A deep quantitative type system

Giulio Guerrieri* Willem Heijltjes* Joe Paulus**

* University of Bath ** Rijksuniversiteit Groningen

CSL 2021

Open deduction

[Guglielmi, Gundersen & Parigot 2010]

Derivations:

$$\begin{array}{c} X \\ \Downarrow \\ Z \end{array} \quad ::= \quad a \quad \Big| \quad \boxed{\begin{array}{c} X_1 \\ \Downarrow \\ Z_1 \end{array}} \star \boxed{\begin{array}{c} X_2 \\ \Downarrow \\ Z_2 \end{array}} \quad \Big| \quad \dfrac{\boxed{\begin{array}{c} X \\ \Downarrow \\ Y_1 \end{array}}}{\boxed{\begin{array}{c} Y_2 \\ \Downarrow \\ Z \end{array}}}\, r$$

Composition:

$$\begin{array}{c} \boxed{\begin{array}{c} X \\ \Downarrow \\ Y \end{array}} \\ \text{- - -} \\ \boxed{\begin{array}{c} Y \\ \Downarrow \\ Z \end{array}} \end{array}$$

For intuitionistic logic:

$$\begin{array}{c} X \\ \Downarrow \\ Z \end{array} \; ::= \; a \;\Big|\; \top \;\Big|\; \boxed{\begin{array}{c} X_1 \\ \Downarrow \\ Z_1 \end{array}} \wedge \boxed{\begin{array}{c} X_2 \\ \Downarrow \\ Z_2 \end{array}} \;\Big|\; \boxed{\begin{array}{c} Z_1 \\ \Uparrow \\ X_1 \end{array}} \rightarrow \boxed{\begin{array}{c} X_2 \\ \Downarrow \\ Z_2 \end{array}} \;\Big|\; \dfrac{\boxed{\begin{array}{c} X \\ \Downarrow \\ Y_1 \end{array}}}{\boxed{\begin{array}{c} Y_2 \\ \Downarrow \\ Z \end{array}}}\, r$$

$$\frac{X}{Y \rightarrow (X \wedge Y)}\,\lambda \qquad \frac{X \wedge (Y \wedge Z)}{(X \wedge Y) \wedge Z}\,=$$

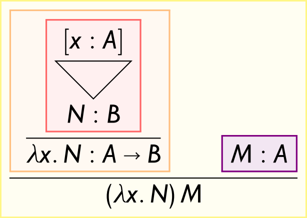
$$\frac{(X \rightarrow Y) \wedge X}{Y}\,@ \qquad \frac{X \wedge Y}{Y \wedge X}\,=$$

$$\frac{X}{X \wedge \cdots \wedge X}\,\Delta \qquad \frac{X \wedge \top}{X}\,=$$

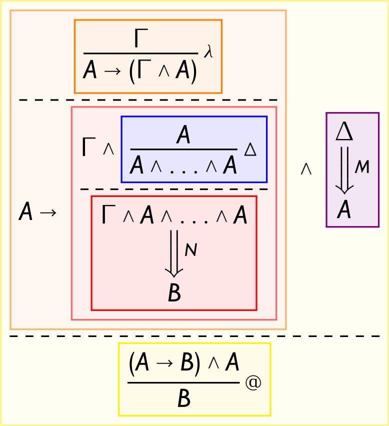
Example: typing a redex

$(\lambda x . N)\ M$

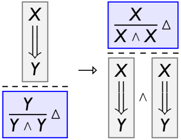
Natural deduction



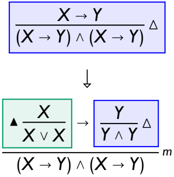
Open deduction



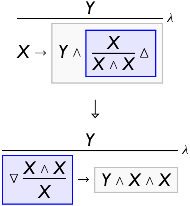
Computational interpretations of deep inference



- Categorical combinators
  [Curien 1985]
  [Brünnler & McKinley 2008]
- Explicit substitution



- Atomic λ-calculus
  [Gundersen, H. & Parigot 2013]
- Sharing graphs?
  [Lamping 1990]
  [Asperti & Guerrini 1998]



