = \{(\sigma, \eta) \in \mathrm{Hom}_{kQ}(f_2, f_1) \times \mathrm{Hom}_{\mathrm{gr}}(f_1, k^{\mathbf{b}}) | \eta \text{ is surjective, } \ker\eta \text{ is a } kQ\text{-module and } \eta\sigma = 0\};$$
$$\mathcal{S}_2 = \{\eta \in \mathrm{Hom}_{\mathrm{gr}}(f_1, k^{\mathbf{b}}) | \eta \text{ is surjective and } \ker\eta \text{ is a } kQ\text{-module}\};$$
$$\mathcal{S}_2' = \{(M, \eta) \in \mathrm{Rep}(\mathbf{b}) \times \mathrm{Hom}_{\mathrm{gr}}(f_1, k^{\mathbf{b}}) | \eta : f_1 \to M \text{ is a } kQ\text{-homomorphism}\};$$
$$\mathrm{Inj}_{A,A'}(f_2, f_1) = \{\sigma \in \mathrm{Inj}(f_2, f_1) | \mathrm{cok}(\sigma) \in \mathcal{E}(M, N)\},$$

where M and N are the modules determined by A and A', respectively.

For convenience we denote by $\mathrm{Inj}(f_2, f_1)$ by $\mathcal{S}_3$. We obtain some fibre bundles as follows.

Lemma 3.1 ([8])**.** *The natural projection $\pi_1 : \mathcal{S}_1' \to \mathcal{S}_2$ is a vector bundle.*

Lemma 3.2 ([8])**.** *The natural projection $\pi_2 : \mathcal{S}_1 \to \mathcal{S}_3$ is a principal $\mathrm{GL}(\mathbf{b})$-bundle.*

Lemma 3.3 ([8])**.** *The natural projection $\pi_3 : \mathcal{S}_2' \to \mathrm{Rep}(\mathbf{b})$ is a vector bundle.*

Note that any $\eta \in S_2$ determines a unique module $M \in \mathrm{Rep}(\mathbf{b})$ and this defines an open embedding of $\mathcal{S}_2$ into $\mathcal{S}_2'$. So we can view $\mathcal{S}_2$ as an open subset of $\mathcal{S}_2'$.

Lemma 3.4. *(1)* $\mathrm{Inj}_{A,A'}(f_2, f_1) = \pi_2(\mathcal{S}_1 \cap \pi_1^{-1}(\mathcal{S}_2 \cap \pi_3^{-1}(\mathcal{E}(M, N))))$.
(2) $\mathrm{Inj}_{A,A'}(f_2, f_1)$ *is irreducible.*

Proof. Following the definitions of π_1 and π_3, $\mathrm{cok}\sigma \in \mathcal{E}(M, N)$ for any $(\sigma, \eta) \in \mathcal{S}_1 \cap \pi_1^{-1}(\mathcal{S}_2 \cap \pi_3^{-1}(\mathcal{E}(M, N)))$. Therefore $\sigma = \pi_2((\sigma, \eta)) \in \mathrm{Inj}_{A,A'}(f_2, f_1)$. On the other hand, suppose that $\sigma \in \mathrm{Inj}_{A,A'}(f_2, f_1)$. Then $\sigma \in \pi_2(\mathcal{S}_1 \cap \pi_1^{-1}(\mathcal{S}_2 \cap \pi_3^{-1}(X)))$, where X is the module determined by η for a preimage $(\sigma, \eta) \in \pi_2^{-1}(\sigma)$. This proves (1). Now (2) follows from (1) and Lemmas 3.1-3.3. □

Let
$$\mathcal{E}'(A,A') = \{(f_1,f) \in \mathcal{E}(A,A') | f \in \mathcal{F}\}.$$
Define
$$\pi : \mathrm{Inj}_{A,A'}(f_2,f_1) \to \mathcal{F} \times \mathcal{F},\ \sigma \mapsto (f_1, \mathrm{Im}\sigma),$$
where $\mathrm{Im}\sigma$ can be viewed as a flag in $\mathcal{F}$ with the last step the natural embedding of $\mathrm{Im}\sigma(f_1)_{n-1}$ into V.

