FILTRATIONS IN ABELIAN CATEGORIES WITH A TILTING OBJECT OF HOMOLOGICAL DIMENSION TWO

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Abstract. We consider filtrations of objects in an abelian category $\mathcal{A}$ induced by a tilting object T of homological dimension at most two. We define three disjoint subcategories with no maps between them in one direction, such that each object has a unique filtation with factors in these categories. This filtration coincides with the the classical two-step filtration induced by torsion pairs in dimension one. We also give a refined filtration, using the derived equivalence between the derived categories of $\mathcal{A}$ and the module category of $End_{\mathcal{A}}(T)^{op}$. The factors of this filtration consist of kernel and cokernels of maps between objects which are quasi-isomorphic to shifts of $End_{\mathcal{A}}(T)^{op}$-modules via the derived equivalence $\mathbb{R}Hom_{\mathcal{A}}(T,-)$.

Introduction

Let k be a field and let $\mathcal{A}$ be either a noetherian abelian k-category with finite homological dimension and hom-finite bounded derived category $\mathcal{D}^b(\mathcal{A})$, or let $\mathcal{A}$ be the module category of a finite dimensional algebra A. In the first case we will assume that there is a locally noetherian abelian Grothendieck k-category $\mathcal{A}'$ with finite homological dimension such that $\mathcal{A} \subseteq \mathcal{A}'$ is the subcategory of noetherian objects. For more information about tilting in noetherian categories, please see for example Baer [Ba], Bondal [Bo] and Keller [K1].

Let $T \in \mathcal{A}$ be a tilting object. That is, $T \in \mathcal{A}$ is an object without self-extensions in non-zero degrees and which induces a pair of mutually inverse derived equivalences

$$F = \mathbb{R}Hom(T,-) : \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{D}^b(B) \text{ and } G : \mathcal{D}^b(B) \longrightarrow \mathcal{D}^b(\mathcal{A}),$$

where $B = End(T)^{op}$ is a finite dimensional algebra and $\mathcal{D}^b(B) = \mathcal{D}^b(modB)$ is the bounded derived category of finite dimensional left B-modules.

Let $\tau_T M$ denote the trace of T in M. If the projective dimension of T is one (i.e. $Ext^i_{\mathcal{A}}(T,-) = 0$ for $i > 1$), then there is a canonical short exact sequence

$$0 \longrightarrow \tau_T M \longrightarrow M \longrightarrow M/\tau_T M \longrightarrow 0, \qquad (1)$$

where $\tau_T M \in FacT$, $M/\tau_T M \in RejT$, $FacT$ is the torsion subcategory of objects N with a surjective homomorphism $T^n \to N$ for some $n > 0$, and $RejT$ is the torsion free subcategory of objects N with $Hom(T,N) = 0$. It is well known that the sequence (1) generalises when T has higher homological dimension to a torsion theory in the derived category, but our interest is in constructing filtrations in the abelian category. Our aim is to generalize the sequence (1) to abelian categories with a tilting object with higher homological dimension. As a first step towards that goal, we present in this paper the solution for homological dimension two, that is, to abelian categories $\mathcal{A}$ with a tilting object T with $H^iFX = Ext^i_{\mathcal{A}}(T,X) = 0$ for all $X \in \mathcal{A}$ and $i \geq 3$. The sequence will be replaced by a filtration with three terms, and will coincide with the sequence (1) when the projective dimension of T is one.

Let $F^i = H^iF = Ext^i_{\mathcal{A}}(T,-)$ and let $G_i = H_iG$. Let $\mathcal{G}_i$ denote the subcategory of $modB$ consisting of modules X with $G_jX = 0$ for $i \neq j$. Similarly, we define $\mathcal{F}^i$ to be the subcategory of $\mathcal{A}$ consisting of objects X with $F^jX = 0$ for $i \neq j$. It is clear that the categories $\mathcal{F}^i$ are closed under extensions and pairwise disjoint, and there are induced equivalences $F_{|\mathcal{F}^i} : \mathcal{F}^i \longrightarrow \mathcal{G}_i[-i]$ and $G_{|\mathcal{G}_i} : \mathcal{G}_i \longrightarrow \mathcal{F}^i[i]$, where $[i]$ denotes the usual shift in the derived category. In other words,

$$\bigcup \mathcal{F}^i \subseteq \mathcal{A}$$

are objects in $\mathcal{A}$ which are B-modules via the derived equivalence. In dimension one we have $FacT = \mathcal{F}^0$ and $RejT = \mathcal{F}^1$, and the short exact sequence (1) shows how to canonically reconstruct $\mathcal{A}$ using the subcategories $\mathcal{F}^i$. A natural thing to try in higher homological dimensions is to consider filtrations with subfactors in the disjoint subcategories $\mathcal{F}^i$, see Tonolo [T] for this approach. Unfortunately, such filtrations fail to exist in general, even in dimension two, and we give an example in Section 4. The problem is that the categories $\mathcal{F}^i$ are two small to filter any object in $\mathcal{A}$. To ensure that any object can be filtered we enlarge the categories $\mathcal{F}^i$ to categories $\mathcal{E}^i$, which are still closed under extensions and pairwise disjoint, and are large enough to allow a unique filtration for any object in $\mathcal{A}$. Moreover, the categories $\mathcal{E}^i$ are explicitely constructed, using the categories $\mathcal{F}^i$.

