Willem’s Adventures in Curry–Howard Land

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Part I: The computational side of deep inference

Open deduction

[Guglielmi, Gundersen & Parigot 2010]

$$\boxed{\begin{array}{c}A\\ \Downarrow\\ C\end{array}} \quad ::= \quad a \quad \Big| \quad \boxed{\begin{array}{c}A_1\\ \Downarrow\\ C_1\end{array}} \star \boxed{\begin{array}{c}A_2\\ \Downarrow\\ C_2\end{array}} \quad \Big| \quad \begin{array}{c}\boxed{\begin{array}{c}A\\ \Downarrow\\ B_1\end{array}}\\ \hline\hline \boxed{\begin{array}{c}B_2\\ \Downarrow\\ C\end{array}}\end{array}\, r$$

A *derivation* from assumption A to conclusion C:

- *Atom a*
- *Horizontal construction* with connective $\star$ with *arity* in $\{+,-\}^*$
- *Vertical construction* with rule r from B_1 to B_2

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An open-deduction *proof system* is given by:

- A signature of *connectives*
- A set of *rules*

Open deduction for intuitionistic logic

Derivations:

$$\boxed{\begin{array}{c} A \\ \Big\Downarrow \\ C \end{array}} \quad ::= \quad a$$

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Connectives: $\rightarrow \ \wedge \ \top$ of arity: $(-+)$ $(++)$ $()$

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Connectives: $\to \;\wedge\; \top$ of arity: $(-+)\;(++)\;()$

Rules:

$$\frac{B}{A \to (B \wedge A)} \qquad \frac{(A \to B) \wedge A}{B} \qquad \frac{A}{A \wedge A} \qquad \frac{A}{\top}$$

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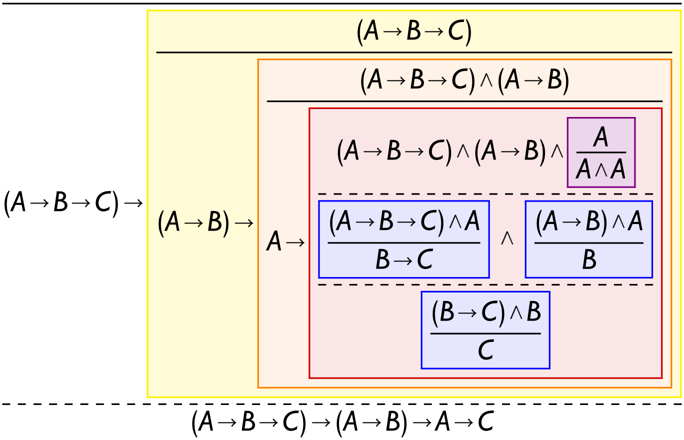
$$\begin{array}{c}A\wedge\top\\ \hdashline A\end{array} \qquad \begin{array}{c}A\\ \hdashline A\wedge\top\end{array} \qquad \begin{array}{c}A\wedge B\\ \hdashline B\wedge A\end{array} \qquad \begin{array}{c}(A\wedge B)\wedge C\\ \hdashline A\wedge(B\wedge C)\end{array} \qquad \begin{array}{c}A\wedge(B\wedge C)\\ \hdashline (A\wedge B)\wedge C\end{array}$$

Example



$(A\to B\to C)\to$

$(A\to B\to C)$

$(A\to B)\to$

$(A\to B\to C)\land(A\to B)$

$A\to$

$(A\to B\to C)\land(A\to B)\land\dfrac{A}{A\land A}$

$\dfrac{(A\to B\to C)\land A}{B\to C}\land\dfrac{(A\to B)\land A}{B}$

$\dfrac{(B\to C)\land B}{C}$

$(A\to B\to C)\to(A\to B)\to A\to C$

Benefits

- Universal framework for proof systems
- *Locality:* correctness of a rule is locally verifiable
- New fine-grained rules such as *medial*:

$$\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

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Results

- Expresses more logics than sequent calculi (BV) [Tiu 2006]
- Quasipolynomial normalization (CPL)
  [Jeřábek 2009; Bruscoli, Guglielmi, Gundersen & Parigot 2016]
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- No *subformula property*

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Remark

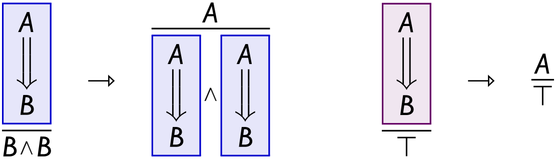
- Same syntax as *category theory* but different aims and techniques

What is the computational meaning of open deduction?