Lemma 3.5. *(1)* $\mathrm{Im}\pi = \mathcal{E}'(A,A')$.
(2) $\mathcal{E}'(A,A')$ *is irreducible.*

Proof. Let $\sigma \in \mathrm{Inj}_{A,A'}(f_2,f_1)$. Then by the following diagram,

$$\begin{array}{ccccc}
f_2 & \longrightarrow & f_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow \\
\ker\xi\eta & \longrightarrow & f_1 & \xrightarrow{\xi\eta} & M, \\
\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow \\
N & \longrightarrow & \mathrm{cok}\sigma & \xrightarrow[\xi]{} & M
\end{array}$$

where each square commutes and all rows and columns are short exact sequences, we know that $(f_1, \mathrm{Im}\sigma) \in \mathcal{E}'(A,A')$. On the other hand by the definition of $\mathcal{E}'(A,A')$, for any $(f_1,f) \in \mathcal{E}'(A,A')$, the natural embedding $f_2 \cong f \subseteq f_1$ is in $\mathrm{Inj}_{A,A'}(f_2,f_1)$. Now the irreducibility of $\mathcal{E}'(A,A')$ follows from that of $\mathrm{Inj}_{A,A'}(f_2,f_1)$. $\square$

As a consequence of Lemma 3.5 we can see the existence of a unique dense open orbits in $\mathcal{E}(A,A')$. Indeed, by Lemma 3.5 and the surjective map $\mathcal{E}'(A,A') \times \mathrm{GL}(r) \to \mathcal{E}(A,A')$, the set $\mathcal{E}(A,A')$ is irreducible. Since there are only finitely many $\mathrm{GL}(r)$-orbits in $\mathcal{E}(A,A')$, there exists a unique dense open orbits in $\mathcal{E}(A,A')$.

Definition 3.6. *Let* $\mathcal{O}_{A''}$ *be the dense open orbit in* $\mathcal{E}(A,A')$. *We say that an injection* $\sigma : f' \to f$ *is generic in* $\mathrm{Inj}_{A,A'}(f',f)$ *if the pair of flags* $(f,\ \mathrm{Im}\sigma)$ *is contained in* $\mathcal{O}_{A''}$.

Proposition 3.7. *Let* $(\sigma,\eta) \in \mathcal{S}_1$, $(f_1,f) \in \mathcal{O}_{A'}$ *and* $(f,f_2) \in \mathcal{O}_{A''}$. *Then* σ *is generic in* $\mathrm{Inj}_{A,A'}(f_2,f_1)$ *if and only if the module determined by* η *is generic in* $\mathcal{E}(M,N)$, *where* M *and* N *are the modules determined by* A *and* A', *respectively.*

Proof. Suppose that $\mathcal{O}_{A''}$ is the dense open orbit in $\mathcal{E}(A,A')$ and that $\mathcal{O}_X$ is the dense open orbit in $\mathcal{E}(M,N)$. By Lemmas 3.1-3.3, $\pi_2(\mathcal{S}_1 \cap \pi_1^{-1}(\mathcal{S}_2 \cap \pi_3^{-1}(\mathcal{O}_X)))$ is open in $\mathrm{Inj}_{A,A'}(f_2,f_1)$. By Lemma 3.5, $\pi^{-1}(\mathcal{O}_{A''} \cap \mathcal{E}'(A,A'))$ is open in $\mathrm{Inj}_{A,A'}(f_2,f_1)$. Since $\mathrm{Inj}_{A,A'}(f_2,f_1)$ is irreducible, the intersection $\pi_2(\mathcal{S}_1 \cap \pi_1^{-1}(\mathcal{S}_2 \cap \pi_3^{-1}(\mathcal{O}_X))) \cap \pi^{-1}(\mathcal{O}_{A''} \cap \mathcal{E}'(A,A'))$ is non-empty. Therefore, X is the module determined by A''. This finishes the proof. $\square$

3.2. **A generic multiplication in** $S_q(n,r)$**.** We now define a multiplication, called a generic multiplication, by
$$e_A \circ e_{A'} = \begin{cases} e_{A''} & \text{if } \mathcal{E}(A,A') \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$
where $\mathcal{O}_{A''}$ is the dense open orbit in $\mathcal{E}(A,A')$.

