Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Rigid representations of a double quiver of type A, and Richardson elements in seaweed Lie algebras

Bernt Tore Jensen, Xiuping Su and Rupert Wei Tze Yu

Abstract

In this paper, we show that there is always an open adjoint orbit in the nilpotent radical of a seaweed Lie algebra in $\mathrm{gl}_n(k)$, thus answering positively in this $\mathrm{gl}_n(k)$ case to a question raised independently by Michel Duflo and Dmitri Panyushev. The proof gives an explicit construction, using Δ-filtered modules of quasi-hereditary algebras arising from quotients of the double of quivers of type A. An example of a seaweed Lie algebra in a simple Lie algebra of type E_8 not admitting an open orbit in its nilpotent radical is given.

1. *Introduction*

Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field k of characteristic zero. A Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called a seaweed Lie algebra if there exists a pair of parabolic subalgebras $(\mathfrak{p}, \mathfrak{p}')$ of $\mathfrak{g}$ such that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ and $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$ (such a pair of parabolic subalgebras is called weakly opposite). For example, take the pair consisting of a Borel subalgebra and its opposite.

Seaweed Lie algebras are introduced by Vladimir Dergachev and Alexandre Kirillov [**5**] in the case $\mathfrak{g} = \mathrm{gl}_n(k)$, and in the above generality by Dmitri Panyushev [**11**]. The set of seaweed Lie algebras in $\mathfrak{g}$ contains clearly all parabolic subalgebras of $\mathfrak{g}$ and their Levi factors. In particular, they provide new examples of index zero Lie algebras (or Frobenius Lie algebras) [**5**, **11**, **14**].

A general formula for the index of a seaweed Lie algebra conjectured in [**14**] was recently proved by Anthony Joseph [**9**]. This is an unexpected and pleasant surprise. Naturally, we would like to know to what extent certain classical results can be generalized to this large class of Lie subalgebras of $\mathfrak{g}$.

Let $\mathbf{G}$ be a connected reductive algebraic group whose Lie algebra is $\mathfrak{g}$. Let $(\mathfrak{p}, \mathfrak{p}')$ be a pair of weakly opposite parabolic subalgebras of $\mathfrak{g}$, and $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ the corresponding seaweed Lie algebra. Denote by $\mathbf{P}$ and $\mathbf{P}'$ the parabolic subgroups of $\mathbf{G}$ corresponding to $\mathfrak{p}$ and $\mathfrak{p}'$. Set $\mathbf{Q} = \mathbf{P} \cap \mathbf{P}'$.

We are interested in the following question raised by Michel Duflo and Dmitri Panyushev independently:

Question 1.1. Is there an open $\mathbf{Q}$-orbit in the nilpotent radical $\mathfrak{n}$ of $\mathfrak{q}$?

Equivalently, we may ask if there is an element $x \in \mathfrak{n}$ verifying $[x, \mathfrak{q}] = \mathfrak{n}$.

In the case where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$, then the answer is yes, and this is a result that is commonly known as Richardson's Dense Orbit Theorem [**13**]. If there is an open $\mathbf{Q}$-orbit in the nilpotent radical of $\mathfrak{q}$, then an element in the open $\mathbf{Q}$-orbit is called a Richardson element of $\mathfrak{q}$.

2000 *Mathematics Subject Classification* 16G20, 17B20.

The first author is supported by NFR project HoGeMetAlg. The second author is supported by Marie-Curie Fellowship IIF

Using some computations with the program GAP4, we are able to check that Richardson elements exist in any seaweed Lie algebra in a simple Lie algebra of rank ≤ 7. However, in type E_8, we found a seaweed Lie algebra whose nilpotent radical does not contain an open orbit.

Our present task is to study the case of seaweed Lie algebras in $\mathrm{gl}_n(k)$, for an algebraically closed field k of any characteristic. We prove that :

THEOREM 1.2. *Any seaweed Lie algebra in* $\mathrm{gl}_n(k)$ *has a Richardson element.*

In this particular case, seaweed Lie algebras can be viewed as the stabilizer of a pair of weakly opposite flags. This provides a nice description of seaweed Lie algebras, and allows us to transfer the problem to a quiver representations setting, extending the one for parabolic subalgebras considered by Thomas Brüstle, Lutz Hille, Claus Michael Ringel and Gerhard Röhrle [**2**].

A seaweed Lie algebra in $\mathrm{gl}_n(k)$ can be realized as the endomorphism algebra of a projective module for a quiver of type A. The corresponding algebraic group is the automorphism group of the projective, the nilpotent radical is the Jacobson radical of the endomorphism ring and the action is by conjugation. So studying $\mathbf{Q}$-orbits in $\mathfrak{n}$ can be translated to studying representations of a double quiver, using results from [**8**].

We show how to directly associate to $\mathfrak{q}$ a certain double quiver $\tilde{Q}$ of type A with relations $\mathcal{I}$, together with a dimension vector $\mathbf{d}$. Then the quotient $\mathbf{D} = k\tilde{Q}/\mathcal{I}$ has a structure of a quasi-hereditary algebra with respect to a partial order on the vertices. This family of quasi-hereditary algebras constructed as quotients of the double of quivers has appeared in different contexts, see [**3**, **4**, **8**, **10**]. These quasi-hereditary algebras are sometimes called the twisted double incidence algebras of posets, see for example [**4**].

The existence of an open $\mathbf{Q}$-orbit in the nilpotent radical of $\mathfrak{q}$ corresponds then to the existence of an open $\mathrm{GL}(\mathbf{d})$-orbit in the set $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ of Δ-filtered modules of dimension vector $\mathbf{d}$ verifying $\mathcal{I}$.

Our main theorem is the following:

THEOREM 1.3. *There exists a unique (up to isomorphism) rigid* Δ*-filtered module for any given* Δ*-dimension vector.*

The idea of the proof consists firstly of constructing rigid Δ-filtered modules with linear Δ-support, and then of constructing a rigid Δ-filtered module by gluing together in a specific way these rigid Δ-filtered modules with linear Δ-support. This construction has the advantage of providing an explicit Richardson element. These constructions extend those used in [**2**].

A consequence of the proof of the above theorem is an explicit construction of all classical (partial) tilting $\mathbf{D}$-modules. This can be understood using the notion of a volume of a tilting module, as introduced by Lutz Hille [**7**].

2. *Quasi-hereditary algebras arising from quotients of the double of quivers*

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows and s, t are maps sending each arrow to its starting vertex and its terminating vertex, respectively. A vertex $i \in Q_0$ is called a sink (resp. source) vertex if there are no arrows starting (resp. terminating) at i. A vertex is *admissible* if it is either a source or a sink. We denote by kQ the path algebra of Q.

Let $\mathcal{J}$ be an ideal in kQ. A representation $M = (\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$ of Q is a representation of $(Q, \mathcal{J})$ if the maps M_α satisfy the relations in $\mathcal{J}$. It is well-known that the category of representations of $(Q, \mathcal{J})$ is equivalent to the category of left modules of $kQ/\mathcal{J}$.

Let $\mathrm{Rep}(Q, \mathcal{J}, \mathbf{d})$ denote the variety of representations of $(Q, \mathcal{J})$ with dimension vector $\mathbf{d} = (d_i)_{i\in Q_0}$. The group $\mathrm{GL}(\mathbf{d}) = \prod_i \mathrm{GL}_{d_i}(k)$ acts on $\mathrm{Rep}(Q, \mathcal{J}, \mathbf{d})$ by conjugation and there is a one-to-one correspondence between the $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{Rep}(Q, \mathcal{J}, \mathbf{d})$ and isomorphism classes of $kQ/\mathcal{J}$-modules with dimension vector $\mathbf{d}$.