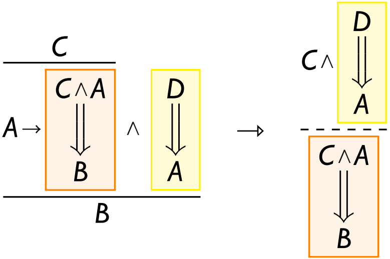
Reduction in open deduction

[Brünnler & McKinley 2008]

duplication/
deletion



beta-reduction



Similar to categorical combinators [Curien 1986]

Preservation of strong normalization (PSN) fails

$$\frac{\boxed{\dfrac{A}{\dfrac{A \wedge \dfrac{A}{\top}}{A}}}}{\dfrac{A \wedge \dfrac{A}{\top}}{A}}$$

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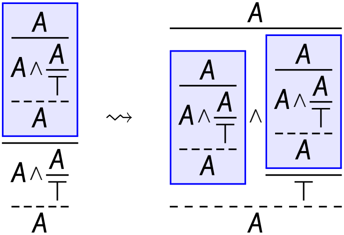

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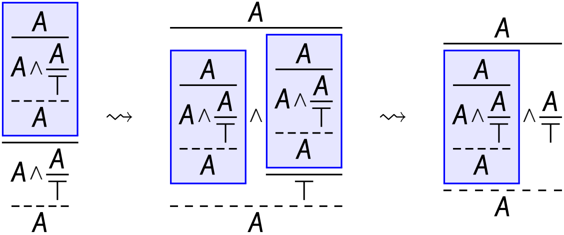

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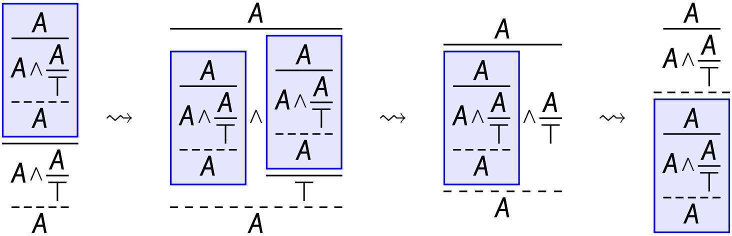

Open deduction as a type system

Terms: $M, N \quad ::= \quad x \mid M\,N \mid \lambda x.M$

Types: $A, B, C \quad ::= \quad a \mid A \to B$

Contexts: $\Gamma^{\bar{x}}, \Delta^{\bar{y}} \quad ::= \quad A_1^{x_1} \wedge \ldots \wedge A_n^{x_n}$

Derivations: $\boxed{\begin{array}{c}\Gamma^{\bar{x}} \\ \Big\| \, M \\ A\end{array}}$

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Derivations: $\boxed{\begin{array}{c}\Gamma^{\bar{x}}\\ \Vert\, M\\ A\end{array}}$

x

$$A^x$$

$M\,N$

$$\frac{\boxed{\begin{array}{c}\Gamma^{\bar{x}}\\ \Vert\, M\\ A\to B\end{array}} \wedge \boxed{\begin{array}{c}\Delta^{\bar{y}}\\ \Vert\, N\\ A\end{array}}}{B}\,@$$

$\lambda x.M$

$$\frac{\Gamma^{\bar{y}}}{A \to \boxed{\begin{array}{c}\Gamma^{\bar{y}} \wedge A^x\\ \Vert\, M\\ B\end{array}}}\,\lambda$$

Open deduction as a type system

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$$\begin{array}{c} A^x \\ \hdashline A^x \wedge \ldots \wedge A^x \end{array}\,\triangle$$

$$\begin{array}{c} A \wedge \top \\ \hdashline A \end{array} \qquad \begin{array}{c} A \\ \hdashline A \wedge \top \end{array} \qquad \begin{array}{c} A \wedge B \\ \hdashline B \wedge A \end{array} \qquad \begin{array}{c} (A \wedge B) \wedge C \\ \hdashline A \wedge (B \wedge C) \end{array} \qquad \begin{array}{c} A \wedge (B \wedge C) \\ \hdashline (A \wedge B) \wedge C \end{array}$$

