Proof nets for bi-intuitionistic linear logic

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$$\frac{A \quad B \quad \Gamma \vdash \Delta}{A \otimes B \quad \Gamma \vdash \Delta} \qquad \frac{\Gamma \quad A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Gamma \vdash \Delta \quad C \quad D}{\Gamma \vdash \Delta \quad C \parr D} \qquad \frac{D \vdash C \quad \Delta}{D - C \vdash \Delta}$$

MLL without negation (linearly distributive categories)

IMLL (symmetric monoidal closed categories)

FILL = MLL + IMLL

BILL = FILL + subtraction

$$\frac{A \;\; B \;\; \Gamma \vdash \Delta}{A \otimes B \;\; \Gamma \vdash \Delta} \qquad \frac{\Gamma \;\; A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Gamma \vdash \Delta \;\; C \;\; D}{\Gamma \vdash \Delta \;\; C \wp D} \qquad \frac{D \vdash C \;\; \Delta}{D - C \vdash \Delta}$$

Problem: FILL/BILL cut-elimination [Schellinx 1991, Bierman 1996]

$$\dfrac{\dfrac{\dfrac{a \vdash a \qquad d \vdash d - c \;\; c}{a \wp d \vdash d - c \;\; a \;\; c}}{a \wp d \vdash d - c \;\; a \wp c} \qquad \dfrac{\dfrac{\dfrac{a \;\; a \multimap b \vdash b \qquad c \vdash c}{a \wp c \;\; a \multimap b \vdash b \;\; c}}{a \wp c \;\; a \multimap b \vdash b \wp c}}{a \wp c \vdash (a \multimap b) \multimap (b \wp c)}}{a \wp d \vdash d - c \;\; (a \multimap b) \multimap (b \wp c)}$$

But the conclusion sequent is not cut-free provable.

$$a \wp d \vdash d - c \;\; (a \multimap b) \multimap (b \wp c)$$

$$\frac{A \quad B \quad \Gamma \vdash \Delta}{A \otimes B \quad \Gamma \vdash \Delta} \qquad \frac{\Gamma \quad A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Gamma \vdash \Delta \quad C \quad D}{\Gamma \vdash \Delta \quad C \parr D} \qquad \frac{D \vdash C \quad \Delta}{D - C \vdash \Delta}$$

Multi-conclusion $\multimap R$ and multi-assumption $-L$ collapse onto MLL

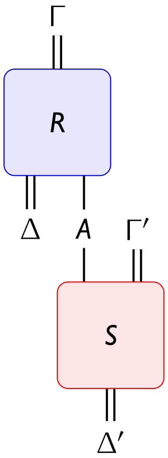
$$\frac{\Gamma \quad A \vdash B \quad \Delta}{\Gamma \vdash A \multimap B \quad \Delta} \qquad \frac{\Gamma \quad D \vdash C \quad \Delta}{\Gamma \quad D - C \vdash \Delta}$$

Solution: annotate sequents with a relation, as $\Gamma \vdash_R \Delta$, to indicate which conclusions depend on which assumptions.

$$\frac{\Gamma \quad A \vdash_R B \quad \Delta}{\Gamma \vdash_S A \multimap B \quad \Delta}\ (A \not{R} \Delta) \qquad \frac{\Gamma \quad D \vdash_R C \quad \Delta}{\Gamma \quad D - C \vdash_S \Delta}\ (\Gamma \not{R} C)$$

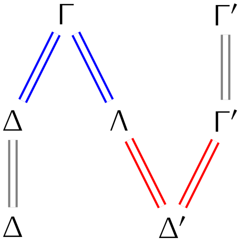
[Hyland & De Paiva 1993, Bräuner & De Paiva 1997, Eades & De Paiva 2016]

$$\frac{\Gamma \vdash_R \Delta \;\; A \qquad A \;\; \Gamma' \vdash_S \Delta'}{\Gamma \;\; \Gamma' \vdash_T \Delta \;\; \Delta'}$$



$$R \subseteq \Gamma \times \Delta\,\Lambda \qquad\qquad S \subseteq \Lambda\,\Gamma' \times \Delta'$$

$$R \star S \;=\; (R \cup id_{\Gamma'})\,;\,(id_{\Delta} \cup S) \;\subseteq\; \Gamma\,\Gamma' \times \Delta\,\Delta'$$