We recall the definition of a family of quasi-hereditary algebras. In the rest of this paper, we assume that the quiver Q is of type A, for more general constructions, see [**3**, **4**, **8**, **10**]. Let $Q_0 = \{1, \ldots, n\}$ be the set of vertices of Q. For $1 \le i \le n-1$, there is a unique arrow connecting vertices i and $i+1$ in Q and we denote it by α_i. We denote by $\tilde{Q}$ the double of Q. That is, $\tilde{Q}$ is obtained from Q by, for each i, adding an arrow β_i in the reverse direction of α_i, that is, $s(\beta_i) = t(\alpha_i)$ and $t(\beta_i) = s(\alpha_i)$. For the sake of convenience, we set $\alpha_0, \beta_0, \alpha_n$ and β_n to be zero arrows. The algebra $\mathbf{D}$ is the quotient of the path algebra of $\tilde{Q}$ by the ideal $\mathcal{I}$ generated by the relations

(1) $\beta_{j-1}\alpha_{j-1}$ and $\beta_j\alpha_j$ for a source j;
(2) $\beta_j\alpha_{j-1}$ and $\beta_{j-1}\alpha_j$ for a sink j;
(3) $\beta_{j-1}\alpha_{j-1} - \alpha_j\beta_j$ for j non-admissible and $t(\alpha_j) = j$;
(4) $\alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j$ for j non-admissible and $s(\alpha_j) = j$.

Note that the inclusion of quivers $Q \subseteq \tilde{Q}$, makes kQ a subalgebra of $\mathbf{D}$. Also, mapping each $\beta_i \in \tilde{Q}_1 \backslash Q_1$ to zero gives us a surjective map of algebras $\mathbf{D} \longrightarrow kQ$. We view any $\mathbf{D}$-module as an kQ-module and any kQ-module as a $\mathbf{D}$-module via these two algebra homomorphisms.

Let $\Delta(i)$ be an indecomposable projective kQ-module with simple top $L(i)$. We define a partial order $\succeq$ on the vertex set Q_0 by $i \succeq j$ if $\Delta(j) \subseteq \Delta(i)$ as kQ-modules. We let $P(i)$ be a projective $\mathbf{D}$-module with simple top $L(i)$. The modules $\Delta(i)$ and the partial order $\succeq$ give $\mathbf{D}$ the structure of a quasi-hereditary algebra [**4**] [**8**]. Moreover $\mathrm{gldim}\, \mathbf{D} \le 2$. Recall that a representation M is Δ-filtered if it has a finite filtration with composition factors the $\Delta(i)$. The following result is easy and well-known to experts.

Proposition 2.1. *The following are equivalent for a representation $M$of $(\tilde{Q}, \mathcal{I})$.*

(i) *The representation M is Δ-filtered.*
(ii) *The projective dimension of M is at most one.*
(iii) *The representation M is projective as a kQ-module.*
(iv) *For any arrow $\alpha \in Q_1$, the k-linear map M_α is injective and*

$$\mathrm{Im}(M_\alpha) \bigcap \sum_{\substack{\beta\in Q_1\backslash\{\alpha\} \\ t(\beta)=t(\alpha)}} \mathrm{Im}(M_\beta) = 0.$$

A proof of the equivalence of (i)-(iii) of Proposition 2.1 can be found in [**8**] and the equivalence of (iii) and (iv) is clear for any quiver.

Let M be a Δ-filtered module. The Δ-dimension vector of M, denoted by $\underline{\dim}_\Delta(M)$, is the vector with its i-th entry $(\underline{\dim}_\Delta(M))_i$ the multiplicity of $\Delta(i)$ as a Δ-composition factor in a Δ-filtration of M. The Δ-support of M, denote by $\mathrm{supp}_\Delta(M)$, is the the full subgraph of Q with the set of vertices $\{i \in Q_0 | (\underline{\dim}_\Delta(M))_i > 0\}$. We will also need the usual support of M, denoted by $\mathrm{supp}(M)$, defined via the ordinary dimension vector $\underline{\dim}(M)$.

We need some properties of the Δ-filtered modules. By the definition of $\mathbf{D}$, it is not difficult to see that $\Delta(i)$ is projective for a source i, that is, $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i), -) = 0$. A similar, but weaker, property holds for sinks. The proof is easy.

LEMMA 2.2. *If M is Δ-filtered and i is a sink in Q, then* $\mathrm{Ext}^1_{\mathbf{D}}(M,\Delta(i)) = 0$.

We need two more easy lemmas on Δ-filtered modules.

LEMMA 2.3. *Let i be a source in Q and let M be Δ-filtered with $d_i = (\underline{\dim}_\Delta(M))_i > 0$, then there exists a unique submodule $\Delta(i)^{d_i} \subseteq M$. Moreover, the quotient module $M/\Delta(i)^{d_i}$ is Δ-filtered.*

LEMMA 2.4. *Let i be a sink in Q and let M be Δ-filtered with $d_i = (\underline{\dim}_\Delta(M))_i > 0$, then there exists a unique quotient module which is isomorphic to $\Delta(i)^{d_i}$. Moreover the submodule Y, satisfying $M/Y \cong \Delta(i)^{d_i}$, is Δ-filtered.*

Let $\mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$ be the subset of elements of $\mathrm{Rep}(\tilde{Q},\mathcal{I},\mathbf{d})$, which are Δ-filtered. Suppose that $\mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$ is not empty. We fix a standard embedding $k^{d_{s(\alpha)}} \subseteq k^{d_{i_0}}$ for each $\alpha \in Q_1$, such that $k^{d_{s(\alpha)}} \cap k^{d_{s(\alpha')}} = 0$ if $t(\alpha) = t(\alpha') = i_0$. Now define $R^\alpha = \{M \in \mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d}) |$ each map M_α is the standard embedding $k^{d_{s(\alpha)}} \subseteq k^{d_{i_0}}\}$.

PROPOSITION 2.5. *Suppose that the set* $\mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$ *is non-empty.*
(1) *The subset R^α is an affine space.*
(2) *The subset* $\mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$ *is open and irreducible in* $\mathrm{Rep}(\tilde{Q},\mathcal{I},\mathbf{d})$.

Proof. By the definition of $\mathcal{I}$, we see that any element in $\mathcal{I}$ is a linear combination of arrows in $\tilde{Q}_1 \backslash Q_1$ when we view arrows in Q_1 as constants. Thus R^α is the solution space of a linear system, and so it is an affine space. The openness of $\mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$ follows from Proposition 2.1 (iv). By Proposition 2.1 (iii), we know that the $\mathrm{GL}(\mathbf{d})$-orbit of any Δ-filtered representation M meets with R^α. So we have a surjective map $\mathrm{GL}(\mathbf{d}) \times R^\alpha \to \mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$, and thus $\mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$ is irreducible. $\square$

3. *Gluing of Δ-filtered modules*

In this section, we explain how to glue two Δ-filtered modules at an admissible vertex and obtain a new Δ-filtered module, which usually has higher dimension.

Let i be a sink or a source of Q. Let M' and M'' be two Δ-filtered modules with $\mathrm{supp}_\Delta(M') \subseteq \{1,\ldots,i\}$ and $\mathrm{supp}_\Delta(M'') \subseteq \{i,\ldots,n\}$, and $(\underline{\dim}_\Delta(M'))_i = (\underline{\dim}_\Delta(M''))_i = d_i > 0$.