Open deduction as a type system

Terms: $M, N ::= x \mid M\,N \mid \lambda x.M \mid M[x \leftarrow N]$

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Derivations: $\boxed{\begin{array}{c}\Gamma^{\bar{x}}\\ \Vert\, M\\ A\end{array}}$

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$$\frac{\boxed{\begin{array}{c}\Gamma^{\bar{x}}\\ \Vert\, M\\ A \to B\end{array}} \wedge \boxed{\begin{array}{c}\Delta^{\bar{y}}\\ \Vert\, N\\ A\end{array}}}{B}\,@$$

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$M[x \leftarrow N]$

$$\begin{array}{c}\Gamma^{\bar{y}} \wedge \boxed{\begin{array}{c}\boxed{\begin{array}{c}\Delta^{\bar{z}}\\ \Vert\, N\\ A\end{array}}\\ \hline A^x \wedge \ldots \wedge A^x\end{array}}^{\vartriangle}\\ \hdashline \boxed{\begin{array}{c}\Gamma^{\bar{y}} \wedge A^x \wedge \ldots \wedge A^x\\ \Vert\, M\\ B\end{array}}\end{array}$$

$$\begin{array}{c}A \wedge \top\\ \hdashline A\end{array} \qquad \begin{array}{c}A\\ \hdashline A \wedge \top\end{array} \qquad \begin{array}{c}A \wedge B\\ \hdashline B \wedge A\end{array} \qquad \begin{array}{c}(A \wedge B) \wedge C\\ \hdashline A \wedge (B \wedge C)\end{array} \qquad \begin{array}{c}A \wedge (B \wedge C)\\ \hdashline (A \wedge B) \wedge C\end{array}$$

In Curry–Howard–Lambek

- Same embedding of natural deduction in deep inference as in cartesian closed categories [Lambek 1972]
- Not an isomorphism, but a correspondence or interpretation
- Many derivations for one λ-term
- Terms guide reduction

Atomic reduction

Medial rules make contractions atomic:

$$\frac{(A \vee B) \to (C \wedge D)}{(A \to C) \wedge (B \to D)}\,\mathsf{m} \qquad \frac{a}{a \wedge a}\,{\scriptstyle\triangle} \qquad \frac{a \vee a}{a}\,{\scriptstyle\blacktriangledown}$$

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$$\boxed{\frac{A\to B}{(A\to B)\wedge(A\to B)}\,\vartriangle} \quad\longrightarrow\quad \frac{\boxed{\blacktriangle\,\frac{A}{A\vee A}}\to\boxed{\frac{B}{B\wedge B}\,\vartriangle}}{(A\to B)\wedge(A\to B)}\,\mathsf{m}$$

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$$\boxed{\frac{A\to B}{(A\to B)\wedge(A\to B)}\,\vartriangle}\quad\longrightarrow\quad\frac{\boxed{\blacktriangle\,\frac{A}{A\vee A}}\ \to\ \boxed{\frac{B}{B\wedge B}\,\vartriangle}}{(A\to B)\wedge(A\to B)}\,\mathsf{m}$$

$$\boxed{\frac{A\wedge B}{(A\wedge B)\wedge(A\wedge B)}\,\vartriangle}\quad\longrightarrow\quad\begin{array}{c}\boxed{\frac{A}{A\wedge A}\,\vartriangle}\ \wedge\ \boxed{\frac{B}{B\wedge B}\,\vartriangle}\\ \hdashline (A\wedge B)\wedge(A\wedge B)\end{array}$$

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Consequences for proof complexity — here, we look at computational meaning

Simplify to a distribution rule to avoid disjunction:

$$\frac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}\,{}^{d} \qquad \boxed{\frac{A \to B}{(A \to B) \wedge (A \to B)}\,{}^{\triangle}} \longrightarrow \frac{A \to \boxed{\frac{B}{B \wedge B}\,{}^{\triangle}}}{(A \to B) \wedge (A \to B)}\,{}^{d}$$

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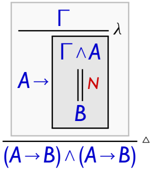
Introduce a corresponding distributor term construct:

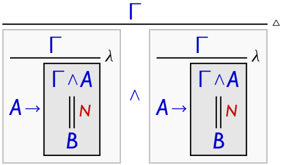
$$M[x \twoheadleftarrow \lambda y.T] \qquad \boxed{\frac{\dfrac{\Delta}{A\to \boxed{\begin{array}{c}\Delta\wedge A^{y}\\ \Vert\, T\\ B\wedge\ldots\wedge B\end{array}}}\,\lambda}{(A\to B)^{x}\wedge\ldots\wedge(A\to B)^{x}}\,d}$$

$$\dfrac{\boxed{\dfrac{\Gamma}{A \to \boxed{\begin{array}{c}\Gamma \wedge A\\ \Big\Vert\, N\\ B\end{array}}}\,\lambda}}{(A \to B) \wedge (A \to B)}\,\triangle$$

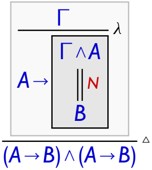
$$M[x \leftarrow \lambda y.N]$$



$M[x \leftarrow \lambda y.N]$

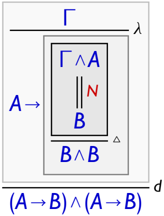


$M\{\lambda y.N/x\}$

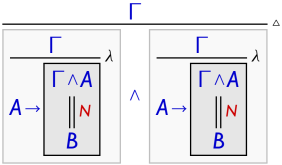


$M[x \leftarrow \lambda y.N]$

$\longrightarrow$



$M[x \twoheadleftarrow \lambda y.\langle z, z\rangle[z \leftarrow N]]$



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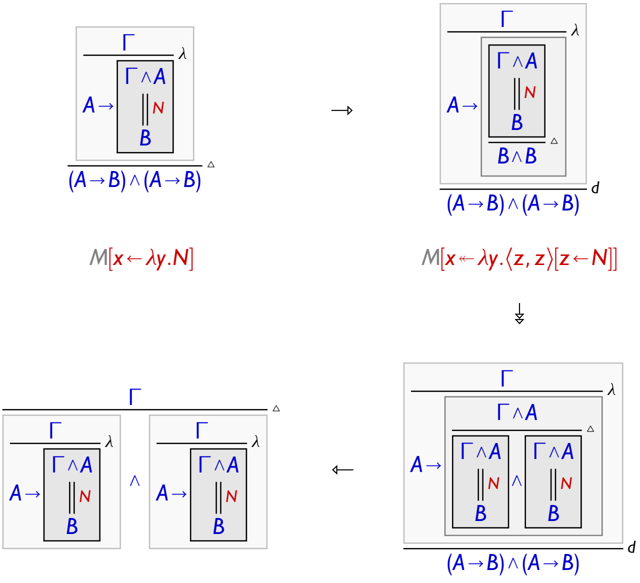


$M[x\leftarrow\lambda y.N]$

$M[x\twoheadleftarrow\lambda y.\langle z,z\rangle[z\leftarrow N]]$

$M[x\twoheadleftarrow\lambda y.\langle N,N\rangle]$

$M\{\lambda y.N/x\}$

The atomic λ-calculus family

Term calculi for:

- Full laziness [Gundersen, H & Parigot 2013]
- Atomic $\lambda\mu$-reduction [He 2018]
- Spinal full laziness [Sherratt, H, Gundersen & Parigot 2020]
- Atomic distance reduction [Kesner, Peyrot & Ventura 2021]

Deep reduction

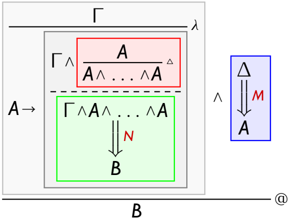
Deep reduction

(Or categorically: dinaturality)

$$\cfrac{\Gamma}{A \to \boxed{\Gamma \wedge \boxed{\cfrac{A}{A \wedge \ldots \wedge A}\,\vartriangle}}}\,\lambda \quad\longrightarrow\quad \cfrac{\Gamma}{\boxed{\triangledown\,\cfrac{A \wedge \ldots \wedge A}{A}} \to \boxed{\Gamma \wedge A}}\,\lambda$$

$$\cfrac{\boxed{\boxed{\triangledown\,\cfrac{A \wedge \ldots \wedge A}{A}} \to B} \wedge A}{B}\,@ \quad\longrightarrow\quad \cfrac{\boxed{A \to B} \wedge \boxed{\cfrac{A}{A \wedge \ldots \wedge A}\,\vartriangle}}{B}\,@$$