$$\frac{\Gamma \vdash_R \Delta\;\; A \qquad A\;\; \Gamma' \vdash_S \Delta'}{\Gamma\;\; \Gamma' \vdash_T \Delta\;\; \Delta'} \qquad\qquad T = R \star S$$

$$\frac{}{A \vdash_T A} \qquad T = \frac{A}{A}$$

$$\frac{\Gamma \vdash_R \Delta \;\; A \qquad A \;\; \Gamma' \vdash_S \Delta'}{\Gamma \;\; \Gamma' \vdash_T \Delta \;\; \Delta'} \qquad T = R \star \frac{A}{A} \star S$$

$$\frac{A \;\; B \;\; \Gamma \vdash_R \Delta}{A \otimes B \;\; \Gamma \vdash_T \Delta} \qquad T = \frac{A \otimes B}{A \qquad B} \star R$$

$$\frac{\Gamma \vdash_R \Delta \;\; A \qquad \Gamma' \vdash_S \Delta' \;\; B}{\Gamma \;\; \Gamma' \vdash_T \Delta \;\; \Delta' \;\; A \otimes B} \qquad T = (R \cup S) \star \frac{A \qquad B}{A \otimes B}$$

$$\frac{C \;\; \Gamma \vdash_R \Delta \qquad D \;\; \Gamma' \vdash_S \Delta'}{C \wp D \;\; \Gamma \;\; \Gamma' \vdash_T \Delta \;\; \Delta'} \qquad T = \frac{C \wp D}{C \qquad D} \star (R \cup S)$$

$$\frac{\Gamma \vdash_R \Delta \;\; C \;\; D}{\Gamma \vdash_T \Delta \;\; C \wp D} \qquad T = R \star \frac{C \qquad D}{C \wp D}$$

$$\frac{\Gamma \vdash_R \Delta \;\; A \qquad B \;\; \Gamma' \vdash_S \Delta'}{\Gamma \;\; A \multimap B \;\; \Gamma' \vdash_T \Delta \;\; \Delta'} \qquad T = R \star \frac{A \multimap B \qquad A}{B} \star S$$

$$\frac{\Gamma \;\; A \vdash_R B \;\; \Delta}{\Gamma \vdash_T A \multimap B \;\; \Delta} A \not{R} \Delta \qquad T = \frac{}{A} \star R \star \frac{B}{A \multimap B}$$

$$\frac{\Gamma \;\; C \vdash_R D \;\; \Delta}{\Gamma \;\; C - D \vdash_T \Delta} \Gamma \not{R} D \qquad T = \frac{D - C}{D} \star R \star \frac{C}{}$$

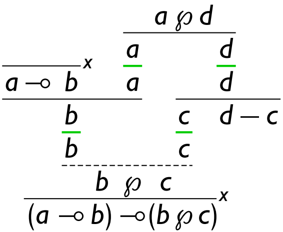
$$\frac{\Gamma \vdash_R \Delta \;\; C \qquad D \;\; \Gamma' \vdash_S \Delta'}{\Gamma \;\; \Gamma' \vdash_T \Delta \;\; C - D \;\; \Delta'} \qquad T = R \star \frac{D}{C \qquad D - C} \star S$$

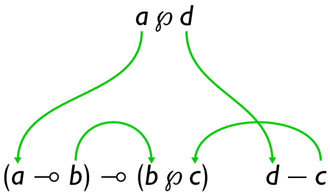
$$\frac{\Gamma}{\Delta} \; := \; \Gamma \times \Delta$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\overline{a \vdash a} \qquad \overline{b \vdash b}}{a \multimap b \;\; a \vdash b}
      \qquad
      \dfrac{\overline{d \vdash d} \qquad \overline{c \vdash c}}{d \vdash c \;\; d - c}
    }{a \multimap b \;\; a \wp d \vdash_R b \;\; c \;\; d - c}
  }{a \multimap b \;\; a \wp d \vdash_S b \wp c \;\; d - c}
}{a \wp d \vdash (a \multimap b) \multimap (b \wp c) \;\; d - c}
$$

$$R = \{\; (a \multimap b\,, b)\;,\; (a \wp d\,, b)\;,\; (a \wp d\,, c)\;,\; (a \wp d\,, d - c) \;\}$$

$$S = \{\; (a \multimap b\,, b \wp c)\;,\; (a \wp d\,, b \wp c)\;,\; (a \wp d\,, d - c) \;\}$$