Assume that i is a sink in Q. By Lemma 2.4, we have short exact sequences

$$0 \longrightarrow \mathrm{Ker}(f') \longrightarrow M' \xrightarrow{f'} \Delta(i)^{d_i} \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ker}(f'') \longrightarrow M'' \xrightarrow{f''} \Delta(i)^{d_i} \longrightarrow 0.$$

Let M be given by the pullback of f' and f'', that is, we have a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \oplus M'' \xrightarrow{(f' \ -f'')} \Delta(i)^{d_i} \longrightarrow 0.$$

Similarly, if i is a source, we have a pushout sequence

$$0 \longrightarrow \Delta(i)^{d_i} \xrightarrow{\binom{f'}{-f''}} M' \oplus M'' \longrightarrow M \longrightarrow 0.$$

In both of these cases, we say that M is obtained by gluing M' and M'' at i.

Proposition 3.1. *Let i be a sink or a source in Q. Then for any Δ-filtered module M with $(\underline{\dim}_\Delta(M))_i > 0$, there exists Δ-filtered modules M' and M'' with $(\underline{\dim}_\Delta(M'))_i = (\underline{\dim}_\Delta(M''))_i = (\underline{\dim}_\Delta(M))_i$ such that M is isomorphic to a module glued from M' and M'' at i. Conversely, if M is glued from M' and M'' at i, then M is Δ-filtered.*

Proof. Let i be a source in Q. By Lemma 2.3, there is a short exact sequence

$$0 \longrightarrow \Delta(i)^{d_i} \xrightarrow{f} M \xrightarrow{g} \mathrm{Cok}(f) \longrightarrow 0,$$

where $\mathrm{Cok}(f)$ is Δ-filtered. Here $\Delta(i)^{d_i}$ is the submodule of M generated by M_i. Then $(\underline{\dim}(\mathrm{Cok}(f)))_i = 0$ and therefore $\mathrm{Cok}(f) = Y' \oplus Y''$, where $\mathrm{supp}(Y')$ is contained in $\{1, \ldots, i-1\}$ and $\mathrm{supp}(Y'')$ is contained in $\{i+1, \ldots, n\}$. We have short exact sequences

$$0 \longrightarrow \Delta(i)^{d_i} \longrightarrow M' \longrightarrow Y' \longrightarrow 0$$

and

$$0 \longrightarrow \Delta(i)^{d_i} \longrightarrow M'' \longrightarrow Y'' \longrightarrow 0,$$

where $M' = g^{-1}(Y')$ and $M'' = g^{-1}(Y'')$. By Proposition 2.1, we see that M' and M'' are Δ-filtered with their Δ-supports contained in $\{1, \ldots, i\}$ and $\{i, \ldots, n\}$, respectively. Now there is a short exact sequence

$$0 \longrightarrow X \longrightarrow M' \oplus M'' \xrightarrow{(\,g'\ g''\,)} M \longrightarrow 0,$$

where g' and g'' are the inclusion maps. We have $\underline{\dim}(X) = \underline{\dim}(\Delta(i)^{d_i})$ and X is projective as an kQ-module. Therefore $X \cong \Delta(i)^{d_i}$. Hence M is glued from M' and M'' at i.

Using Proposition 2.1 (ii) we can easily prove the converse.

The proof in the case where i is a sink is similar and is left to the reader. □

We construct a morphism $h : M \longrightarrow N$ from maps $h' : M' \longrightarrow N'$ and $h'' : M'' \longrightarrow N''$ provided there exists a map $w : \Delta(i)^{d_i} \longrightarrow \Delta(i)^{d_i}$, where $d_i = \underline{\dim}_\Delta(M)_i = \underline{\dim}_\Delta(N)_i$, such that the first square of the diagram of gluing sequences

$$\begin{tikzcd}
0 \arrow[r] & \Delta(i)^{d_i} \arrow[r] \arrow[d, "w"] & M' \oplus M'' \arrow[r] \arrow[d, "h' \oplus h''"] & M \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \Delta(i)^{d_i} \arrow[r] & N' \oplus N'' \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

commutes. Now h is a morphism $M \longrightarrow N$, which makes the second square of the above diagram commute. In this case, we say that h is obtained by gluing h' and h'' at i. Note that any morphism $h : M \longrightarrow N$ is obtained by gluing morphisms h' and h''. We similarly define gluing of morphisms at a sink i.

Lemma 3.2. *Let M be obtained by gluing M' and M'' at i. If M is rigid, then M' and M'' are rigid.*

Proof. Assume that M is rigid. We first consider the case where i is a sink. We have a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \oplus M'' \longrightarrow \Delta(i)^c \longrightarrow 0,$$

where $c = (\underline{\dim}_\Delta(M'))_i = (\underline{\dim}_\Delta(M''))_i$. By applying the functor $\mathrm{Hom}_{\mathbf{D}}(M,-)$ to this sequence and using Lemma 2.2, we see that $\mathrm{Ext}^1_{\mathbf{D}}(M, M' \oplus M'') = 0$. Then by applying $\mathrm{Hom}_{\mathbf{D}}(-, M')$, we get a surjection $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i)^c, M') \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M' \oplus M'', M')$. From Lemma 2.4, we have a short exact sequence

$$0 \longrightarrow X \longrightarrow M'' \longrightarrow \Delta(i)^c \longrightarrow 0.$$

Note that $\mathrm{supp}(\mathrm{top}(X)) \cap \mathrm{supp}(M') = \emptyset$, where $\mathrm{top}(X)$ is the top of X. Thus $\mathrm{Hom}_{\mathbf{D}}(X, M') = 0$. We get an injection $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i)^c, M') \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M'', M')$ by applying $\mathrm{Hom}_{\mathbf{D}}(-, M')$. Hence $\mathrm{Ext}^1_{\mathbf{D}}(M', M') = 0$. By symmetry, $\mathrm{Ext}^1_{\mathbf{D}}(M'', M'') = 0$.

The proof in the case where i is a source is similar and is left to the reader. □

4. *Rigid Δ-filtered modules with a linear Δ-support*

Let us assume for the moment that Q is linearly oriented with vertex 1 a sink and vertex n a source. Following [**2**], a module M is isomorphic to a nonzero submodule of $P(1)$ if and only if the socle of M is $L(1)$. Such a module M is indecomposable, rigid and Δ-filtered. Moreover, the map sending a submodule M of $P(1)$ to the set of vertices of its Δ-support affords a bijection between the set of submodules of $P(1)$ and the set of the subsets of $\{1, \dots, n\}$. We denote by $\Delta(I)$ the submodule of $P(1)$ with I as the set of vertices of its Δ-support. Given $\mathbf{d}$ a non-zero vector in $\mathbb{N}^n$, define $I(\mathbf{d}) = \mathrm{supp}(\mathbf{d})_0$, the set of vertices of $\mathrm{supp}(\mathbf{d})$. We denote by $\mathbf{d}_{I(\mathbf{d})}$ the dimension vector with

$$(\mathbf{d}_{I(\mathbf{d})})_i = \begin{cases} 1 & \text{if } i \in I(\mathbf{d}), \\ 0 & \text{otherwise.} \end{cases}$$

Now define inductively a module $\Delta(\mathbf{d})$ as follows: $\Delta(\mathbf{d}) = \Delta(I(\mathbf{d})) \oplus \Delta(\mathbf{d} - \mathbf{d}_{I(\mathbf{d})})$. In this way we also obtain a descending sequence of subsets $I(\mathbf{d}) \supseteq I(\mathbf{d} - \mathbf{d}_{I(\mathbf{d})}) \supseteq \cdots$.