Proposition 3.8. *Let* $A,\ A',\ A'' \in \Theta_r^u$. *Then* $(e_A \circ e_{A'}) \circ e_{A''} = e_A \circ (e_{A'} \circ e_{A''})$

Proof. By the definition of the multiplication $\circ$, we see that $(e_A \circ e_{A'}) \circ e_{A''} = 0$ implies that $e_A \circ (e_{A'} \circ e_{A''}) = 0$ and vice versa. So we may assume that neither of them is zero. Let M, N, L be the module determined by A, A', A'', respectively. By Lemma 3.1 in [10], we know that $(M * N) * L = M * (N * L)$. Now the proof follows from Proposition 3.7. $\square$

We can now state the main result of this section.

Theorem 3.9. *$\mathbb{Q}(\{e_A|A \in \Theta_r^u\}, +, \circ)$ is an algebra with unit $\sum_{D\in D_r} e_D$.*

Proof. We need only to show that $\sum_{D\in D_r} e_D$ is the unit. Let D be a diagonal matrix and let $(f_1, f_2) \in \mathcal{O}_D$. Then $f_1 = f_2$. For any $A \in \Theta_r^u$, by the definition of the generic mulitiplication,

$$e_A \circ \sum_{D\in D_r} e_D = e_A \circ e_C.$$

where C is the diagnal matrix $\text{diag}(\sum_j a_{1j}, \cdots, \sum_j a_{nj})$. Since $e_A e_C = \sum_B g_{A,C,B} e_B$, where $(f_1, f) \in \mathcal{O}_B$ and $g_{A,C,B} = |\{f|(f_1, f) \in \mathcal{O}_A, (f, f) \in \mathcal{O}_C\}|$, we see that $e_A e_C = e_A$. Therefore $e_A \circ e_C = e_A$. Similarly, $(\sum_{D\in D_r} e_D) \circ e_A = e_A$. Therefore $\sum_{D\in D_r} e_D$ is the unit. □

We denote the algebra $\mathbb{Q}(\{e_A|A \in \Theta_r^u\}, +, \circ)$ in Theorem 3.9 by S_0^+.

4. The algebra S_0^+ as a quotient and 0-Schur algebras

We have two tasks in this section. We will first prove that a certain subalgebra of S_0^+ is a quotient of the monoid algebra defined in Section 2. We will then prove that this subalgebra gives a geometric realisation of a positive part of 0-Schur algebras.

It is well-known that the specialisation of $S_q(n, r)$ at $q = 1$ gives us the classical Schur algebra $S(n, r)$ of type $\mathbb{A}$. Much about the structure and representation theory of $S(n, r)$ is known, see for example [7]. A natural question is to consider the specialisation of $S_q(n, r)$ at $q = 0$, which is called 0-Schur algebra and denoted by $S_0(n, r)$. 0-Schur algebras have been studied in [4, 9, 13]. In this section the 0-Schur algebras will be studied from a different point of view, that is, via a modified version $\theta : \mathbf{H}_q(Q) \to S_q(n, r)$ of the map $\theta : U_{\mathcal{A}}(\text{gl}_n) \to S_v(n, r)$.

We call $\theta(\mathbf{H}_q(Q))$ the positive part of the q-Schur algebra, and denote it by $S_q^+(n, r)$. Denote the specialisation of $S_q^+(n, r)$ at $q = 0$ by $S_0^+(n, r)$. Denote by S_0^{++} the subalgebra of S_0^+, generated by $l_{A,r} = \sum_D e_{A+D}$, where A is a strict upper triangle matrix with its entries non-negative intergers and the sum is taken over all diagnal matrices in $D_{r-\sum_{i,j} a_{ij}}$.

4.1. **A modified version of θ.** For convenience we denote by E_i the element $l_{E_{i,i+1},r}$ in $S_q(n, r)$. We have the following result.