BILL proof nets are graphs satisfying a correctness condition

- Nodes are links with a premise-sequent and conclusion-sequent
- Formulas on links are ports
- Edges connect a conclusion-port A to a premise-port A

$$\frac{A_1 \quad \ldots \quad A_n}{B_1 \quad \ldots \quad B_m}$$

$$\dfrac{A^-}{A^+}\,ax \qquad\qquad \dfrac{A^+}{A^-}\,cut$$

$$\dfrac{A^+ \qquad B^+}{(A \otimes B)^+}\otimes I \qquad \dfrac{\begin{array}{c}\overline{A^-}^{\,x}\\ \vdots \\ B^+\end{array}}{(A \multimap B)^+}\multimap I,x \qquad \dfrac{A^+ \qquad B^+}{(A \wp B)^+}\wp I\ \text{(dashed)} \qquad \dfrac{B^+}{A^- \qquad (B - A)^+}-I$$

$$\dfrac{(A \otimes B)^-}{A^- \qquad B^-}\otimes E\ \text{(dashed)} \qquad \dfrac{(A \multimap B)^- \qquad A^+}{B^-}\multimap E \qquad \dfrac{(A \wp B)^-}{A^- \qquad B^-}\wp E \qquad \dfrac{(B - A)^-}{\begin{array}{c}B^-\\ \vdots \\ \underline{A^+}_{\,x}\end{array}}-E,x$$

Correctness 1: Contractibility

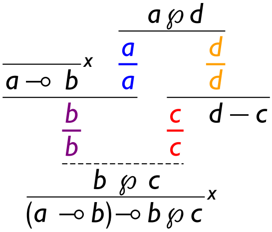
Contractibility [Danos 1990, Lafont 1995, Guerrini & Masini 2001]

- Correctness and sequentialization by local rewriting
- Contraction steps correspond to sequent rules
- Efficient (linear-time for MLL)

$$\text{sequent:} \quad \Gamma \vdash_R \Delta \qquad\qquad \text{link:} \quad \frac{\Gamma}{\Delta}\,R$$

$$\frac{\Gamma \;\; A \vdash_R B \;\; \Delta}{\Gamma \vdash_T A \multimap B \;\; \Delta}\, A \not{R} \Delta \qquad\qquad T = \frac{}{A} \star R \star \frac{B}{A \multimap B}$$

$$\cfrac{\cfrac{\cfrac{}{A}\,x \qquad \Gamma}{B \qquad\quad \Delta}\,R}{A \multimap B}\,x \quad \underset{A \not{R} \Delta}{\rightsquigarrow} \quad \frac{\qquad\qquad\qquad \Gamma}{A \multimap B \qquad \Delta}\,T$$



$$\overline{a \vdash a} \qquad \overline{b \vdash b} \qquad \overline{d \vdash d} \qquad \overline{c \vdash c}$$

$$
\dfrac{\dfrac{\dfrac{}{a \multimap b}{}^{x} \quad \dfrac{a \wp d}{a \qquad d}}{b} \qquad \dfrac{d}{c \quad d - c}}{\dfrac{b \wp c}{(a \multimap b) \multimap b \wp c}{}^{x}}
$$

$$
\dfrac{\dfrac{}{a \vdash a} \qquad \dfrac{}{b \vdash b}}{a \multimap b \quad a \vdash b} \qquad \qquad \dfrac{\dfrac{}{d \vdash d} \qquad \dfrac{}{c \vdash c}}{d \vdash c \quad d - c}
$$

$$\frac{\dfrac{\dfrac{\overline{a \multimap b}^{\,x} \qquad a \wp d}{b \qquad c \qquad d - c}\,R}{b \wp c}}{(a \multimap b) \multimap b \wp c}\,x$$

$$\frac{\dfrac{\overline{a \vdash a} \qquad \overline{b \vdash b}}{a \multimap b \;\; a \vdash b} \qquad \dfrac{\overline{d \vdash d} \qquad \overline{c \vdash c}}{d \vdash c \;\; d - c}}{a \multimap b \;\; a \wp d \vdash_R b \;\; c \;\; d - c}$$

$$R = \{\ (a \multimap b\,,\, b)\ ,\ (a \wp d\,,\, b)\ ,\ (a \wp d\,,\, c)\ ,\ (a \wp d\,,\, d - c)\ \}$$

$$\dfrac{\dfrac{\dfrac{}{a \multimap b}\,x \qquad a \wp d}{b \wp c \qquad d - c}\,S}{(a \multimap b) \multimap b \wp c}\,x$$