Let $\mathcal{K}^0$ be the full subcategory of objects which are cokernels of monomorphisms from objects in $\mathcal{F}^2$ to objects in $\mathcal{F}^0$, let $\mathcal{K}^2$ be the full subcategory of objects which are kernels of epimorphisms from objects in $\mathcal{F}^2$ to objects in $\mathcal{F}^0$, and let $\mathcal{K}^1 = \mathcal{F}^1$. Note that, $\mathcal{F}^0 \subseteq \mathcal{K}^0 \subseteq KerF^2$ and $\mathcal{F}^2 \subseteq \mathcal{K}^2 \subseteq KerF^0$. Let $\mathcal{E}^i$ be the extension closure of $\mathcal{K}^i$, that is, $\mathcal{E}^i \subseteq \mathcal{A}$ is the smallest subcategory closed under extensions, and containing $\mathcal{K}^i$. Note that $\mathcal{E}^1 = \mathcal{K}^1 = \mathcal{F}^1$. The categories $\mathcal{E}^i$ have the following key properties.

Theorem 1. *If $j > i$, then $Hom(\mathcal{E}^i, \mathcal{E}^j) = 0$. In particular, the subcategories $\mathcal{E}^0$, $\mathcal{E}^1$ and $\mathcal{E}^2$ are pairwise disjoint.*

In abelian categories with tilting object of homological dimension two, we have the following filtration, generalizing the short exact sequence (1).

Theorem 2. *Let $T \in \mathcal{A}$ be a tilting object with homological dimension at most two and let $X \in \mathcal{A}$. Then there is a unique and functorial filtration $0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 = X$ with $X_{i+1}/X_i \in \mathcal{E}^i$ for $i = 0, 1, 2$.*

If the homological dimension of T is one, then $\mathcal{E}^0 = \mathcal{F}^0$ and $\mathcal{E}^2 = 0$, and so we recover the sequence (1) as a corollary.

Corollary 3. *Let $T \in \mathcal{A}$ be a tilting object with homological dimension one and let $X \in \mathcal{A}$. Then there is a canonical short exact sequence $0 \longrightarrow X_1 \longrightarrow X \longrightarrow X/X_1 \longrightarrow 0$ with $X_1 \in \mathcal{E}^0$ and $X/X_1 \in \mathcal{E}^1$.*

We also prove the existence of the following refined filtration. The proof is constructive and allows us to compute the filtration in concrete examples.

Theorem 4. *Let $T \in \mathcal{A}$ be a tilting module with homological dimension two and let $X \in \mathcal{A}$. Then there is a filtration $(0) = Z_0 \subseteq ... \subseteq Z_n \subseteq Y_n \subseteq ... \subseteq Y_0 = X$ with $Y_i/Y_{i+1} \in \mathcal{K}^2$, $Z_{i+1}/Z_i \in \mathcal{K}^0$ and $Y_n/Z_n \in \mathcal{K}^1$.*

By the uniqueness in Theorem 2, we remark that $X_1 = Z_n$ and $X_2 = Y$. Also, the theorem and its proof show how the the objects in $\mathcal{E}^i$ are constructed from the categories $\mathcal{F}^i$. An important consequence is an explicit and canonical reconstruction of $\mathcal{A}$ using the subcategories $\mathcal{F}^i$.

Examples of tilting objects T satisfying the hypothesis in the theorems above include the k-dual DA for an algebra of global dimension two. Another class of examples are tilting sheaves over smooth projective surfaces, for example over the projective plane.

A natural question to ask is when the two filtrations in the theorems coincide. That is, when is $\mathcal{K}^i$ closed under extensions. We discuss the case of $\mathcal{K}^0$ in the last section of this paper. This question is related to the notion of a compatible t-structure, introduced by Keller-Vossieck [KV1] (see also Section 7 in [K1]). If T induces a compatible t-structure, then in particular $\mathcal{F}^2 = \mathcal{K}^2 = \mathcal{E}^2$.

The remainder of this paper is organized as follows. In Section 1 we analyse the equivalence $GF \cong 1_{\mathcal{A}}$. The proof of Theorem 4 is given in Section 2, and the proof of Theorem 2 is given in Section 3. In Section 4 we discuss the extension closure of $FacT$ where T is a tilting module in a category $\mathcal{A}$ of modules of a finite dimensional algebra.

1. Some homological algebra

Let $T \in \mathcal{A}$ be a tilting object. We will assume that T has homological dimension at most two, that is $F^iX = Ext^i_{\mathcal{A}}(T, X) = 0$ for all $X \in \mathcal{A}$ and $i \geq 3$. In this section we prove some general homological properties for tilting objects with homological dimension two.

Lemma 5. *If T has homological dimension two, then $G_iM = 0$ for all $M \in modB$ and $i \geq 3$.*

Proof. Let $M \in modB$. There is an exact triangle

$$\tau_{\geq 3}GM \longrightarrow GM \longrightarrow \tau_{<3}GM \longrightarrow \tau_{\geq 3}GM[1],$$

where τ denotes truncation. If we apply F we get the triangle

$$F\tau_{\geq 3}GM \longrightarrow M \longrightarrow F\tau_{<3}GM \longrightarrow F\tau_{\geq 3}GM[1]$$

Since T has homological dimension 2, we see that $H^i F\tau_{\geq 3}GM = 0$ for $i \geq 0$. So the map $M \longrightarrow F\tau_{<3}GM$ induces isomorphisms in cohomologies, and is therefore a quasi-isomorphism. This shows that $F\tau_{\geq 3}GM = 0$ and therefore $\tau_{\geq 3}GM = 0$. Hence $G_jM = 0$ for all $j \geq 3$. □

We let $J^i_j = G_jF^i$. Using the quasi-isomorphism $GF(X) \cong X$ for $X \in \mathcal{A}$ we get the following double complex.