Proposition 4.1. *The elements $E_1, \ldots, E_{n-1}$ satisfy the following modified quantum Serre relations:*

$$E_i^2 E_j - (q + 1)E_i E_j E_i + qE_j E_i^2 = 0 \text{ for } |i - j| = 1 \text{ and}$$
$$E_i E_j - E_j E_i = 0 \text{ for } |i - j| > 1.$$

Proof. We only prove the first equation for $j = i + 1$. The remaining part can be done in a similar way. By Lemma 1.1, we have the following.

$$E_i E_{i+1} = l_{E_{i,i+2},r} + l_{E_{i,i+1}+E_{i+1,i+2},r},$$
$$E_{i+1} E_i = l_{E_{i,i+1}+E_{i+1,i+2},r},$$
$$E_i E_i = (q + 1)l_{2E_{i,i+1},r},$$
$$E_i l_{E_{i,i+2},r} = q l_{E_{i,i+1}+E_{i,i+2},r},$$
$$E_i l_{E_{i,i+1}+E_{i+1,i+2},r} = l_{E_{i,i+1}+E_{i,i+2},r} + (q + 1)l_{2E_{i,i+1}+E_{i+1,i+2},r},$$
$$E_{i+1} E_i E_i = (q + 1)l_{2E_{i,i+1}+E_{i+1,i+2},r}.$$

Therefore,

$$E_i^2 E_{i+1} - (q + 1)E_i E_{i+1} E_i + qE_{i+1} E_i^2 = E_i(E_i E_{i+1} - (q + 1)E_{i+1} E_i) + qE_{i+1} E_i^2 = 0.$$

□

Following Proposition 4.1 and [12], we can now modify the restriction $\theta|_{\mathbf{H_q(Q)}}$ as follows.

$$\theta : \mathbf{H}_q(Q) \to S_q(n,r),$$
$$S_i \mapsto E_i$$

From now on, unless stated otherwise, by θ we mean the modified map $\theta|_{\mathbf{H}_q(Q)}$. The following result is a modified version of Proposition 2.3 in [6] and we will give a direct proof. For any two modules M, N, recall that Hall multiplication of M and N is given by

$$MN = \sum_X F^X_{MN} X,$$

where $F^X_{MN} = |\{U \subseteq X | U \cong N, X/U \cong N\}|$ and where the sum is taken over all the isomorphism classes of modules.

Proposition 4.2. *Let A be a strict upper triangular matrix with entries non-negative integers and let M be the module determined by A. Then*

$$\theta(M) = \begin{cases} l_{A,r} & \textit{if } \sum_{i,j} a_{ij} \le r, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Note that $\mathcal{B} = \{\Pi_{i,j} M(ij)^{x_{ij}} | x_{ij} \in \mathbb{Z}_{\ge 0}\}$ is a PBW-basis of $\mathbf{H}_q(Q)$, where the product is ordered as follows: $M(ij)$ is on the left hand side of $M(st)$ if either $i = s$ and $j > t$, or $i > s$. Let $M \in \mathcal{B}$ be the module determined by A. Supose that $M(st)$ is the left most term with $x_{st} > 0$. We can write $M = M(st) \oplus M'$.

First consider the case $M = M(st)$, that is, M is indecomposable. We may suppose that $t - s > 1$. Then $M = M(s, t-1)M(t-1, t) - M(t-1, t)M(s, t-1)$. By induction on the lenght of M, Lemma 1.1 and a dual version of it, we have

$$\theta(M(s,t-1)M(t-1,t)) = l_{E_{s,t},r} + l_{E_{s,t-1}+E_{t-1,t},r},$$
$$\theta(M(t-1,t)M(s,t-1)) = l_{E_{s,t-1}+E_{t-1,t},r}.$$

Therefore $\theta(M) = \theta(M(s,t-1)M(t-1,t)) - \theta(M(t-1,t)M(s,t-1)) = l_{E_{st},r}$.

Now consider the case that M is decomposable. We use induction on the number of indecomposable direct summands of M. By the assumption we have $M = \frac{q-1}{q^{x_{st}}-1} M(st)M'$. Therefore

$$\begin{aligned} \theta(M) &= \tfrac{q-1}{q^{x_{st}}-1} \theta(M(st))\theta(M') \\ &= \tfrac{q-1}{q^{x_{st}}-1} \textstyle\sum_{D \in D_{r-1}} e_{E_{st}+D} e_{A-E_{st}+D'}, \end{aligned}$$

where D' is the diagonal matrix with non-negative integers as entries such that $\mathrm{co}(E_{st}+D) = \mathrm{ro}(A - E_{st} + D')$. Suppose that e_B appears in the multiplication of $e_{E_{st}+D} e_{A-E_{st}+D'}$ and $(f, h) \in \mathcal{O}_B$. Note that in the minimal projective resolution $Q \to P$ of M', the projective module P_t is not a direct summand of P. Therefore by the definition of the multiplication $e_{E_{st}+D} e_{A-E_{st}+D'}$, we have $f/h \cong M$, that is $B = A$, and the coefficient of e_B is about the possibilities of choosing a submodule, which is isomorphic to P_s, of $P_s^{x_{st}}$. Hence the coefficient is $\frac{q^{x_{st}}-1}{q-1}$ and so $\theta(M) = \sum_D e_{A+D} = l_{A,r}$. This finishes the proof. □