- Sharing graphs (linear logic)
  [Gimenez & Moser 2013]
- Intersection types/
  Resource calculus

Aims

Use deep inference to give a principled, robust, structural, uniform account of

typed λ-calculi more widely, and of

intersection types and resource λ-calculi in this work specifically

Intersection types: [Coppo & Dezani 1978, 1980]

$$\dfrac{\begin{array}{c}[x : A_1] \cdots [x : A_n] \\ \vdots \\ N : B\end{array}}{\lambda x.\, N : (A_1 \cap \cdots \cap A_n) \to B} \qquad \dfrac{N : (A_1 \cap \cdots \cap A_n) \to B \qquad M : A_1 \quad \cdots \quad M : A_n}{N\, M : B}$$

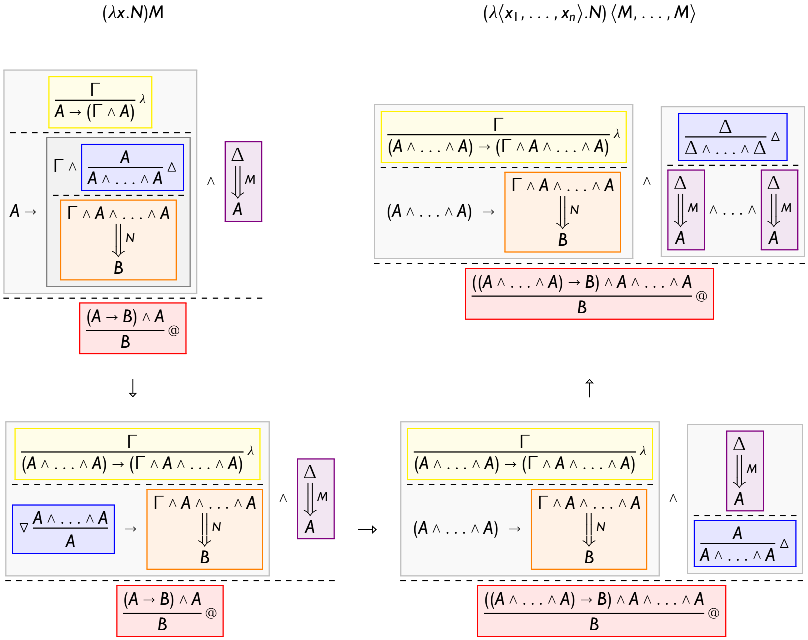
Characterizes strong normalization of λ-terms by typeability

Resource λ-calculus: [Boudol 1993]

$$M, N \ ::= \ x \ \mid \ N \langle M_1, \ldots, M_n \rangle \ \mid \ \lambda \langle x_1, \ldots, x_n \rangle .\, N$$

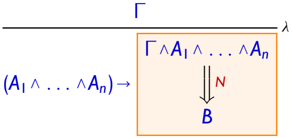
Characterizes strong normalization of λ-terms by expansion

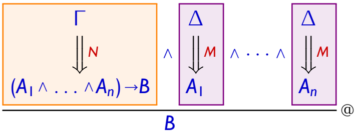
$$(\lambda x.\, N)\, M \quad \mapsto \quad (\lambda \langle x_1, \ldots, x_n \rangle .\, N)\, \langle M, \ldots, M \rangle$$

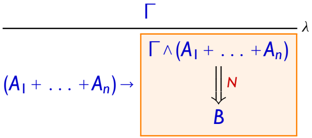

(λx.N)M
(λ⟨x1, . . . , xn⟩.N) ⟨M, . . . , M⟩

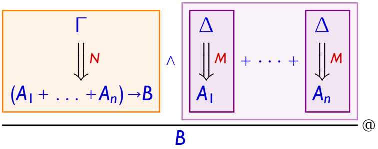
$$\frac{\begin{array}{c}[x : A_1] \cdots [x : A_n] \\ \vdots \\ N : B\end{array}}{\lambda x. N : (A_1 \cap \cdots \cap A_n) \to B}$$

$$\frac{N : (A_1 \cap \cdots \cap A_n) \to B \qquad M : A_1 \qquad \cdots \qquad M : A_n}{N M : B}$$