Lemma 6. *Let $X \in \mathcal{A}$. Then X is quasi-isomorphic to the total complex of a double complex*

$$\begin{array}{ccccccccccccc}
 & & & & & & \cdots & \longrightarrow & T^2_2 & \longrightarrow & T^2_1 & \longrightarrow & T^2_0 \longrightarrow 0 \\
 & & & & & & & /\!\!/ & \downarrow & & & & \\
 & & & \cdots & \longrightarrow & T^1_2 & \longrightarrow & T^1_1 & \longrightarrow & T^1_0 & \longrightarrow & 0 & \\
 & & & & & \downarrow & /\!\!/ & & & & & & \\
\cdots & \longrightarrow & T^0_2 & \longrightarrow & T^0_1 & \longrightarrow & T^0_0 & \longrightarrow & 0 & & & &
\end{array}$$

where the horizontal homology is J^i_jX for all $0 \leq i \leq 2$ and $j \geq 0$, the double lines indicate total degree 0, and T^i_j is a finite direct sum of summands of T.

Proof. The complex FX has cohomology in degrees 0, 1 and 2. By taking projective resolutions of the cohomology, we construct a double complex

$$\begin{array}{ccccccccccccc}
 & & & & & & \cdots & \longrightarrow & P^2_2 & \longrightarrow & P^2_1 & \longrightarrow & P^2_0 \longrightarrow 0 \\
 & & & & & & & /\!\!/ & \downarrow & & & & \\
 & & & \cdots & \longrightarrow & P^1_2 & \longrightarrow & P^1_1 & \longrightarrow & P^1_0 & \longrightarrow & 0 & \\
 & & & & & \downarrow & /\!\!/ & & & & & & \\
\cdots & \longrightarrow & P^0_2 & \longrightarrow & P^0_1 & \longrightarrow & P^0_0 & \longrightarrow & 0 & & & &
\end{array}$$

with horizontal cohomology H^*FX and total complex quasi-isomorphic to FX. Then the lemma follows by applying the functor G_0 and using the isomorphism $GHom_{\mathcal{A}}(T,T) = G_0Hom_{\mathcal{A}}(T,T) \cong T$. □

We show that the horizontal homology of the double complex of Lemma 6 vanishes almost everywhere.

Lemma 7. *We have $J^i_jX = 0$ for*

a) $j > 2$,
b) $i = 0, j = 1, 2$, *and*
c) $i = 2, j = 0, 1$.

Proof. Part a) follows since $G_i = 0$ for $i \geq 3$, by Lemma 5.

Let $i = 0$ and $j > 0$. Let Y be the total complex of the two top rows of the double complex in Lemma 6, let Z be the bottom row, let W be the middle row, and let V be the top row. We have a triangle

$$Z \longrightarrow X \longrightarrow Y \longrightarrow Z[1]$$

and therefore $H_{j+1}Y \cong H_jZ$ since $H_jX = 0$ for $j \neq 0$. Moreover, there is the triangle

$$W \longrightarrow Y \longrightarrow V \longrightarrow W[1]$$

and therefore an exact sequence

$$H_{j+1}W \longrightarrow H_{j+1}Y \longrightarrow H_{j+1}V.$$

By Part a) we have $H_{j+1}W = J^j_{j+2}X = 0 = J^0_{j+3}X = H_{j+1}V$ and therefore $H_{j+1}Y \cong H_jZ = J^0_jX = 0$. This proves Part b).

Part c) is similar and is left to the reader. □

In the cases that the horizontal homology does not vanish, we get the following edge effect.

Lemma 8. *We have a complex*

$$0 \longrightarrow J^1_2X \longrightarrow J^0_0X \longrightarrow X \longrightarrow J^2_2X \longrightarrow J^1_0X \longrightarrow 0,$$

with homology at X equal to J^1_1X, and vanishing homology elsewhere.

Proof. Let Y, Z, V and W be as in the proof of the previous lemma. By computing long exact sequence for the triangle

$$Z \longrightarrow X \longrightarrow Y \longrightarrow Z[1]$$

we get an exact sequence

$$0 \longrightarrow H_1Y \longrightarrow H_0Z \longrightarrow X \longrightarrow H_0Y \longrightarrow 0,$$

and also $H_{-1}Y = 0$. By computing long exact sequence for the triangle

$$W \longrightarrow Y \longrightarrow V \longrightarrow W[1]$$

we get an exact sequence

$$0 \longrightarrow H_0W \longrightarrow H_0Y \longrightarrow H_0V \longrightarrow H_{-1}W \longrightarrow 0,$$

and an isomorphism $H_1W \cong H_1Y$. By splicing these sequences together we get the complex

$$0 \longrightarrow H_1Y \longrightarrow H_0Z \longrightarrow X \longrightarrow H_0V \longrightarrow H_{-1}W \longrightarrow 0$$

with homology at X equal to H_0W and vanishing homology elsewhere. The lemma follows since $H_1Y = J^1_2X$, $H_0Z = J^0_0X$, $H_0W = J^1_1X$, $H_0V = J^2_2X$ and $H_{-1}W = J^1_0X$. □

We note the following consequence.