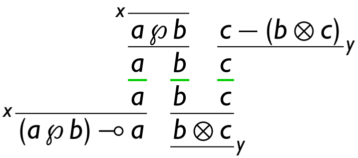
$$\dfrac{\dfrac{\dfrac{}{a \vdash a} \qquad \dfrac{}{b \vdash b}}{a \multimap b \;\; a \vdash b} \qquad \dfrac{\dfrac{}{d \vdash d} \qquad \dfrac{}{c \vdash c}}{d \vdash c \;\; d - c}}{\dfrac{a \multimap b \;\; a \wp d \vdash_R b \;\; c \;\; d - c}{a \multimap b \;\; a \wp d \vdash_S b \wp c \;\; d - c}}$$

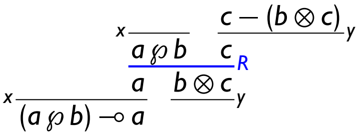
$$S = \{\ (a \multimap b\,,\, b \wp c)\ ,\ (a \wp d\,,\, b \wp c)\ ,\ (a \wp d\,,\, d - c)\ \}$$

$$\frac{a \wp d}{(a \multimap b) \multimap b \wp c \quad d - c}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a} \qquad \dfrac{}{b \vdash b}}{a \multimap b \quad a \vdash b} \qquad \dfrac{\dfrac{}{d \vdash d} \qquad \dfrac{}{c \vdash c}}{d \vdash c \quad d - c}}{a \multimap b \quad a \wp d \vdash_R b \quad c \quad d - c}}{a \multimap b \quad a \wp d \vdash_S b \wp c \quad d - c}}{a \wp d \vdash (a \multimap b) \multimap (b \wp c) \quad d - c}$$

An example of an incorrect net that fails to contract:

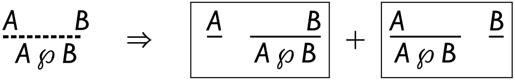




$R = \{\ (a \wp b\,, a)\ ,\ (a \wp b\,, b \otimes c)\ ,\ (c\,, b \otimes c)\ \}$

Correctness 2: Geometric

MLL correctness: switching [Danos & Regnier 1989]



- A switching is a choice of disconnecting one premise of each ⅋-link.
- Each resulting switching graph must be a tree (acyclic + connected).

IMLL correctness: functionality [Lamarche 2008]

$$\dfrac{\begin{matrix}\overline{A}^{x} \\ \vdots \\ B\end{matrix}}{A \multimap B}\multimap I,x$$

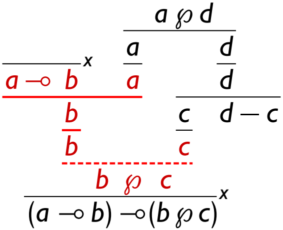
- Any downward path from an assumption A^x to the conclusion must pass through the closing $\multimap I, x$ rule.

$$\dfrac{\Gamma \;\; A \vdash_R B \;\; \Delta}{\Gamma \vdash_S A \multimap B \;\; \Delta}\;(A \not R \Delta)$$

BILL correctness:

$$\frac{A \quad B}{A \wp B}\,\wp I \qquad \frac{A \otimes B}{A \quad B}\,\otimes E \qquad \frac{\begin{array}{c}\overline{A}^{\,x}\\ \vdots \\ B\end{array}}{A \multimap B}\,{\multimap} I,x \qquad \begin{array}{c}\dfrac{B - A}{B}\,{-}E,x \\ \vdots \\ \underline{A}_{\,x}\end{array}$$

- The targets of a switched link are:
  - $\wp I$: its premises
  - $\otimes E$: its conclusions
  - $\multimap I$: any link downward from its assumption (but not from itself)
  - $-E$: any link upward from its conclusion (but not from itself)
- A switching graph connects each switched link to exactly one target
- Each switching graph must be a tree (acyclic + connected)