Theorem 4.1. *[**2**] Given $\mathbf{d}$ a non-zero vector in $\mathbb{N}^n$. The module $\Delta(\mathbf{d})$ constructed above is the unique (up to isomorphism) rigid Δ-filtered module with Δ-dimension vector $\mathbf{d}$.*

We return to the case where Q has arbitrary orientation. Let $\mathbf{d}$ be a non-zero vector in $\mathbb{N}^n$ with a linear support. Suppose that the set of vertices of $\mathrm{supp}_\Delta(\mathbf{d})$ is $\{i, i+1, \cdots, j\}$, where any vertex in $\{i+1, \cdots, j-1\}$ is a non-admissible vertex of Q. We may suppose that i is a source in $\mathrm{supp}(\mathbf{d})$ and j is a sink in $\mathrm{supp}(\mathbf{d})$. Consider the following subquiver of Q,

$$l \longleftarrow \cdots \longleftarrow i' \longrightarrow \cdots \longrightarrow i \longrightarrow i+1 \longrightarrow \cdots \longrightarrow j \longrightarrow \cdots \longrightarrow j' \ ,$$

where i' is a source vertex, l and j' are sink vertices and all other vertices are non-admissible vertices of Q. We denote by Q' the subquiver with the set of vertices $Q'_0 = \{i', i'+1 \dots, j, \cdots, j'\}$. Let $\tilde{Q}'$ be the double of Q' and $\mathcal{I}' = k\tilde{Q}' \cap \mathcal{I}$. Then $k\tilde{Q}'/\mathcal{I}'$ is the double of the linearly oriented quiver Q'. By Theorem 4.1, there is a unique rigid Δ-filtered $k\tilde{Q}'/\mathcal{I}'$-module $\Delta(\mathbf{d})$ with Δ-dimension vector $\mathbf{d}$.

We construct a $\mathbf{D}$-module $M(\mathbf{d})$ as follows. If i is not an interior source of Q, we let $M(\mathbf{d}) = \Delta(\mathbf{d})$. We now consider the case where $i = i'$ is an interior source of Q. We write

$$\Delta(\mathbf{d}) = \bigoplus_{s \geq 1} \Delta(I_s),$$

where $I(\mathbf{d}) = I_1 \supseteq I_2 = I(\mathbf{d} - \mathbf{d}_{I(\mathbf{d})}) \supseteq \cdots$ and each $\Delta(I_s)$ is an indecomposable rigid Δ-filtered $k\tilde{Q}'/\mathcal{I}'$-module.

For each I_s, define $M_{I_s} = \Delta(I_s)$ if $(\underline{\dim}_\Delta(\Delta(I_s)))_i = 0$; otherwise define M_{I_s} as the extension of $\Delta(I_s)$ by $\Delta(i-1)$, with $(M_{I_s})_{\beta_{i-1}} = 0$ and $(M_{I_s})_{\alpha_{i-1}} = 1$. By the construction, we see that

M_{I_s} is Δ-filtered and the set of vertices of $\mathrm{supp}_\Delta(M_{I_s})$ is I_s. Now let

$$M(\mathbf{d}) = \bigoplus_{s \geq 1} M_{I_s}.$$

One can check that non-trivial self-extensions of $M(\mathbf{d})$ induce non-trivial self-extensions of $\Delta(\mathbf{d})$. So the following proposition follows from [**2**].

Proposition 4.2. *Let $\mathbf{d}$ be a non-zero vector in $\mathbb{N}^n$ with linear support as above. Then the module $M(\mathbf{d})$ constructed above is the unique (up to isomorphism) rigid Δ-filtered module with Δ-dimension vector $\mathbf{d}$.*

Now we will define an order on Δ-filtered modules and calculate this order in the case of linear Δ-support. Let M and N be two Δ-filtered modules with $(\underline{\dim}_\Delta(M))_i = 1 = (\underline{\dim}_\Delta(N))_i$ for an admissible vertex i. If i is a source, there is an inclusion $\Delta(i) \longrightarrow M$ by Lemma 2.3, and we define $M \geq_i N$ if the map $\mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(\Delta(i), N)$ is surjective.

Similarly, if i is a sink, there is a quotient map $N \longrightarrow \Delta(i)$ by Lemma 2.4, and we define $M \geq_i N$, if $\mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, \Delta(i))$ is surjective. By $M >_i N$ we mean that $M \geq_i N$ and $N \not\geq_i M$.

We compute these orders for the indecomposable rigid Δ-filtered modules M_J with J contained in the linearly oriented subquiver Q'. To simplify the notation we may assume that $i = i'$ and $j = j'$ are a source vertex and a sink vertex of Q, respectively.

Lemma 4.3. *Let $J = \{j_1 \succ j_2 > \cdots \succ j_s\}$ and $J' = \{j'_1 \succ j'_2 \succ \cdots \succ j'_t\}$ be two subsets of $\{i, \ldots, j\}$, where $j_1 = i = j'_1$ is a source in Q. Then $M_J \geq_i M_{J'}$ if and only if $s \leq t$ and $j_r \preceq j'_r$ for $r = 1, \ldots, s$.*

Proof. We have $M_J \geq_i M_{J'}$ if and only if there exists a monomorphism $M_J \longrightarrow M_{J'}$, since for any morphism f from M_J to $M_{J'}$, $\mathrm{Soc}(M_J) \subseteq \Delta(i)$ and $\mathrm{Ker}(f) \cap \mathrm{Soc}(M_J) \neq 0$ if $\mathrm{Ker}(f) \neq 0$. Now the lemma follows from our construction of M_J and $M_{J'}$ and Lemma 4 in [**2**]. □

We state the analogous result for sinks.

Lemma 4.4. *Let $J = \{j_1 \prec j_2 \prec \cdots \prec j_s\}$ and $J' = \{j'_1 \prec j'_2 \prec \cdots \prec j'_t\}$ be two subsets of $\{i, \ldots, j\}$, where $j_1 = j = j'_1$ is a sink in Q. Then $M_J \geq_j M_{J'}$ if and only if $s \geq t$ and $j_r \preceq j'_r$ for $r = 1, \ldots, t$.*

As a corollary of Proposition 4.2 and Lemmas 4.3 and 4.4, we have the following:

Corollary 4.5. *Let $\mathbf{d}$ be a non-zero vector in $\mathbb{N}^n$ with $\mathrm{supp}(\mathbf{d})_0 \subseteq \{i, \cdots, j\}$. Then the indecomposable direct summands X of the rigid Δ-filtered module $M(\mathbf{d})$ with $(\underline{\dim}_\Delta(X))_i = 1$ are totally ordered using $\geq_i$. Similarly, we have a total order using $\geq_j$.*

5. *Rigid Δ-filtered modules*

This section is devoted to proving the following main result.

THEOREM 5.1. *Given a non-zero vector $\mathbf{d} \in \mathbb{N}^n$, there exists a unique (up to isomorphism) rigid Δ-filtered $\mathbf{D}$-module $M(\mathbf{d})$ with Δ-dimension vector $\mathbf{d}$.*

Let $\mathbf{d}$ be a non-zero vector in $\mathbb{N}^n$. We construct a rigid Δ-filtered representation $M(\mathbf{d})$, with Δ-dimension vector $\mathbf{d}$. Following Lemma 3.2, we see that any indecomposable rigid module is obtained by gluing rigid modules with linear Δ-support. In the following we show how to glue the rigid Δ-filtered modules with linear Δ-support to obtain rigid Δ-filtered modules with arbitrary Δ-support.

Let $i_1 < i_2 \cdots < i_t$ be a complete list of interior admissible vertices in Q. Let $i_0 = 1$ and $i_{t+1} = n$ be the end vertices of Q. Let M and N be two indecomposable Δ-filtered modules obtained by successively gluing $M^a, \ldots, M^u$ and $N^b, \ldots, N^v$, respectively, where $1 \leq a \leq u \leq t+1$, $1 \leq b \leq v \leq t+1$, and for each l, there exist subsets $J_l, J'_l \subseteq \{i_{l-1}, \cdots, i_l\}$ such that $M^l = M_{J_l}$ and $N^l = M_{J'_l}$. Assume that $M^l \oplus N^l$ is rigid for each l, that is, either $J_l \subseteq J'_l$ or $J'_l \subseteq J_l$. We will find a necessary condition for the vanishing of $\mathrm{Ext}^1_{\mathbf{D}}(M, N)$.