Lemma 9. *Let $X \in \mathcal{A}$.*

a) *If $X \in KerF^0$ then there is an exact sequence $0 \longrightarrow J^1_1X \longrightarrow X \longrightarrow J^2_2X \longrightarrow J^1_0X \longrightarrow 0$.*

b) *Any* $X \in KerF^1$ *decomposes uniqely as* $X = J_0^0 X \oplus J_2^2 X$.
c) *If* $X \in KerF^2$ *then there is an exact sequence* $0 \longrightarrow J_2^1 X \longrightarrow J_0^0 X \longrightarrow X \longrightarrow J_1^1 X \longrightarrow 0$.

Proof. If $X \in KerF^0$ then $J_0^0 X = 0$ and the exact sequence in Part *a*) is a special case of the exact sequence in Lemma 8. Part *c*) is similar. If $X \in KerF^1$, then the middle row of the double complex in Lemma 6 vanishes, and the total complex of the double complex decomposes into the direct sum of the top and bottom row. So by taking homology, we see that $X = J_0^0 X \oplus J_2^2 X$. □

We point out that the analysis we have done in this section is considerably more complicated for higher homological dimension. Also, several of the proofs can be done quite efficiently using spectral sequences, we have however chosen more conceptual arguments using elementary properties of derived categories. For a systematic study of tilting using double complexes and spectral sequences, see Keller-Vossieck [KV2] [KV1] and Brenner-Butler [BB].

2. Proof of Theorem 4

We start by computing the subcategories $\mathcal{F}^i$ and $\mathcal{G}_i$ for $i \neq 1$.

Lemma 10. *For* $i \neq 1$, *the image of the functor* $F^i : \mathcal{A} \longrightarrow modB$ *is dense in* $\mathcal{G}_i$. *Similarly, the image of the functor* $G_i : modB \longrightarrow \mathcal{A}$ *is dense in* $\mathcal{F}^i$.

Proof. We prove the lemma for F^0. The proofs for the other functors are similar, and are left to the reader.

Since the homology in degree 1 and 2 in the total complex of the double complex in Lemma 6 is zero, we see that $GF^0X \cong G_0F^0X$ for any $X \in \mathcal{A}$, and so GF^0X can only have homology in degree 0. Therefore $F^0X \in \mathcal{G}_0$. Similarly, we have $G_0M \in \mathcal{F}^0$ for a B-module M. Let $M \in \mathcal{G}_0$. Then $M \cong FGM \cong FG_0M \cong F^0G_0M$, which shows that the imageof F^0 is dense in $\mathcal{G}_0$. □

The following lemma is an easy application of long exact sequence in homology.

Lemma 11. *If* $X \in \mathcal{K}^0$ *is obtained as a cokernel of an injection*

$$0 \longrightarrow J_2 \longrightarrow J_0 \longrightarrow X \longrightarrow 0,$$

$J_2 \in \mathcal{F}^2, J_0 \in \mathcal{F}^0$, *then* $F^0X \cong F^0J_0$, $F^1X \cong F^2J_2$ *and* $F^2X = 0$. *Similarly, if* $Y \in \mathcal{K}^2$ *is obtained as a kernel of a surjection*

$$0 \longrightarrow Y \longrightarrow L_2 \longrightarrow L_0 \longrightarrow 0,$$

$L_2 \in \mathcal{F}^2, L_0 \in \mathcal{F}^0$, *then* $F^0Y = 0$, $F^1Y \cong F^0L_0$ *and* $F^2Y \cong F^2L_2$.

We are ready to prove the main inductive step in the construction of the filtration.

Lemma 12. *Let $X \in \mathcal{A}$. We have a filtration $Z \subseteq Y \subseteq X$ with $Z \in \mathcal{K}^0$, $X/Y \in \mathcal{K}^2$ and $Y/Z \cong J_1^1 X$.*

Proof. Let Z be the image of the map $J_0^0 X \longrightarrow X$, and Y the kernel of the map $X \longrightarrow J_2^2 X$ in the complex in Lemma 8. Then from Lemma 8 we see that $Y/Z \cong J_1^1 X$. Furthermore, $Z \in \mathcal{K}^0$, $X/Y \in \mathcal{K}^2$ by Lemma 10. □

For $X \in \mathcal{A}$ let $d(X)$ be the dimension of $F^1 X$. We have $d(X) = d(J_1^1 X)$ if $X \in \mathcal{F}^1$.

Lemma 13. *Let $X \in \mathcal{A}$. Then $d(J_1^1 X) \leq d(X)$. Moreover, if $d(J_1^1 X) = d(X)$ then $J_1^1 X \in \mathcal{F}^1$.*

Proof. Let $X \notin \mathcal{F}^1$ with $d(X) > 0$. Let $Z \subseteq Y \subseteq X$ be the canonical filtration of Lemma 12. There are long exact sequences in homology

$$F^1 Z \longrightarrow F^1 Y \longrightarrow F^1(Y/Z) \longrightarrow 0$$

and

$$0 \longrightarrow F^1 Y \longrightarrow F^1 X \longrightarrow F^1(X/Y)$$

which shows that $d(J_1^1 X) = d(Y/Z) \leq d(Y) \leq d(X)$.