$(\lambda x.N)M$



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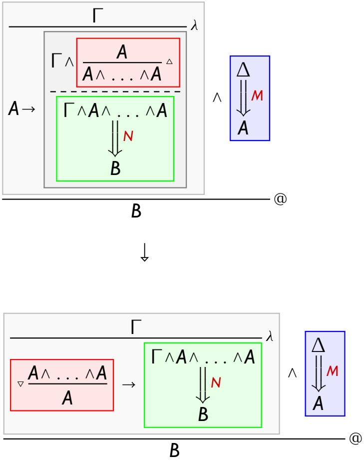

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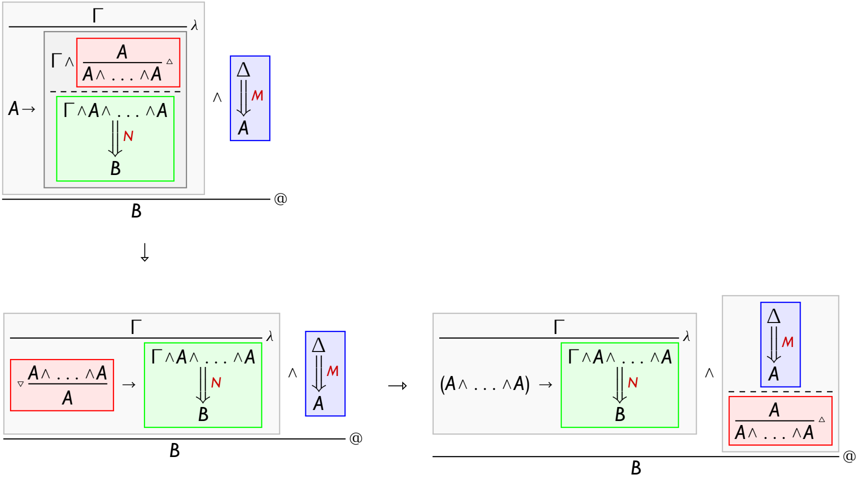
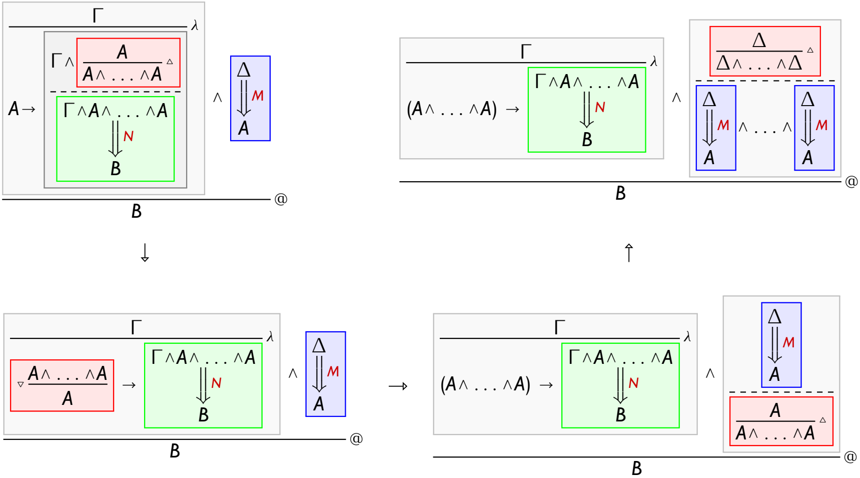


$(\lambda x.N)M$

$(\lambda\langle x_1,\ldots,x_n\rangle.N)\,\langle M,\ldots,M\rangle$



Deep reduction takes a simply-typed term to a resource λ-term

Resource λ-calculus: [Boudol 1993]

$$M, N \ ::= \ x \ \mid \ \lambda\langle x_1, \ldots, x_n\rangle.M \ \mid \ M\,\langle N_1, \ldots, N_n\rangle$$