Let $M^{\leq l} = 0$ if $l < a$; $M^{\leq l} =$ the module obtained by gluing $M^a, \ldots, M^l$ if $a \leq l \leq u$; and $M^{\leq l} = M$ if $l > u$. Similarly, we define $N^{\leq l}$.

LEMMA 5.2. $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, N^{\leq l}) = 0$, *for $l \leq \max\{a, b\}$.*

Proof. The statement is clearly true if $l < \max\{a, b\}$. So we may assume $l = \max\{a, b\}$. We consider the case where i_{l-1} is a source. The case where i_{l-1} is a sink is similar and is left to the reader. Assume $a \leq b$. We have $l = b$ and $N^{\leq l} = N^l$ and a short exact sequence

$$0 \longrightarrow M^l \longrightarrow M^{\leq l} \longrightarrow X \longrightarrow 0\cdot$$

Applying $\mathrm{Hom}_{\mathbf{D}}(-, N^{\leq l})$ gives us a surjection

$$\mathrm{Ext}^1_{\mathbf{D}}(M^l, N^l) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, N^l) \longrightarrow 0,$$

where the left term is zero, by construction. This shows that $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, N^{\leq l}) = 0$. Assume $b \leq a$. We have $l = a$ and $M^{\leq l} = M^l$ and a short exact sequence

$$0 \longrightarrow N^l \longrightarrow N^{\leq l} \longrightarrow Y \longrightarrow 0\cdot$$

Applying $\mathrm{Hom}_{\mathbf{D}}(M^l, -)$ gives us a short exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(M^l, N^l) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M^l, N^{\leq l}) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M^l, Y).$$

Now, there is no map from the syzygy of M^l to Y hence $\mathrm{Ext}^1_{\mathbf{D}}(M^l, Y) = 0$ and therefore $\mathrm{Ext}^1_{\mathbf{D}}(M^l, N^{\leq l}) = 0$. □

We need the following special case of Lemma 5.2.

LEMMA 5.3. *We have* $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, \Delta(i_l)) = 0 = \mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_l), N^{\leq l})$ *for* $\max\{a, b\} \leq l < \min\{u, v\}$.

LEMMA 5.4. *We have* $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, N^l) = 0 = \mathrm{Ext}^1_{\mathbf{D}}(M^l, N^{\leq l-1})$ *for* $\max\{a, b\} < l \leq \min\{u, v\}$.

Proof. We only give the proof in the case where i_{l-1} is a source. We have an exact sequence

$$0 \longrightarrow \Delta(i_{l-1}) \longrightarrow N^{\leq l-1} \longrightarrow X \longrightarrow 0.$$

By applying $\mathrm{Hom}_{\mathbf{D}}(M^l, -)$, we have an exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(M^l, \Delta(i_{l-1})) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M^l, N^{\leq l-1}) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M^l, X).$$

Since M^l has linear support, we have $\mathrm{Ext}^1_{\mathbf{D}}(M^l, \Delta(i_{l-1})) = 0$. There is no map from the syzygy of M^l to X, therefore $\mathrm{Ext}^1_{\mathbf{D}}(M^l, X) = 0$. Thus $\mathrm{Ext}^1_{\mathbf{D}}(M^l, N^{\leq l-1})) = 0$.

By applying $\mathrm{Hom}_{\mathbf{D}}(M^{\leq l-1}, -)$ to the exact sequence

$$0 \longrightarrow \Delta(i_{l-1}) \longrightarrow N^l \longrightarrow Y \longrightarrow 0,$$

we have an exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, \Delta(i_{l-1})) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, N^l) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, Y).$$

By Lemma 5.3, we have $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, \Delta(i_{l-1})) = 0$. By the construction of $M^{\leq l-1}$, we have $\mathrm{Hom}_{\mathbf{D}}(\Omega(M^{\leq l-1}), Y) = 0$, where $\Omega(M^{\leq l-1})$ denotes the syzygy of $M^{\leq l-1}$. Therefore $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, Y) = 0$, and so $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, N^l) = 0$. □

Lemma 5.5. *Assume* $\max\{a, b\} < l \leq \min\{u, v\}$. *If* $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, N^{\leq l-1}) = 0$, *and either* $M^{\leq l-1} \geq_{i_{l-1}} N^{\leq l-1}$ *or* $M^l \geq_{i_{l-1}} N^l$, *then* $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, N^{\leq l}) = 0$.

Proof. We only give the proof in the case where i_{l-1} is a source. The other case is similar and is left to the reader. By applying $\mathrm{Hom}_{\mathbf{D}}(-, N^{\leq l})$ to the gluing sequence of $M^{\leq l}$ at i_{l-1},

$$0 \longrightarrow \Delta(i_{l-1}) \longrightarrow M^{\leq l-1} \oplus M^l \longrightarrow M^{\leq l} \longrightarrow 0,$$

we see that $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, N^{\leq l}) = 0$ if (1) $\mathrm{Hom}_{\mathbf{D}}(M^{\leq l-1} \oplus M^l, N^{\leq l}) \to \mathrm{Hom}_{\mathbf{D}}(\Delta(i_{l-1}), N^{\leq l})$ is surjective and (2) $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1} \oplus M^l, N^{\leq l}) = 0$. It is easy to see that (1) holds if either $M^{\leq l-1} \geq_{i_{1-1}} M^{\leq l-1}$ or $M^l \geq_{i_{l-1}} N^l$.

By Lemmas 5.3 and 5.4 and by applying $\mathrm{Hom}_{\mathbf{D}}(M^{\leq l-1} \oplus M^l, -)$ to the gluing sequence of $N^{\leq l}$, we get that $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1} \oplus M^l, N^{\leq l}) = 0$ if $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l-1}, N^{\leq l-1}) = 0 = \mathrm{Ext}^1_{\mathbf{D}}(M^l, N^l)$. Now $\mathrm{Ext}^1_{\mathbf{D}}(M^l, N^l) = 0$ by assumption, which completes the proof in the case where i_{l-1} is a source. □

Lemma 5.6. *If* $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, N^{\leq l}) = 0$, *for* $l = \min\{u, v\}$, *then* $\mathrm{Ext}^1_{\mathbf{D}}(M^{\leq l}, N^{\leq l}) = 0$, *for* $l > \min\{u, v\}$.

Proof. The proof is similar to the proof of 5.2. □

Lemma 5.7. *Assume* $\max\{a, b\} \leq l < \min\{u, v\}$. *Then* $M^{\leq l} \geq_{i_l} N^{\leq l}$ *if and only if, either*

(1) $M^{\leq l} \cong N^{\leq l}$, *or*

(2) *there exists an* $\max\{a, b\} \leq l' \leq l$ *such that* $M^{l'} >_{i_{l'}} N^{l'}$ *and* $M^j \cong N^j$ *for* $j = l' + 1, \ldots, l$.

Proof. If $M^{\leq l} \cong N^{\leq l}$, then clearly $M^{\leq l} \geq_{i_l} N^{\leq l}$, so we may assume that $M^{\leq l} \not\cong N^{\leq l}$. Then there exists an l' such that $M^{l'} \not\cong N^{l'}$ and $M^s \cong N^s$ for $s = l' + 1, \ldots, l$. To prove the lemma we need only show that $M \geq_{i_l} N$ if and only if $M^{l'} >_{i_{l'}} N^{l'}$.