Now assume that $d(J_1^1 X) = d(X)$. Then from the above long exact sequences we see that $F^1 Z \longrightarrow F^1 Y$ is the zero map. Since the inclusion $Z \subseteq X$ factors as $Z \subseteq Y \subseteq X$, we also have that $F^1 Z \longrightarrow F^1 X$ is zero. So there is the long exact sequence

$$0 \longrightarrow F^0 Z \longrightarrow F^0 X \longrightarrow F^0(X/Z) \longrightarrow F^1 Z \longrightarrow 0$$

Now $F^0 Z \cong F^0 J_0^0 X$ by Lemma 11 and $F^0 J_0^0 X \cong F^0 X$ by the essential surjectivity of F^0. Hence $F^1 Z \cong F^0(X/Y) \in \mathcal{G}_0$. But we also have $F^1 Z \cong F^2 J_2^1 X \in \mathcal{G}_2$, by Lemma 11. But then $F^1 Z \in \mathcal{G}_0 \cap \mathcal{G}_2 = 0$, and so $F^1 Z = 0$, and therefore $F^2 J_2^1 X = 0$. Hence $J_2^1 X = 0$ since $J_2^1 X \in \mathcal{F}^2$. Similarly, $J_0^1 X = 0$. Therefore $GF^1 X$ has homology $J_1^1 X$ concentrated in degree 1, and so $J_1^1 X \in \mathcal{F}^1$. □

We can now prove the second theorem stated in the introduction.

Theorem 14. *Let $T \in \mathcal{A}$ be a tilting object with projective dimension two. For any $X \in \mathcal{A}$ there is a filtration $(0) = Z_0 \subseteq ... \subseteq Z_n \subseteq Y_n \subseteq ... \subseteq Y_0 = X$ with $Y_i/Y_{i+1} \in \mathcal{K}^2$, $Z_{i+1}/Z_i \in \mathcal{K}^0$ and $Y_n/Z_n \in \mathcal{K}^1$.*

Proof. Let $Z_1 = Z$ and $Y_1 = Y$ be given by Lemma 12. Given Z_i and Y_i, with $Y_i/Z_i \notin \mathcal{F}^1$ we construct Z_{i+1} and Y_{i+1} by applying Lemma 12 to Y_i/Z_i, and obtain the filtration

$$(0) = Z_0 \subseteq ... \subseteq Z_{i+1} \subseteq Y_{i+1} \subseteq ... \subseteq Y_0 = X$$

with $Y_i/Y_{i+1} \in \mathcal{K}^2$, $Z_{i+1}/Z_i \in \mathcal{K}^0$, and $Y_{i+1}/Z_{i+1} = J_1^1 Y_i/Z_i$. By Lemma 13, this procedure must eventually stop. □

As a consequence of the construction we see that $Y_t/Z_t \cong (J_1^1)^t X$ and that for any X, there exists a smallest integer $t > 0$ such that $(J_1^1)^t X = (J_1^1)^{t+1} X$. It would be interesting to find a method to compute this number in general.

We end this section by showing the existence of the filtration in Theorem 2. A different construction, which also shows uniqueness and functoriality, will be given in the next section.

Corollary 15. *There is a filtration* $0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 = X$ *with* $X_{i+1}/X_i \in \mathcal{E}^i$ *for* $i = 0, 1, 2$.

Proof. With the notation of the previous theorem, let $X_1 = Z_n, X_2 = Y_n$ and $X_3 = X$. Then X_3/X_2 is in the extension closure of $\mathcal{K}^2$, $X_2/X_1 \in \mathcal{K}^1 = \mathcal{E}^1$, and X_1/X_0 is in the extension closure of $\mathcal{K}^0$. The proof follows. □

We will give a different construction of this filtration in the next section.

3. Proof of Theorem 2

We will now give an alternative description of the filtration from Corollary 15, which will show that the filtration is functorial, and so Theorem 2 follows.

We show that $\mathcal{K}^0 = FacT$. That is, $\mathcal{E}^0$ consists of objects having filtrations with subfactors from $FacT$.

Lemma 16. $\mathcal{K}^0 = FacT$

Proof. Assume that $X \in FacT$. Then there is an approximation

$$0 \longrightarrow Y \longrightarrow T^n \longrightarrow X \longrightarrow 0$$

with $Y \in KerF^1$. By Lemma 9 and Lemma 10, Y decomposes as $Y = Y_0 \oplus Y_2$ where $Y_0 \in \mathcal{F}^0$ and $Y_2 \in \mathcal{F}^2$. The pushout of the projection $Y \longrightarrow Y_2$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_0 & = & Y_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \longrightarrow & T^n & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_2 & \longrightarrow & T' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

gives us a short exact sequence

$$0 \longrightarrow Y_2 \longrightarrow T' \longrightarrow X \longrightarrow 0$$

with $T' \in \mathcal{F}^0$ which shows that $X \in \mathcal{K}^0$.

Conversely, assume that $X \in \mathcal{K}^0$. Then there is an epimorphism $T' \longrightarrow X$ with $T' \in \mathcal{F}^0$. The proof is complete if we can show that $T' \in FacT$. Let $T^n \longrightarrow T'$ be an approximation. Then $Hom_{\mathcal{A}}(T, T^n) \longrightarrow Hom_{\mathcal{A}}(T, T')$ is surjective, and so

$$T^n \cong G_0 Hom_{\mathcal{A}}(T, T^n) \longrightarrow G_0 Hom_{\mathcal{A}}(T, T') \cong T'$$

is surjective, since G_0 is right exact, and the lemma follows. □

Lemma 17. *$\mathcal{E}^0$ is closed under factors.*

Proof. Any quotient of an object filtered in $FacT$ is also filtered in $FacT$, and so $\mathcal{E}^0$ is closed under factors. □

We consider the subcategories $KerF^i$. First note that $KerF^i$ is closed under extensions, $KerF^0$ is closed under subobjects and $KerF^2$ is closed under factors. For a subcategory $\mathcal{C} \subseteq \mathcal{A}$ and $X \in \mathcal{A}$, let $\tau_{\mathcal{C}} X$ denote the trace of $\mathcal{C}$ in X.