Equivalently, deep reduction takes a simple type derivation to a (non-idempotent) intersection-type derivation

Intersection types: [Coppo & Dezani 1978]

$$A, B \ ::= \ a \ \mid \ (A_1 \cap \ldots \cap A_n) \rightarrow B$$

Open-deduction intersection types

[Guerrieri, H & Paulus 2021]

Types: $A, B \quad ::= \quad a \mid I \rightarrow A$

Collections: $I, J \quad ::= \quad A \mid I \cap J$

Contexts: $\Gamma^{\bar{x}}, \Delta^{\bar{y}} \quad ::= \quad I_1^{x_1} \wedge \ldots \wedge I_n^{x_n}$

Derivations: $\boxed{\begin{array}{l} \Gamma^{\bar{x}} \\ \,\Big\| \, M \\ A \end{array}}$

Open-deduction intersection types

[Guerrieri, H & Paulus 2021]

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Contexts: $\Gamma^{\overline{x}}, \Delta^{\overline{y}} \quad ::= \quad I_1^{x_1} \wedge \ldots \wedge I_n^{x_n}$

Derivations: $\boxed{\begin{array}{c}\Gamma^{\overline{x}} \\ \Big\| M \\ A\end{array}}$

$$\boxed{\begin{array}{c}\Gamma^{\overline{x}} \\ \Big\| M \\ I\end{array}} \cap \boxed{\begin{array}{c}\Delta^{\overline{x}} \\ \Big\| M \\ J\end{array}}$$

Open-deduction intersection types

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Derivations: $\boxed{\begin{array}{l}\Gamma^{\bar{x}} \\ \quad \Big\| \, M \\ A\end{array}}$

$$\boxed{\begin{array}{l}\Gamma^{\bar{x}} \\ \quad \Big\| \, M \\ I\end{array}} \cap \boxed{\begin{array}{l}\Delta^{\bar{x}} \\ \quad \Big\| \, M \\ J\end{array}} \qquad\qquad \frac{(I \cap J)^{x}}{I^{x} \wedge J^{x}}\,\triangle$$

Open-deduction intersection types

[Guerrieri, H & Paulus 2021]

Types: $A, B \quad ::= \quad a \mid I \rightarrow A$

Collections: $I, J \quad ::= \quad A \mid I \cap J$

Contexts: $\Gamma^{\overline{x}}, \Delta^{\overline{y}} \quad ::= \quad I_1^{x_1} \wedge \ldots \wedge I_n^{x_n}$

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$$\frac{(A \to B) \land C}{A \to (B \land C)}$$

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[Hendriks & Van Oostrom 2003; Sherratt et al 2020]

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$$\boxed{\begin{array}{c} A \\ \Vert \\ C \end{array}} \; a \; \boxed{\begin{array}{c} B \\ \Vert \\ D \end{array}} \qquad\qquad \frac{(A \mathbin{a} B) \to (C \mathbin{a} D)}{(A \to C) \mathbin{a} (B \to D)}$$

Subatomic logic: may be interpreted as conditionals or decision trees

[Barrett & Guglielmi 2022; Dal Lago, Guerrieri & H. 2020]

Summary

Deep inference is explicit $\Longrightarrow$ fine-grained computational notions

- atomic λ-calculus
- deep intersection types
- switch as end-of-scope
- subatomic logic as conditionals

All these extend the standard Curry–Howard–Lambek correspondence.

But . . .

Part II: The logical side of the Functional Machine Calculus

Effects

The problem: how to combine λ-calculus with computational effects
(I/O, store, non-determinism, error handling, concurrency, etc.)

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Call–by–value with thunks	[Landin 1964, Plotkin 1975]
Monads (Haskell)	[Moggi 1989]
Call–by–push–value	[Levy 1999]
Effect handlers	[Plotkin & Pretnar 2009]

Approach

- effects require sequentiality
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Simple abstract machine: cf. KAM [Krivine 2007]

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application is PUSH $\quad M\,N \;=\; [N].M \quad \dfrac{(\,S\quad,\ [N].M\,)}{(\,S\,N\ ,\quad M\,)}$