First assume that $M^{\leq l} \geq_{i_l} N^{\leq l}$. We prove the case where i_l is a source. The case where i_l is a sink is similar and is left to the reader. Let $f : M \longrightarrow N$ be a morphism with the property that an injective morphism $\Delta(i_l) \longrightarrow N^{\leq l}$ factors through f.

We have a map $g = f|_{M^l} : M^l \longrightarrow N^l$, the restriction of f to M^l, which shows that $M^l \geq_{i_l} N^l$. If $M^l \not\cong N^l$, then $M^l >_{i_l} N^l$ and we are done. If $M^l \cong N^l$, then g is an isomorphism since it is injective. Hence $M^{\leq l-1} \geq_{i_{l-1}} N^{\leq l-1}$. Using induction we get that $M^{l'} >_{i_{l'}} N^{l'}$ and we are done.

For the converse, assume that $M^{l'} >_{i_{l'}} N^{l'}$ and $M^j \cong N^j$ for $j = l'+1, \ldots, l$. If $l = l'$, let $g : M^l \longrightarrow N^l$ be a morphism with the property that an injective morphism $\Delta(i_l) \longrightarrow N^l$ factors through g. Since $M^l \ncong N^l$ we see that the top isomorphic to $\Delta(i_{l-1})$ is not in the image of g. We get a morphism $f : M^{\leq l} \longrightarrow N^{\leq l}$ by gluing g and the zero map $0 : M^{\leq l-1} \longrightarrow N^{\leq l-1}$ at i_{l-1}. This shows that $M^{\leq l} \geq_{i_l} N^{\leq l}$.

If $l' < l$, then $M^{\leq l-1} \geq_{i_{l-1}} N^{\leq l-1}$ by induction and there exists a morphism $g : M^{\leq l-1} \longrightarrow N^{\leq l-1}$ such that a surjective morphism $M^{\leq l-1} \longrightarrow \Delta(i_{l-1})$ factors through g. We may glue g with an isomorphism $M^l \longrightarrow N^l$ at i_{l-1}, and get a map $f : M^{\leq l} \longrightarrow N^{\leq l}$. This shows that $M^{\leq l} \geq_{i_l} N^{\leq l}$. □

Proof Proof of Theorem 5.1. Let $\mathbf{d}^s$, for $s = 1, \ldots, t+1$, be the vector given by $(\mathbf{d}^s)_j = d_j$ if $j \in \{i_{s-1}, \ldots, i_s\}$ and zero elsewhere. Note that each support $\mathrm{supp}(\mathbf{d}^s)$ is a linearly oriented subquiver of Q.

Let $M(\mathbf{d}^s) = \bigoplus_l M_{J_l^s}$ be the rigid Δ-filtered module from Proposition 4.2. We fix an ordering on the indecomposable direct summands of $M(\mathbf{d}^s)$ as follows. For $s = 1$, we assume that $(\underline{\dim}_\Delta(M_{J_l^1}))_{i_1} = 1$ for $l = 1, \ldots, d_{i_1}$ and that $M_{J_l^1} \geq_{i_1} M_{J_{l+1}^1}$ for $l = 1, \ldots, d_{i_1} - 1$. For each $s > 1$, we assume that $(\underline{\dim}_\Delta(M_{J_l^s}))_{i_{s-1}} = 1$ for $l = 1, \ldots, d_{i_{s-1}}$ and that $M_{J_l^s} \geq_{i_{s-1}} M_{J_{l+1}^s}$ for $l = 1, \ldots, d_{i_{s-1}}$. This is possible by Corollary 4.5.

Let $\mathbf{c}^s$ be the Δ-dimension vector given by $(c^s)_j = d_j$ for $j \in \{1, \ldots, i_s\}$ and zero elsewhere. Here $\mathbf{c}^{t+1} = \mathbf{d}$ and $\mathbf{c}^1 = \mathbf{d}^1$. We will inductively construct a rigid Δ-filtered module $M(\mathbf{c}^s)$ with Δ-dimension vector $\mathbf{c}^s$ for all $s = 1, \ldots, t+1$. If $t = 0$, then Q is linearly oriented, and we let $M(\mathbf{d}) = \Delta(\mathbf{d})$ as in [**2**]. Now suppose that $t > 0$. For $s = 1$, we let $M(\mathbf{c}^1) = M(\mathbf{d}^1)$. Suppose that we have $M(\mathbf{c}^s)$. We construct $M(\mathbf{c}^{s+1})$ by gluing $M(\mathbf{c}^s)$ and $M(\mathbf{d}^{s+1})$ at vertex i_s as follows.

First, we decompose $M(\mathbf{c}^s)$ into indecomposable direct summands

$$M(\mathbf{c}^s) = \bigoplus_l M_{K_l^s},$$

where $K_l^s \subseteq \{1, \ldots, i_s\}$ and $(\underline{\dim}_\Delta(M_{K_l^s}))_j = 1$ if $j \in K_l^s$ and zero elsewhere. Moreover, we reorder the indecomposable direct summands of $M(\mathbf{c}^s)$ such that $(\underline{\dim}_\Delta(M_{K_l^s}))_{i_s} = 1$ for $l = 1, \ldots, d_{i_s}$ and $M_{K_l^s} \geq_{i_s} M_{K_{l+1}^s}$ for $l = 1, \ldots, d_{i_s} - 1$. This is possible, by Lemma 5.7. We define $M(\mathbf{c}^{s+1})$ as

$$M(\mathbf{c}^{s+1}) = \bigoplus_l M_{K_l^{s+1}},$$

where $M_{K_l^{s+1}}$ is the module obtained by gluing $M_{K_l^s}$ and $M_{J^{s+1}_{d_{i_s}+1-l}}$ at i_s for $l = 1, \ldots, d_{i_s}$, and $\bigoplus_{l > d_{i_s}} M_{K_l^{s+1}}$ is the direct sum of all the terms $M_{K_l^s}$ and $M_{J_l^{s+1}}$ with $l > d_{i_s}$.

By the construction and the properties of gluing, we get that the representation $M(\mathbf{c}^{s+1})$ is Δ-filtered and each of its direct summands $M_{K_l^{s+1}}$ is indecomposable with $\mathrm{supp}_\Delta(M_{K_l^{s+1}})_0 = K_l^{s+1}$. Moreover, by Lemmas 5.2, 5.4, 5.5 and 5.6 we see that $\mathrm{Ext}^1_{\mathbf{D}}(M, N) = 0$ for any pair of indecomposable direct summands M and N in $M(\mathbf{c}^{s+1})$. Thus $M(\mathbf{c}^{s+1})$ is rigid.

Finally we have $M(\mathbf{d}) = M(\mathbf{c}^t)$ and we have completed the proof of the existence of the rigid module $M(\mathbf{d})$.

Let $\mathbf{b} = \sum_i d_i \underline{\dim}(\Delta(i))$. By Voigt's lemma [**17**] (see also [**6**]) we know that the $\mathrm{GL}(\mathbf{b})$-orbit of $M(\mathbf{d})$ in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{b})$ is open, and therefore dense, by Proposition 2.5. Hence $M(\mathbf{d})$ is unique up to isomorphism. □

Corollary 5.8. *Let N be a Δ-filtered module with Δ-dimension vector $\mathbf{d}$. Then N is contained in the Zariski closure of the* $\mathrm{GL}(\mathbf{b})$*-orbit of the rigid Δ-filtered module $M(\mathbf{d})$, where $\mathbf{b} = \sum_i d_i \underline{\dim}(\Delta(i))$.*

Two elements $j, j' \in \{1, \dots, n\}$ are called non-comparable if $j \not\preceq j'$ and $j' \not\preceq j$.