Lemma 18. *The following hold.*

a) *For $X \in KerF^0$ there is a canonical exact sequence $0 \longrightarrow X'' \longrightarrow X \longrightarrow X' \longrightarrow 0$ with $X'' = \tau_{\mathcal{E}^1} X \in \mathcal{E}^1$ and $X' \in \mathcal{E}^2$.*

b) *For $X \in KerF^2$ there is a canonical exact sequence $0 \longrightarrow X'' \longrightarrow X \longrightarrow X' \longrightarrow 0$ with $X'' = \tau_{\mathcal{E}^0} X \in \mathcal{E}^0$ and $X' \in \mathcal{E}^1$.*

Proof. Let $X \in KerF^0$. Then $\tau_{\mathcal{E}^1} X \in KerF^0$, since $X \in KerF^0$, and $\tau_{\mathcal{E}^1} X \in KerF^2$ since there is a surjection $E \longrightarrow \tau_{\mathcal{E}^1} X$ with $E \in \mathcal{E}^1 \subseteq KerF^2$. This shows that $\tau_{\mathcal{E}^1} X \in \mathcal{E}^1$. There is a short exact sequence

$$0 \longrightarrow \tau_{\mathcal{E}^1} X \longrightarrow X \longrightarrow Z \longrightarrow 0,$$

where $Z = X/\tau_{\mathcal{E}^1} X$. By taking pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_{\mathcal{E}^1} X & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tau_{\mathcal{E}^1} X & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

along any nonzero map $E \longrightarrow Z$ from $\mathcal{E}^1$ to Z and using that $\mathcal{E}^1$ is closed under extensions we see that $\tau_{\mathcal{E}^1} Z = 0$. We now show that $Z \in KerF^0$. There is an exact sequence

$$0 \longrightarrow F^0 Z \longrightarrow F^1 \tau_{\mathcal{E}^1} X \longrightarrow Z' \longrightarrow 0,$$

where Z' is the cokernel of the inclusion $F^0 Z \longrightarrow F^1 \tau_{\mathcal{E}^1} X$. Using the functor G we get a long exact sequence with zero terms except for

$$0 \longrightarrow \tau_{\mathcal{E}^1} X \longrightarrow G_1 Z' \longrightarrow J_0^0 Z \longrightarrow 0.$$

In particular, $G_i Z' = 0$ for $i \neq 1$ and so $G_1 Z' \in \mathcal{E}^1$. If $J_0^0 Z$ is nonzero, then we have a nonzero map $G_1 Z' \to J_0^0 Z \to Z$ contradicting that $\tau_{\mathcal{E}^1} Z = 0$. Hence $J_0^0 Z = 0$ and so $F^0 Z = 0$ since $F^0 Z \in \mathcal{G}_0$. This proves that $Z \in KerF^0$.

Let $Y_0 = Z$ and let $Y_n = J_1^1 Y_{n-1}$ for all $n > 0$. By Lemma 9 there is an exact sequence

$$0 \longrightarrow J_1^1 Y_n \longrightarrow Y_n \longrightarrow J_2^2 Y_n \longrightarrow J_0^1 Y_n \longrightarrow 0,$$

and so by Lemma 13 there exists a $t > 0$ such that $Y_t = Y_{t+1} \in \mathcal{E}^1$. But then $Y_t = 0$ since $Y_t \subseteq Z$ and $\tau_{\mathcal{E}^1} Z = 0$. Then $Y_{t-1} \in \mathcal{K}^2$, since $J_2^2 Y_{t-1} \in \mathcal{F}^2$ and $J_0^1 Y_{t-1} \in \mathcal{F}^0$, by Lemma 10. Then by induction on t we see that $Z = Y_0$ has a filtration with factors in $\mathcal{K}^2$, and therefore $Z \in \mathcal{E}^2$. This completes the proof of Part a).

Now let $X \in KerF^2$. Since $\mathcal{E}^0$ is closed under factors we see that $\tau_{\mathcal{E}^0} X \in \mathcal{E}^0$. There is a short exact sequence

$$0 \longrightarrow \tau_{\mathcal{E}^0} X \longrightarrow X \longrightarrow Z \longrightarrow 0,$$

where Z is the cokernel of the inclusion $\tau_{\mathcal{E}^0} X \longrightarrow X$. Then $Z \in KerF^2$, since $X \in KerF^2$ and $KerF^2$ is closed under factors. By taking pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_{\mathcal{E}^0} X & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tau_{\mathcal{E}^0} X & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

along a non-zero map $T \longrightarrow Z$ we get a map from $\mathcal{E}^0$ to X which does not factor through the inclusion $\tau_{\mathcal{E}^0} X \longrightarrow X$, which is a contradiction, and so $Z \in KerF^0$. Therefore $Z \in \mathcal{E}^1 = KerF^0 \cap KerF^2$. This completes the proof of Part b). □

For $X \in \mathcal{A}$, let $X_0 = 0$, $X_1 = \tau_{\mathcal{E}^0} X$ and let $X_2 \subseteq X$ be the preimage in X of $\tau_{\mathcal{E}^1}(X/X_1)$ for the quotient map $X \longrightarrow X/X_1$, and $X_3 = X$.