Basic types:	Collection types:	Context types:
A, B, C, D	I, J, K, L	$\Gamma, \Delta, \Lambda, \Sigma$
$::=$	$::=$	$::=$
$a \mid I \to A$	$\langle\rangle \mid A \mid I + J$	$\top \mid I \mid \Gamma \wedge \Delta$

Typed terms:	Typed variables:	Premises:
$t : A$	I^x	$\Gamma^{\vec{x}} \overset{t}{\Downarrow} A$

$$\frac{(I+K)^x \;\wedge\; (J+L)^y}{(I^x \wedge J^y) \;+\; (K^x \wedge L^y)}\, m$$

$$\left(I^x \wedge J^y \overset{N}{\Downarrow} A\right) \;+\; \left(K^x \wedge L^y \overset{M}{\Downarrow} B\right)$$

medial

$$\frac{(W + X) \wedge (Y + Z)}{(W \wedge Y) + (X \wedge Z)}\, m$$

contraction

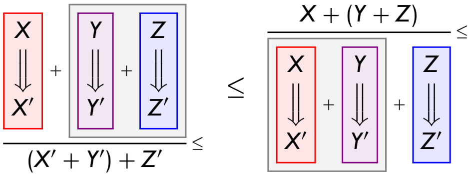
$$\frac{X + Y}{X \wedge Y}\, \Delta$$

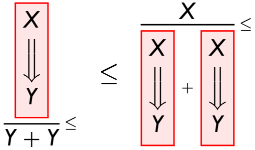
$$\frac{(A \to B \;+\; A)^x}{(A \to B)^x \wedge A^x}\, \Delta$$

$$\frac{(A \to B)^x \wedge A^x}{B}\, @$$

Associativity	$X+(Y+Z) \;=\; (X+Y)+Z$	Redundancy	$X \;\leq\; \langle\rangle$
Unitality	$\langle\rangle+X \;=\; X \;=\; X+\langle\rangle$	Duplicability	$X \;\leq\; X+X$
Symmetry	$X+Y \;=\; Y+X$	Idempotence	$X \;=\; X+X$

$$\boxed{\frac{X}{Y}\leq} \qquad \text{if } X \leq Y$$





A deep quantitative proof system

$$\begin{array}{c} X \\ \Downarrow \\ Z \end{array} \;::=\; a \;\mid\; \top \;\mid\; \boxed{\begin{array}{c} X_1 \\ \Downarrow \\ Z_1 \end{array}} \wedge \boxed{\begin{array}{c} X_2 \\ \Downarrow \\ Z_2 \end{array}} \;\mid\; \boxed{\begin{array}{c} Z_1 \\ \Uparrow \\ X_1 \end{array}} \rightarrow \boxed{\begin{array}{c} X_2 \\ \Downarrow \\ Z_2 \end{array}} \;\mid\; \dfrac{\boxed{\begin{array}{c} X \\ \Downarrow \\ Y_1 \end{array}}}{\boxed{\begin{array}{c} Y_2 \\ \Downarrow \\ Z \end{array}}}\,r \;\mid\; \langle\rangle \;\mid\; \boxed{\begin{array}{c} X_1 \\ \Downarrow \\ Z_1 \end{array}} + \boxed{\begin{array}{c} X_2 \\ \Downarrow \\ Z_2 \end{array}}$$

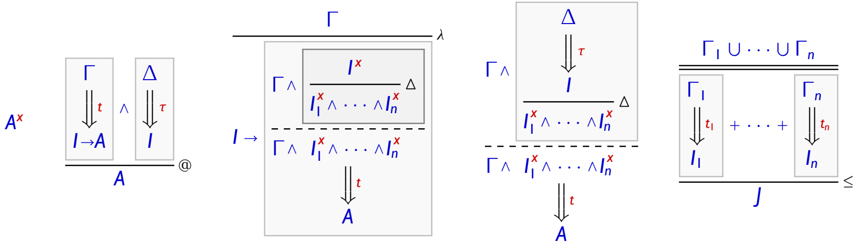
$$\frac{X}{Y \rightarrow (X \wedge Y)}\,\lambda \qquad \frac{(X \rightarrow Y) \wedge X}{Y}\,@ \qquad \frac{X \wedge (Y \wedge Z)}{(X \wedge Y) \wedge Z}\,= \qquad \frac{X \wedge Y}{Y \wedge X}\,= \qquad \frac{X \wedge \top}{X}\,=$$