Corollary 5.9. *The map $J \mapsto M_J$ defines a bijection between isomorphism classes of indecomposable rigid Δ-filtered modules and subsets $J \subseteq \{1, \dots, n\}$ satisfying the following conditions:*
(1) *If $j \succeq i$ and $j' \succeq i$ for two non-comparable $j, j' \in J$, then $i \in J$.*
(2) *If $j \preceq i$ and $j' \preceq i$ for two non-comparable $j, j' \in J$, then $i \in J$.*

6. *Richardson elements in seaweed Lie algebras in* $\mathrm{gl}(V)$

We shall explain in this section how Theorem 5.1 proves the existence of Richardson elements in seaweed Lie algebras in $\mathrm{gl}(V)$.

Let $(\mathfrak{p}, \mathfrak{p}')$ be a pair of weakly opposite parabolic subalgebras in $\mathfrak{g} = \mathrm{gl}(V)$, where V is a k-vector space of dimension $n > 0$. It is proved in [**11**], [**14**], [**15**, Chapter 40] that since $\mathfrak{p} \cap \mathfrak{p}'$ contains a Cartan subalgebra of $\mathfrak{g}$, for a suitable choice of basis $B = (e_1, \dots, e_n)$ of V, $\mathfrak{p}$ is a standard parabolic subalgebra and $\mathfrak{p}'$ is an opposite standard parabolic subalgebra. So we can find two sequences of integers

$$0 = a_0 < a_1 < \cdots < a_r = n \ , \ 0 = b_0 < b_1 < \cdots < b_{r'} = n$$

such that $\mathfrak{p}$ (resp. $\mathfrak{p}'$) is the stabilizer in $\mathfrak{g}$ of the partial flag

$$\mathbf{F} : 0 = V_0 \subset V_1 \subset \cdots \subset V_r = V \ , (\text{resp. } \mathbf{F}' : 0 = W_0 \subset W_1 \subset \cdots \subset W_{r'} = V)$$

where $V_i = \mathrm{Vect}(e_1, \dots, e_{a_i})$ for $1 \le i \le r$, and $W_j = \mathrm{Vect}(e_{n-b_j+1}, \dots, e_n)$ for $1 \le j \le r'$.

Let $0 = m_0 < m_1 < \cdots < m_l = n$ be integers such that $\{m_0, \dots, m_l\}$ is the union of the a_i and $n - b_j$, $0 \le i \le r$, $0 \le j \le r'$. So for each $0 \le s \le l$, either $\mathrm{Vect}(e_1, \dots, e_{m_s})$ is some V_i or $\mathrm{Vect}(e_{m_s+1}, \dots, e_n)$ is some W_j.

For $1 \le s \le l$, set

$$E_s = \mathrm{Vect}(e_{m_{s-1}+1}, \dots, e_{m_s}), \text{ and } Y_s = V_{u(s)} \cap W_{\ell(s)},$$

where $u(s) = \min\{i; E_s \subseteq V_i\}$ and $\ell(s) = \min\{j; E_s \subseteq W_j\}$.

By definition, any V_i (resp. W_j) is of the form $E_1 \oplus \cdots \oplus E_s$ (resp. $E_{s+1} \oplus \cdots \oplus E_l$) for some $1 \le s \le l$. So there exist $p \le s \le q$ such that $Y_s = E_p \oplus \cdots \oplus E_q$, and the next lemma follows easily from the definitions.

Lemma 6.1. *Set $Y_0 = Y_{l+1} = \{0\}$. For $1 \le s \le l$, we have one of following four cases :*

$$Y_s = Y_{s-1} \oplus E_s \ , \ Y_s = Y_{s-1} \oplus E_s \oplus Y_{s+1} \ , \ Y_s = Y_{s+1} \oplus E_s \ , \ Y_s = E_s.$$

We define a quiver Q whose vertices are the integers $\{1, \dots, l\}$, and for $i = 1, \dots, l-1$, we have an arrow

$$i \xrightarrow{\alpha_i} i+1 \ \text{ if } Y_i \subset Y_{i+1} \text{ or } \ i \xleftarrow{\alpha_i} i+1 \ \text{ if } Y_i \supset Y_{i+1}.$$

Note that Q is not necessarily connected, however each of its connected components is of type A.

Now let $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ be the associated seaweed Lie algebra in gl(V). Then its Levi factor $\mathfrak{l}$ is gl$(E_1) \times \cdots \times$ gl(E_l). For $x \in \mathfrak{n}$ the nilpotent radical of $\mathfrak{q}$, we have that $x(E_s) \subseteq \bigoplus_{i \neq s} E_i$ for all $s = 1, \ldots, l$. Since elements of $\mathfrak{q}$ stabilize the two flags, it follows that

$$x(Y_s) \subseteq Y_{s-1} \ , \ x(Y_s) \subseteq Y_{s-1} \oplus Y_{s+1} \ , \ x(Y_s) \subseteq Y_{s+1} \ , \ x(Y_s) = \{0\}$$

according to the four cases in Lemma 6.1. Clearly, any $x \in$ gl(V) verifying these conditions belongs to $\mathfrak{n}$.

We define a representation M^x of the double $\tilde{Q}$ of the quiver Q as follows : $M^x_i = Y_i$, $M^x_{\alpha_i}$ is just the canonical injection, and $M^x_{\beta_i}$ is the projection to $Y_{s(\alpha_i)}$ with respect to the direct sum decompositions in Lemma 6.1 of the restriction of x to $Y_{t(\alpha_i)}$. Clearly, we have $M^x \in \mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ where $\mathbf{d} = (\dim Y_i)_{1 \le i \le l}$.

Let $\mathbf{P}$ and $\mathbf{P}'$ be the parabolic subgroups in $\mathbf{G} = \mathrm{GL}(V)$ corresponding to $\mathfrak{p}$ and $\mathfrak{p}'$. Set $\mathbf{Q} = \mathbf{P} \cap \mathbf{P}'$. Thus $\sigma \in \mathbf{Q}$ if and only if σ stabilizes both flags, or equivalently, σ stabilizes Y_s for $s = 1, \ldots, l$.

Theorem 6.2. *There is a bijection between the $\mathbf{Q}$-orbits in $\mathfrak{n}$ and the* GL$(\mathbf{d})$*-orbits in* $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$*. In particular, $\mathfrak{n}$ contains a unique open $\mathbf{Q}$-orbit.*

Proof. Let R^α be as defined just before Proposition 2.5. By the construction above, we have an injective map

$$\Phi : \mathfrak{n} \longrightarrow R^\alpha \ , \ x \mapsto M^x.$$

Given $M \in R^\alpha$. By the relation of $\mathcal{I}$, we may define an endomorphism x of V, given by for $v \in E_i$, $x(v) = M_{\beta_j}(v)$ if $E_i \subseteq Y_{s(\beta_j)}$, for some β_j, and $x(v) = 0$ otherwise. We can check that $x \in \mathfrak{n}$, and $\Phi(x) = M$. Thus Φ is a bijection.

The restrictions to Y_i of $\tau \in \mathbf{Q}$ determine an element of GL$(\mathbf{d})$ which stabilizes R^α. We see readily that we may identify $\mathbf{Q}$ with the stabilizer of R^α in GL$(\mathbf{d})$. It follows that there is a bijection between $\mathbf{Q}$-orbits in R^α and GL$(\mathbf{d})$-orbits in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$.