Theorem 19. *There is a functorial filtration* $0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 = X$ *with* $X_{i+1}/X_i \in \mathcal{E}^i$ *for* $i = 0, 1, 2$.

Proof. First, $\mathcal{E}^0$ is closed under factors, and so $X_1 \in \mathcal{E}^0$. Also, $X/X_1 \in KerF^0$ since by taking pullback, any nonzero map $T \longrightarrow Z$ induces a map $\mathcal{E}^0$ to X which does not factor through the inclusion $\tau_{\mathcal{E}^0} X \longrightarrow X$. The existence of the filtration then follows from Part a) of Lemma 18. We conclude the proof by showing functoriality. Given $f : X \longrightarrow Y$, since trace is a functor, there are induced maps $f : X_1 \longrightarrow Y_1$, and therefore maps $X/X_1 \longrightarrow Y/Y_1$, by the functoriality of trace again, there are maps $X_2/X_1 \longrightarrow Y_2/Y_1$, since these are induced by f, we must have $f(X_2) \subseteq Y_2$. Hence, the filtration is functorial. □

This theorem completes proof of the existence of the filtration in Theorem 2. The uniqueness will be shown at the end of this section.

Lemma 20. $KerHom(\mathcal{E}^0, -) = KerF^0$

Proof. $KerHom(\mathcal{E}^0, -) \subseteq KerF^0$ since $T \in \mathcal{E}^0$ and $F^0 = Hom(T, -)$. Assume there is a nonzero map $E \longrightarrow Z$ for $E \in \mathcal{E}^0$. We may assume that

$E \to Z$ is an inclusion, since $\mathcal{E}^0$ is closed under factors. But E has a non-zero subobject from $FacT$, and therefore there is a nonzero map $T \to E \to Z$. This proves the other inclusion. □

We describe the category $\mathcal{E}^2$ as follows.

Lemma 21. $\mathcal{E}^2 = KerF^0 \bigcap KerHom(\mathcal{E}^1, -)$

Proof. The inclusion $KerF^0 \bigcap KerHom(\mathcal{E}^1, -) \subseteq \mathcal{E}^2$ follows from part a) of Lemma 18. Let $X \in \mathcal{E}^2$. Then X has a filtration with factors isomorphic to subobjects of objects in $\mathcal{F}^2 \subseteq KerF^0$, and so $X \in KerF^0$. We show that $Hom(\mathcal{E}^1, \mathcal{K}^2) = 0$. Let $Y \in \mathcal{F}^2$. Then $Y \in KerF^0$ and Lemma 18 a) gives us an injection $F^1\tau_{\mathcal{E}^1}Y \longrightarrow F^1Y$, but then $F^1\tau_{\mathcal{E}^1}Y = 0$ since $Y \in KerF^1$. Thus $\tau_{\mathcal{E}^1}Y = 0$ and therefore $Hom(\mathcal{E}^1, Y) = 0$. Any object in $\mathcal{K}^2$ is a subobject of an object in $\mathcal{F}^2$, therefore $Hom(\mathcal{E}^1, \mathcal{K}^2) = 0$.

Now let $X \in \mathcal{E}^2$. Then there is a short exact sequence

$$0 \longrightarrow L \longrightarrow X \longrightarrow K \longrightarrow 0$$

$K \in \mathcal{K}^2$ and $L \in \mathcal{E}^2$. Then any map from $\mathcal{E}^1$ to X must factor through L, and so by induction on the length of a filtration of X we are done. □

We end this section by summarizing some facts about the subcategories $\mathcal{E}^i$. We have already seen that $\mathcal{E}^0$ is closed under factors.

Lemma 22. *The following is true.*

a) $\mathcal{E}^2$ *is closed under submodules.*
b) $\mathcal{E}^1$ *is closed under images of maps to* $KerF^0$.
c) $\mathcal{E}^i$ *is closed under summands.*

Proof. a) follows from Lemma 21.

Let $E \longrightarrow Y$ with $E \in \mathcal{E}^1$ and $Y \in KerF^0$, and let Z be the image of the map. Then $Z \in KerF^0$, since $KerF^0$ is closed under submodules. Also $Z \in KerF^2$ since $\mathcal{E}^1 \subseteq KerF^2$, and $E \longrightarrow Z$ is surjective. This proves b).

From Lemma 17 we see that $\mathcal{E}^0$ is closed under direct summands. Since $\mathcal{E}^1 = \mathcal{F}^1$ and $\mathcal{F}^1$ is defined by the vanishing of the two additive functors F^0 and F^2, we may conclude that $\mathcal{E}^1$ is closed under summands. That $\mathcal{E}^2$ is closed under summands follows from part a). □

We also note the following.