Remark 4.3. *Proposition 2.3 in* [6] *has a minor inaccuracy. Indeed, there is a coefficient missing in front of the image $\theta(M)$. For example, let $n = 3$, $r = 2$ and let M be the module determined by the elementary matrix E_{13}. Then $\theta(M) = v y_{X,r}$, but not $y_{X,r}$, as stated in Proposition 2.3 in* [6]*, here θ is the original map from the quantised enveloping algebra to the Schur algebra $S_v(3,2)$ and $y_{X,r} = \sum_{D \in D_1} [X + D]$.*

4.2. **A homomorphism of algebras $\Gamma : \mathbb{Q}\mathcal{M} \to S_0^+$.** For a given module M, denote by $|M|_{\rm dir}$ the number of indecomposable direct summands of M. Let $\Gamma : \mathbb{Q}\mathcal{M} \to S_0^+$ be the map given by

$$\Gamma(M) = \begin{cases} \theta(M) & \text{if } |M|_{\rm dir} \leq r, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Let X be a module and let $D = \mathrm{diag}(d_1, \cdots, d_n)$ be a diagonal matrix. Write $X = \oplus_{i,j} M(ij)^{x_{ij}}$. By e_{X+D} we mean the basis element in $S_q(n, r)$ corresponding to the matrix with its entry at (i, j) given by $x_{ij} + \delta_{ij} d_i$ for $i \leq j$ and 0 elsewhere, where δ_{ij} are the Kronecker data. Let $\sigma = (\sigma_i)_i : Q \to P$ be an injection of Q into P, where P and Q are projective modules. Then (P, Q) gives a pair of flags (f_1, f_2) with f_2 a subflag of f_1. More precisely, the i-th step of f_1 is given by $\mathrm{Im} P_{\alpha_{n-2}} \cdots P_{\alpha_i}$ for $i \leq n-2$ and the $(n-1)$-th step is given by the vector space associated to vetex $n-1$ of P, where P_{α_j} is the linear map on the arrow α_j from j to $j+1$ for the module P. The i-th step of f_2 is given by $\mathrm{Im} P_{\alpha_{n-2}} \cdots P_{\alpha_i} \sigma_i$ for $i \leq n-2$ and the $(n-1)$-th step is given by $\mathrm{Im}\sigma_{n-1}$. We have the following result.

Theorem 4.4. *The map Γ is a morphism of algebras.*

Proof. The unit in $\mathcal{M}$ is the zero module. By the definition of Γ, it is clear that $\Gamma(0) = \sum_{D \in D_r} e_D$, the unit of $(\Theta_r^u, \circ)$.

We now need only to show

$$\Gamma(M * N) = \Gamma(M) \circ \Gamma(N). \quad (1)$$

Let X be a generic point $\mathcal{E}(M, N)$. Denote the number of indecomposable direct summands of X, M, N by a, b and c, respectively. By the definition of Γ, we can write

$$\Gamma(X) = \sum_{D \in D_{r-a}} e_{X+D},$$

$$\Gamma(M) = \sum_{D' \in D_{r-b}} e_{M+D'},$$

$$\Gamma(N) = \sum_{D'' \in D_{r-c}} e_{N+D''}.$$

We first consider the case where $a > r$. Clearly, in this case $\Gamma(M * N) = 0$ and we claim that $e_{M+D'} e_{N+D''} = 0$ for any $D' \in D_{r-b}$ and $D'' \in D_{r-c}$. In fact suppose that $e_{L+D'''}$ appears in the multiplication, where L is a module and D''' is a diagonal matrix. Then $|L|_{\rm dir} \leq r$, and L is a degeneration of X. Since Q is linearly oriented, $|L|_{\rm dir} \geq a$. This is a contradiction. Therefore $e_{M+D'} e_{N+D''} = 0$, and so

$$\Big(\sum_{D' \in D_{r-b}} e_{M+D'}\Big) \circ \Big(\sum_{D'' \in D_{r-c}} e_{N+D''}\Big) = 0.$$

This proves the equation (1) for the case $a > r$.