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application is PUSH $\quad MN = [N].M \quad \dfrac{(\,S\;,\;[N].M\,)}{(\,S\,N\;,\;M\,)}$

abstraction is POP $\quad \lambda x.M = \langle x\rangle.M \quad \dfrac{(\,S\,N\;,\;\langle x\rangle.M\,)}{(\,S\;,\;\{N/x\}M\,)}$

From λ-calculus to FMC

[H 2022; Barrett, H & McCusker 2023]

$$M, N \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad M\,N$$

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- λ-terms as sequential processes

$\star$	identity, skip, empty
$M; N$	composition, sequencing

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$(S, \star)$ $\qquad$ $[M].[N].\star$ $\quad$ (shorten to $[M].[N]$)

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- similar ideas have appeared before:
  - kappa-calculus [Hasegawa 1995, Power & Thielecke 1999]
  - compiler calculi [Douence & Fradet 1998]
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  - concatenative programming [Pestov et al 2010]
- one more trick gives confluence with state, IO, probabilities

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$M, N ::= x \mid \langle x\rangle.M \mid [N].M$	

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Simple types indicate

- function inputs — ultimate output is o

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Still (conjunction–implication) intuitionistic logic:

$$A_1 \ldots A_n \Rightarrow B_1 \ldots B_m \quad = \quad (A_1 \wedge \cdots \wedge A_n) \to (B_1 \wedge \cdots \wedge B_m)$$

Types as logic/categories

Types defined with vector notation; empty vector ε, concatenation $\overline{A} \cdot \overline{B}$

$$A, B ::= \overline{A} \Rightarrow \overline{B} \qquad\qquad \overline{A} ::= A_1 \ldots A_n$$

Type vectors are formulas, terms are derivations

$$\begin{tikzcd}
 & \overline{A} \arrow[d, Rightarrow, "M"'] \\
\overline{A} & \overline{B} \arrow[d, Rightarrow, "N"'] \\
 & \overline{C}
\end{tikzcd}$$

$$\star \qquad\qquad M; N$$

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$$\frac{\overline{A}\cdot\overline{B}}{\overline{B}\cdot\overline{A}}$$

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\overline{A} \arrow[d, Rightarrow, "M"'] \\
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\end{tikzcd}$$

$$\frac{\overline{A}}{\varepsilon}$$

$$\star \qquad\qquad M; N \qquad\qquad\qquad\qquad \langle\overline{x}\rangle$$

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Type vectors are formulas, terms are derivations

$$\langle\overline{x}\rangle.\langle\overline{y}\rangle.[\overline{x}].[\overline{y}] \qquad \langle\overline{x}\rangle.[\overline{x}].[\overline{x}] \qquad \langle x\rangle.x$$

$$\frac{\overline{A}\cdot\overline{B}}{\overline{B}\cdot\overline{A}} \qquad \frac{\overline{A}}{\overline{A}\cdot\overline{A}} \qquad \frac{\overline{A}\cdot(\overline{A}\Rightarrow\overline{B})}{\overline{B}}$$

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\overline{A} \arrow[d, Rightarrow, "M"'] \\
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\overline{C}
\end{tikzcd}$$

$$\overline{A}$$

$$\frac{\overline{A}}{\varepsilon} \qquad \frac{\overline{B}}{\overline{A}\Rightarrow(\overline{A}\cdot\overline{B})}$$

$$\star \qquad M; N \qquad \langle\overline{x}\rangle \qquad \langle\overline{x}\rangle.[[\overline{x}]]$$

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Types defined with vector notation; empty vector ε, concatenation $\bar{A}\cdot\bar{B}$

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Type vectors are formulas, terms are derivations

$$\begin{array}{ccc}
\langle\bar{x}\rangle.\langle\bar{y}\rangle.[\bar{x}].[\bar{y}] & \langle\bar{x}\rangle.[\bar{x}].[\bar{x}] & \langle x\rangle.x \\
\dfrac{\bar{A}\cdot\bar{B}}{\bar{B}\cdot\bar{A}} & \dfrac{\bar{A}}{\bar{A}\cdot\bar{A}} & \dfrac{\bar{A}\cdot(\bar{A}\Rightarrow\bar{B})}{\bar{B}}
\end{array}$$

$$\begin{array}{ccccc}
\bar{A} & \begin{array}{c}\bar{A}\\ M\Downarrow\\ \bar{B}\\ N\Downarrow\\ \bar{C}\end{array} & \begin{array}{c}\bar{A}\\ M\Downarrow\ \cdot\bar{C}\\ \bar{B}\end{array} & \dfrac{\bar{A}}{\varepsilon} & \dfrac{\bar{B}}{\bar{A}\Rightarrow(\bar{A}\cdot\bar{B})} \\
\star & M;N & \langle\bar{x}\rangle.M.[\bar{x}] & \langle\bar{x}\rangle & \langle\bar{x}\rangle.[[\bar{x}]]
\end{array}$$

Curry–Howard–Lambek?