Theorem 23. *If* $j > i$*, then* $Hom(\mathcal{E}^i, \mathcal{E}^j) = 0$. *In particular, the subcategories* $\mathcal{E}^0$*,* $\mathcal{E}^1$ *and* $\mathcal{E}^2$ *are pairwise disjoint.*

Proof. We have $\mathcal{E}^1, \mathcal{E}^2 \subseteq KerF^0 = KerHom(\mathcal{E}^0, -)$. The other case follows from Lemma 21. That the categories are disjoint is then a trivial consequence. □

Using this theorem we may show the following uniqueness property of the filtration in Theorem 19. In particular, this shows that the filtrations in Corollary 15 and Theorem 19 coincide, and concludes the proof of Theorem 2.

Lemma 24. *If X has a filtration $0 = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq Y_3 = X$ with $Y_{i+1}/Y_i \in \mathcal{E}^i$ for $i = 0, 1, 2$, then $Y_i = X_i$ for $i = 0, 1, 2, 3$.*

Proof. Clearly, $Y_0 = X_0$ and $Y_3 = X_3$. By the definition of X_1 as the trace of $\mathcal{E}^0$ in X, we have $Y_1 \subseteq X_1$. If the inclusion is proper, then there is a non-zero map $E \longrightarrow X/Y_1$ from $\mathcal{E}^0$, but this is impossible since X/X_1 is filtered by $\mathcal{E}^1$ and $\mathcal{E}^2$ and there are no maps from $\mathcal{E}^0$ to $\mathcal{E}^1$ and $\mathcal{E}^2$ by the previous Lemma. Thus $Y_1 = X_1$. Similarly, $Y_2 = X_2$. This finishes the proof. □

4. The extension closure of $FacT$

In this section let $\mathcal{A} = modA$ for a finite dimensional algebra A. For any $X \in \mathcal{A}$ there is a natural map

$$\phi_X : J_0^0 X = T \otimes_B Hom_{\mathcal{A}}(T, X) \longrightarrow X$$

given by $t \otimes f \mapsto f(t)$, with image equal to the trace $\tau_T X$ of T in X. From Lemma 8 there is the short exact sequence

$$0 \longrightarrow J_2^1 X \longrightarrow J_0^0 X \xrightarrow{\phi} X$$

with image $im\phi = Z_1$ equal to the first term Z_1 in the filtration of Theorem 4.

Lemma 25. $Z_1 = \tau_T X$.

Proof. If $X \in FacT$, then $X \in \mathcal{K}^0$ by Lemma 16, and so there is a short exact sequence

$$0 \longrightarrow X_2 \longrightarrow X_0 \longrightarrow X \longrightarrow 0$$

for $X_2 \in \mathcal{F}^2$ and $X_0 \in \mathcal{F}^0$. But then by Lemma 11, $X_0 \cong J_0^0 X$ and $X_2 \cong J_2^1 X$, and so ϕ is surjective by comparing dimensions and therefore $Z_1 = X = \tau_T X$.

Now, for arbitrary X, let $i : \tau_T X \longrightarrow X$ be the inclusion map. There is a commutative diagram

$$\begin{array}{ccc} J_0^0 \tau_T X & \xrightarrow{J_0^0 i} & J_0^0 X \\ \downarrow{\scriptstyle \phi_{\tau_T X}} & & \downarrow{\scriptstyle \phi_X} \\ \tau_T X & \xrightarrow{\quad i \quad} & X \end{array} .$$

Since $\phi_{\tau_T X}$ is surjective, we have $\tau_T X \subseteq im\phi_X$. Then since $J_0^0 X \in FacT$ and therefore $im\phi_X \in FacT$ we have $\tau_T X = im\phi_X = Z_1$. □

We give an example showing that $\mathcal{K}^0 = FacT \subsetneq \mathcal{E}^0$. In particular, this example shows that the filtration of Theorem 4 can be more refined than the filtration of Theorem 2. Consider the following quiver.

$$\Delta : \ 1 \rightrightarrows 2 \longrightarrow 3$$

Let $A = k\Delta/rad(k\Delta)^2$ and let $T = DA$. Then $FacT$ consists of direct sums of direct summands of $T \oplus S_2$, where S_2 is the simple at vertex 2. The representation

$$k \underset{1}{\overset{1}{\rightrightarrows}} k \longrightarrow 0$$

is an extension of $S_1 \in FacT$ by S_2, but is clearly not in $FacT$. Hence $FacT$ is not closed under extensions and so $FacT \subsetneq \mathcal{E}^0$.

However, we recall the following positive result. An indecomposable injective A-module I is called maximal if any surjective map $J \longrightarrow I$ with J injective is split. The following proposition follows from the argument preceding Proposition 6.9 of [AS].

Proposition 26. [AS] *$FacDA$ is closed under extensions if and only if any maximal injective module has projective dimension at most one.*

Let A be a Nakayama algebra. That is, $A = kQ/I$, where Q is an oriented cycle, or a quiver of type $\mathbb{A}$ with linear orientation.

Proposition 27. *Let A be a Nakayama algebra. Then $FacDA$ is closed under extensions.*

Proof. A maximal injective A-module is projective. The proposition then follows from Proposition 26. □

Finailly, we given an example showing that filtrations with factors in $\mathcal{F}^i$ do not exist in general. Let

$$\Delta : \ 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$$

and let A be the path algebra of Δ with a relation equal to the path from vertex 2 to 4. Then A has global dimension two. Let $T = DA$. Then $S_4 \in \mathcal{F}^2$, $I_4 \in \mathcal{F}^0$ and S_3 is the cokernel of the inclusion $S_4 \longrightarrow I_4$, and so $S_3 \in \mathcal{K}^0$, but not in $\mathcal{F}^0$.

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