Now suppose that $a \leq r$. Note that if $(D', D'') \neq (C', C'')$, where $D', C' \in D_{r-b}$ and $D'', C'' \in D_{r-c}$, then $e_{M+D'} \circ e_{N+D''} \neq e_{M+C'} \circ e_{N+C''}$. By Proposition 3.7, we know that if $e_{M+D'} \circ e_{N+D''} \neq 0$, then $e_{M+D'} \circ e_{N+D''} = e_{X+D}$ for some $D \in D_{r-a}$.

On the other hand, we can show that for any e_{X+D} appearing in the image of X under Γ, there exist $D' \in D_{r-b}$ and $D'' \in D_{r-c}$ such that $e_{M+D'} \circ e_{N+D''} = e_{X+D}$. Suppose that

$$0 \longrightarrow Q \xrightarrow{\sigma} P \xrightarrow{\tau} X \longrightarrow 0$$

is the minimal projective resolution of X. Write $D = \mathrm{diag}(d_1, \cdots, d_n)$ and let Y be the projective module $\oplus_i P_i^{d_i}$. Then the pair of flags in $\mathcal{F} \times \mathcal{F}$, determined by $(P \oplus Y, Q \oplus Y)$, is in the orbit $\mathcal{O}_{X+D}$. We have the following diagram where each square commutes and all rows and columns are short exact sequences,

$$\begin{array}{ccccc}
\mathrm{Ker}\lambda & \longrightarrow & Q\oplus Y & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\begin{pmatrix}\sigma & 0\\ 0 & I\end{pmatrix}} & & \downarrow \\
K & \longrightarrow & P\oplus Y & \xrightarrow{(p\tau\ 0)} & M, \\
\downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle(\tau\ 0)} & & \downarrow \\
N & \xrightarrow[i]{} & X & \xrightarrow[p]{} & M
\end{array}$$

where I is the identity map on Y, $K = \mathrm{Ker}\,(\,p\tau\ 0\,)$ and $\lambda = (\tau 0)|_K$. Let $Z' \xrightarrow{I} Z'$ be the maximal contractible piece of the projective resolution

$$0 \longrightarrow K \longrightarrow P\oplus Y \xrightarrow{(p\tau\ 0)} M \longrightarrow 0$$

of M, and let $Z'' \xrightarrow{I} Z''$ be the maximal contractible piece of the projective resolution

$$0 \longrightarrow \mathrm{Ker}\lambda \longrightarrow K \longrightarrow N \longrightarrow 0$$

of N. Write

$$Z' = \oplus_i P_i^{d_i'} \text{ and } Z'' = \oplus_i P_i^{d_i''},$$

and let

$$D' = \mathrm{diag}(d_1', \cdots, d_n') \text{ and } D'' = \mathrm{diag}(d_1'', \cdots, d_n'').$$

Then $e_{M+D'} \circ e_{N+D''} = e_{X+D}$. Therefore,

$$\sum_{D'\in D_{r-b}} e_{M+D'} \circ \sum_{D''\in D_{r-c}} e_{N+D''} = \sum_{D\in D_{r-a}} e_{X+D}.$$

This proves the equations (1), and so the proof is done. □

The following result is a direct consequence of Theorem 4.4.

Corollary 4.5. $\mathrm{Ker}\Gamma = \mathbb{Q}\text{-Span}\{M||M|_{\mathrm{dir}} > r\}$.