This is a different term interpretation of intuitionistic logic and cartesian closed categories

	standard	FMC
elements:	open terms	closed terms
premisses:	(types of) free variables	(type of) input stack
conclusion:	(type of) term	(type of) output stack

Fragments

Cartesian closed: higher-order, non-linear

$$\overline{A}$$

$$\star$$

$$\begin{tikzcd} \overline{A} \arrow[d, Rightarrow, "N"'] \\ \overline{B} \arrow[d, Rightarrow, "M"'] \\ \overline{C} \end{tikzcd}$$

$$N; M$$

$$\langle\overline{x}\rangle.\langle\overline{y}\rangle.[\overline{x}].[\overline{y}]$$

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$$\langle x\rangle.x$$

$$\frac{\overline{A}\cdot(\overline{A}\Rightarrow\overline{B})}{\overline{B}}$$

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Fragments

Cartesian: first-order, non-linear

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$$\langle\overline{x}\rangle.[\overline{x}].[\overline{x}]$$

$$\frac{\overline{A}}{\overline{A}\cdot\overline{A}}$$

$$\frac{\overline{A}}{\varepsilon}$$

$$\langle\overline{x}\rangle$$

Fragments

Symmetric monoidal closed: higher-order, linear, symmetric

$$\overline{A}$$

$$\star$$

$$\begin{tikzcd}
\overline{A} \arrow[d, Rightarrow, "N"'] \\
\overline{B} \arrow[d, Rightarrow, "M"'] \\
\overline{C}
\end{tikzcd}$$

$$N; M$$

$$\langle \overline{x} \rangle . \langle \overline{y} \rangle . [\overline{x}] . [\overline{y}]$$

$$\frac{\overline{A} \cdot \overline{B}}{\overline{B} \cdot \overline{A}}$$

$$\begin{tikzcd}
\overline{A} \arrow[d, Rightarrow, "M"', "\cdot \overline{C}"] \\
\overline{B}
\end{tikzcd}$$

$$\langle \overline{x} \rangle . M . [\overline{x}]$$

$$\langle x \rangle . x$$

$$\frac{\overline{A} \cdot (\overline{A} \Rightarrow \overline{B})}{\overline{B}}$$

$$\frac{\overline{B}}{\overline{A} \Rightarrow (\overline{A} \cdot \overline{B})}$$

$$\langle \overline{x} \rangle . [[\overline{x}]]$$

Fragments

Symmetric monoidal: first-order, linear, symmetric

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$$\star$$

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$$N; M$$

$$\langle\overline{x}\rangle.\langle\overline{y}\rangle.[\overline{x}].[\overline{y}]$$

$$\frac{\overline{A}\cdot\overline{B}}{\overline{B}\cdot\overline{A}}$$

$$\begin{tikzcd} \overline{A} \arrow[d, Rightarrow, "M"'] & \\ \overline{B} & \cdot\,\overline{C} \end{tikzcd}$$

$$\langle\overline{x}\rangle.M.[\overline{x}]$$

Fragments

Monoidal: first-order, linear, asymmetric

$$\overline{A}$$

$$\star$$

$$\begin{tikzcd}
\overline{A} \arrow[d, Rightarrow, "N"'] \\
\overline{B} \arrow[d, Rightarrow, "M"'] \\
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Perspective

What does this mean for logic (if anything)?

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What does this mean for logic (if anything)?

- Like Levy's CBPV, values A vs computations $\varepsilon \Rightarrow A$
- Types for state, I/O, and probabilities.
- May give types for error handling, data types, and co-recursion (loops).
- Classical linear logic may give types for (message-passing) concurrency

  (similar to session types [Honda 1993, Caires & Pfenning 2010])

Thank you!