Since the map Φ is clearly $\mathbf{Q}$-equivariant, we have the required bijection, and by Theorem 5.1, $\mathfrak{n}$ has a unique open $\mathbf{Q}$-orbit. □

Remark 6.3. Via the bijection in Theorem 6.2, we obtain readily that the modality of the $\mathbf{Q}$-action on $\mathfrak{n}$ is the same as the modality of the GL$(\mathbf{d})$-action on $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ (See [**16**] for a definition of the modality of an action).

Example 6.4. From the construction of the explicit rigid Δ-filtered module with a given Δ-dimension vector in section 5, we obtain a procedure for constructing an explicit Richardson element, imitating the construction given in [**2**] for parabolic subalgebras (see also [**1**]). Let us illustrate this by an example. Take $n = 9$, $(a_1, a_2) = (3, 4)$, $(b_1, b_2) = (2, 8)$. Then with respect to the basis B, $\mathfrak{q}$ consists of matrices of the form

$$\left(\begin{array}{c|cc|c|ccc|cc}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
* & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 \\ \hline
0 & 0 & 0 & * & * & * & * & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & *
\end{array}\right)$$

and $\mathfrak{n}$ consists of those elements of $\mathfrak{q}$ whose diagonal blocks are zero. We have $l = 5$, $(m_1, m_2, m_3, m_4) = (1, 3, 4, 7)$, $E_1 = \mathrm{Vect}(e_1)$, $E_2 = \mathrm{Vect}(e_2, e_3)$, $E_3 = \mathrm{Vect}(e_4)$, $E_4 =$

$\mathrm{Vect}(e_5, e_6, e_7)$, $E_5 = \mathrm{Vect}(e_8, e_9)$ and $Y_1 = E_1 \oplus E_2$, $Y_2 = E_2$, $Y_3 = E_2 \oplus E_3$, $Y_4 = E_2 \oplus E_3 \oplus E_4 \oplus E_5$, $Y_5 = E_5$ are just the sum of blocks appearing in each column. The corresponding quiver Q is :

$$\begin{tikzcd} 1 & 2 \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"'] & 3 \arrow[r, "\alpha_3"'] & 4 & 5 \arrow[l, "\alpha_4"'] \end{tikzcd}$$

The construction of the rigid module $M(\mathbf{e})$ in section 5, where $\mathbf{e} = (\dim E_i)_{i=1,\dots,5}$, consists of gluing the following 3 rigid modules with linear support :

$$\begin{tikzcd} & & & & & \Delta(4) & \Delta(4) & \Delta(5) \arrow[l] \\ \Delta(1) & \Delta(2) \arrow[l] & \Delta(2) \arrow[rrr] & & & \Delta(4) & \Delta(4) & \Delta(5) \arrow[l] \\ & \Delta(2) & \Delta(2) \arrow[r] & \Delta(3) \arrow[rr] & & \Delta(4) & \Delta(4) & \end{tikzcd}$$

Notice that we have lined up the indecomposable summands from small to big according to $>_2$ in the first column, from big to small in the second column according to $>_2$ but small to big according to $>_4$, and finally in the third column from big to small according to $>_4$. The direction of the arrow is from big to small relative to the order $\prec$ on the vertices.

Thus $M(\mathbf{e})$ can be represented by the following arrowed diagram :

$$\begin{tikzcd} & & & & \Delta(4) & \Delta(5) \arrow[l] \\ \Delta(1) & \Delta(2) \arrow[l] \arrow[rrr] & & & \Delta(4) & \Delta(5) \arrow[l] \\ & \Delta(2) \arrow[r] & \Delta(3) \arrow[rr] & & \Delta(4) & \end{tikzcd}$$

To obtain an explicit Richardson element in $\mathfrak{n}$, we replace column-wise the $\Delta(i)$'s by the numbers 1 to 9, starting on the left and from the bottom to the top.

Then

$$x = \sum_{i \to j} E_{ij} = E_{31} + E_{24} + E_{45} + E_{36} + E_{86} + E_{97} = \left(\begin{array}{c|cc|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

is a Richardson element of $\mathfrak{q}$.

7. *Remarks on seaweed Lie algebras of other types*

In this section the field has characteristic zero. By considering generic elements, we were able to check by using GAP4 that Richardson elements exist in any seaweed Lie algebra in a simple Lie algebra $\mathfrak{g}$ associated to an irreducible root system of rank ≤ 7. Also, by a type by type consideration and the fact that abelian ideals of parabolic subalgebras are spherical [**12**], we can show that if the nilpotent radical of a seaweed Lie algebra is abelian, then it has a Richardson element.

Let Π denote the set of simple roots of the irreducible root system of type E_8. In the numbering of simple roots of [**15**, Chapter 18], set $S = \Pi \setminus \{\alpha_8\}$, $T = \Pi \setminus \{\alpha_4, \alpha_5\}$. Denote by $\mathfrak{p}_T^+$ (resp. $\mathfrak{p}_S^-$) the standard parabolic (resp. opposite parabolic) subalgebra associated to T (resp. S).

The seaweed Lie algebra $\mathfrak{q} = \mathfrak{p}_T^+ \cap \mathfrak{p}_S^-$ contains the standard Cartan subalgebra $\mathfrak{h}$, and therefore $\mathfrak{q} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}^+$ where $\mathfrak{l}$ is the $\mathfrak{h}$-stable Levi factor, $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) the subspace spanned by the remaining positive (resp. negative) root vectors. In particular, $\mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{n}^+$

is the nilpotent radical of $\mathfrak{q}$. Observe that $\mathfrak{l} \oplus \mathfrak{n}^+$ (resp. $\mathfrak{l} \oplus \mathfrak{n}^-$) is a parabolic subalgebra of reductive Levi factor $\mathfrak{l}_S$ of $\mathfrak{p}_S^-$ (resp. $\mathfrak{l}_T$ of $\mathfrak{p}_T^+$).

We see immediately that if $x_\pm \in \mathfrak{n}^\pm$ are elements such that $x = x_- + x_+$ is a Richardson element of $\mathfrak{q}$, then $x_\pm$ is a Richardson element of $\mathfrak{l} \oplus \mathfrak{n}^\pm$, and we have $[x_\mp, \mathfrak{q}^{x_\pm}] = \mathfrak{n}^\mp$. Using this fact, we were able to check that $\mathfrak{q}$ does not have any Richardson element.

Note that if we take arbitrary Richardson elements of $\mathfrak{l} \oplus \mathfrak{n}^\pm$, their sum need not be a Richardson element of $\mathfrak{q}$. This explains in part why there is a constraint in how to glue rigid Δ-filtered modules with linear Δ-support in Section 5.

Acknowledgements. This work was done during the authors' stay at Universität zu Köln and Université de Poitiers, and the authors thank both institutes for their hospitality and support, in particular, Abderrazak Bouaziz, Steffen König and Peter Littelmann. The authors thank the referee for useful comments on an earlier version of this manuscript, and for pointing out the connection with the volume of a tilting module [**7**], which was also mentioned to us by Lutz Hille. The authors would like to thank Michel Duflo, Lutz Hille, Steffen König, Dmitri Panyushev, Patrice Tauvel and Changchang Xi for interesting questions and discussions.

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Bernt Tore JENSEN
Department of Mathematical Sciences,
Norwegian University of Science and Technology,
N-7034 Trondheim,
Norway.

berntj@math.ntnu.no

Rupert Wei Tze YU
UMR 6086 CNRS,
Département de Mathématiques,
Université de Poitiers,
Téléport 2 - BP 30179,
Boulevard Marie et Pierre Curie,
86962 Futuroscope Chasseneuil cedex,
France.

yuyu@math.univ-poitiers.fr

Xiuping SU
Mathematisches Insitut,
Universität zu Köln,
Weyertal 86-90, 50931 Köln,
Germany.

xsu@math.uni-koeln.de