4.3. A geometric realisation of 0-Schur algebras.

Theorem 4.6. *$S_0^+(n,r) \cong \mathbb{Q}S_0^{++}$ as algebras.*

Proof. Denote by θ_0 the specialisation of θ to 0, that is, $\theta_0 : \mathbf{H}_0(Q) \to S_0(n,r)$. We have $\mathrm{Ker}\theta_0 = \mathbb{Q}\text{-Span}\{M||M|_{\mathrm{dir}} > r\} = \mathrm{Ker}\Gamma$, where Γ is as in Theorem 4.4. Now the proof follows from the following commutative diagram.

$$\begin{array}{ccccc}
\mathrm{Ker}\theta_0 & \longrightarrow & \mathbf{H}_0(Q) & \xrightarrow{\theta_0} & S_0^+(n,r) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \vdots{\scriptstyle\cong} \\
\mathrm{Ker}\Gamma & \longrightarrow & \mathcal{M} & \xrightarrow{\Gamma} & S_0^+.
\end{array}$$

□

As a direct consequence of Theorem 4.6, we obtain a multiplicative basis of the positive part of 0-Schur algebras, in the sense that the multiplication of any two basis elements is either a basis element or zero.

Corollary 4.7. *The elements in $\{l_{A,r}|\ A$ is an strictly upper triangular matrix in $\bigcup_{s\le r}\Theta_s^u\}$ form a multiplicative basis of $S_0^+(n,r)$.*

Under the map Γ, this multiplicative basis $\{l_{A,r} |$ for any $A \in \bigcup_{s \leq r} \Theta_s^u\}$ is the image of the multiplicative basis for $\mathbf{H}_0(Q)$ studied in [10]. By Theorem 7.2 in [10], the multiplicative basis for $\mathbf{H}_0(Q)$ is the specialisation of Lusztig's canonical basis for a two-parameter quantization of the universal enveloping algebra of gl_n given in [14]. Thus we can consider the basis $\{l_{A,r} |\ A$ is a strictly upper triangular matrix in $\bigcup_{s \leq r} \Theta_s^u\}$ as a subset of a specilization of the canonical basis.

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References

[1] Beilinson, A. A., Lusztig, G. and MacPherson, R., *A geometric setting for the quantum deformation of* GL_n, Duke Math. J. 61 (1990), no. 2, 655–677.

[2] Dipper, R., James, G., *The q-Schur algebra*, Proc. London Math. Soc. (3) 59 (1989), no. 1, 23–50

[3] Dipper, R. and James, G., *q-tensor space and q-Weyl modules*, Trans. Amer. Math. Soc. 327 (1991), no. 1, 251–282.

[4] Donkin, S., *The q-Schur algebra*, London Mathematical Society Lecture Note Series, 253. Cambridge University Press, Cambridge, 1998. x+179 pp. ISBN: 0-521-64558-1.

[5] Du, J., *A note on quantised Weyl reciprocity at roots of unity*, Algebra Colloq. 2 (1995), no. 4, 363–372.

[6] Green, R. M., *q-Schur algebras as quotients of quantised enveloping algebras*, J. Algebra 185 (1996), no. 3, 660–687.

[7] Green, J. A., *Polynomial representations of* GL_n. Algebra, Carbondale 1980 (Proc. Conf., Southern Illinois Univ., Carbondale, Ill., 1980), pp. 124–140, Lecture Notes in Math., 848, Springer, Berlin, 1981.

[8] Jensen, B. T. and Su, X., *Singularities in derived categories*, Manuscripta math. 117, 475-490(2005).

[9] Krob, D. and Thibon, J. Y., *Noncommutative symmetric functions IV*, Quantum linear groups and Hecke algebras at $q = 0$. J. Algebraic Combin. 6 (1997), no. 4, 339–376.

[10] Reineke, M., *Generic extensions and multiplicative bases of quantum groups at* $q = 0$, Represent. Theory 5 (2001), 147–163.

[11] Ringel, C. M., *Hall algebras*, Topics in algebra, Part 1 (Warsaw, 1988), 433–447, Banach Center Publ., 26, Part 1, PWN, Warsaw, 1990.

[12] Ringel, C. M., *Hall algebras and quantum groups*, Invent. Math. 101 (1990), no. 3, 583–591.

[13] Solomon, L., *A decomposition of the group algebra of a finite Coxeter group*, J. Algebra 9 (1968), 220–239.

[14] Takeuchi, M., *A two parameter quantization of* Gl_n, Proc. Japan, Acad. 66 (1990), 112-114.

[15] Zwara, G., *Degenerations of finite-dimensional modules are given by extensions*, Compositio Math. 121 (2000), no. 2, 205–218.

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