$$\frac{X_1 + \cdots + X_n}{X_1 \wedge \cdots \wedge X_n}\,\Delta \qquad \frac{(X_1^1 + \cdots + X_n^1) \wedge \cdots \wedge (X_1^m + \cdots + X_n^m)}{(X_1^1 \wedge \cdots \wedge X_1^m) + \cdots + (X_n^1 \wedge \cdots \wedge X_n^m)}\,\mathrm{m} \qquad \frac{X}{Y}\,\leq \quad (X \leq Y)$$

$$\frac{\langle\rangle}{\top}\,\Delta \qquad \frac{\top}{\langle\rangle}\,\mathrm{m} \qquad \frac{\langle\rangle \wedge \langle\rangle}{\langle\rangle}\,\mathrm{m} \qquad \frac{\top}{\top + \top}\,\mathrm{m} \qquad \frac{(W + X) \wedge (Y + Z)}{(W \wedge Y) + (X \wedge Z)}\,\mathrm{m}$$

The collection calculus

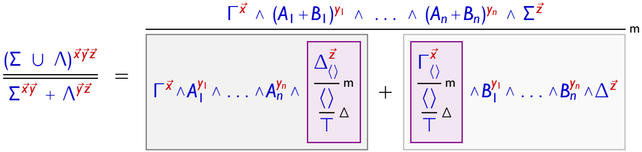
$$r, s, t \; ::= \; x \mid t\tau \mid \lambda x.t \mid t[x \leftarrow \tau] \qquad \rho, \sigma, \tau \; ::= \; \langle\rangle \mid \langle t \rangle \mid \sigma + \tau$$



x $\quad$ $t\tau$ $\quad$ $\lambda x.t$ $\quad$ $t[x \leftarrow \tau]$ $\quad$ $\tau = \langle t_1, \ldots, t_n \rangle$

where:

$$I = I_1 + \cdots + I_n$$
$$I \leq J$$
$$\Gamma_{\langle\rangle} = \langle\rangle \wedge \cdots \wedge \langle\rangle$$



Reduction

Associativity	$\rho + (\sigma + \tau) = (\rho + \sigma) + \tau$	Redundancy	$\tau \leq \langle\rangle$
Unitality	$\langle\rangle + \tau = \tau = \tau + \langle\rangle$	Duplicability	$\tau \leq \tau + \tau$
Symmetry	$\sigma + \tau = \tau + \sigma$	Idempotence	$\tau = \tau + \tau$

$$(\lambda x.t)[\Phi]\,\tau \longrightarrow_{\mathrm{b}} t[x \leftarrow \tau][\Phi] \qquad \text{where } [\Phi] = [x_1 \leftarrow t_1] \cdots [x_n \leftarrow t_n]$$

$$t\{x/y\}[x \leftarrow \tau] \longrightarrow_{\mathrm{c}} \sum_{\tau \leq \rho + \sigma} t[x \leftarrow \rho][y \leftarrow \sigma] \qquad \text{if } |t|_x, |t|_y \geq 1$$

$$t[x \leftarrow \tau] \longrightarrow_{\mathrm{e}} \sum_{\tau \leq \langle s \rangle} t\{s/x\} \qquad \text{if } |t|_x = 1$$

$$t[x \leftarrow \tau] \longrightarrow_{\mathrm{d}} \sum_{\tau \leq \langle\rangle} t \qquad \text{if } |t|_x = 0$$

Subject reduction holds

Results

- The collection calculus is confluent (allowing for non-determinism)
- The typed collection calculus is strongly normalizing
- Adding or removing the algebraic laws of redundancy (weakening) and duplicability (contraction) gives exact, lower, or upper bounds on reduction paths
- A uniformity requirement on collection terms gives intersection types for weak normalization
- A further strength law gives intersection types for strong normalization
- Many existing resource calculi and intersection type systems can be encoded in the collection calculus

Thank you