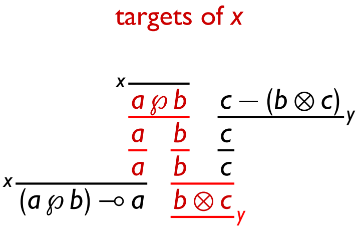
a ℘ d
a
d
x
a ⊸ b
a
d
b
c
d — c
b
c
b ℘ c
x
(a ⊸ b) ⊸ (b ℘ c)

Some details:

- $\multimap I, x$ and x must be considered one link
- $-E, y$ and y must be considered one link
- $\otimes E$-links must be added to collect all open assumptions
- $\wp I$-links must be added to collect all open conclusions

OR

- a path from x to an open conclusion must pass by $\multimap I, x$
- a path from an open assumption to y must pass by $-E, y$
- a path from x to y must pass by $\multimap I, x$ or $-E, y$



targets of x
targets of y
x
a ℘ b
c — (b ⊗ c)
y
a
b
c
a
b
c
x
(a ℘ b) ⊸ a
b ⊗ c
y

Theorem A proof net contracts (i.e. sequentializes) if and only if it is geometrically correct.

Kingdoms in MLL

$$\frac{}{B \quad B^\perp} \qquad \frac{B \qquad C}{B \otimes C} \qquad \frac{B \qquad C}{B \wp C}$$

- A switching path is a path in a switching graph
- $A \ll (B \wp C)$: A is on a switching path from B to C

The kingdom kA is the smallest subgraph such that $A \in kA$ and:

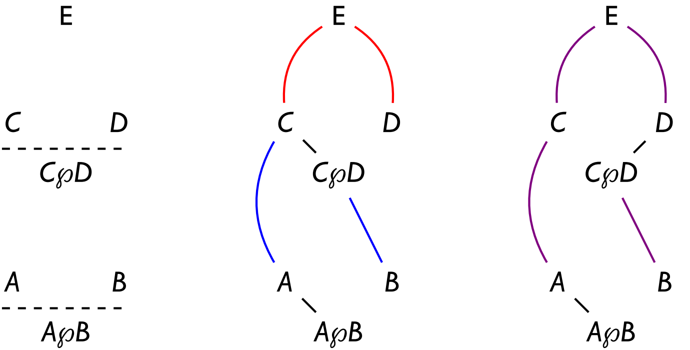
- if $B \in kA$ and B is in an axiom link with $B^\perp$, then $B^\perp \in kA$
- if $B \otimes C \in kA$ then $B \in kA$ and $C \in kA$
- If $B \wp C \in kA$ and $D \ll B \wp C$ then $D \in kA$.

$$\frac{}{\vdash B\ B^\perp} \qquad \frac{\vdash \Gamma\ B \qquad \vdash C\ \Delta}{\vdash \Gamma\ B \otimes C\ \Delta} \qquad \frac{\vdash \Gamma\ B\ C}{\vdash \Gamma\ B \wp C}$$

[Bellin & Van de Wiele 1995]

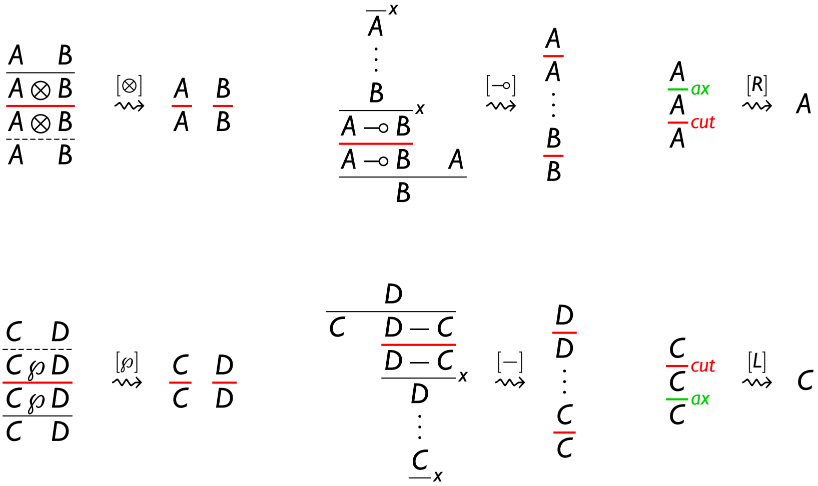
Lemma: Switching-correctness means $\ll$ is transitive.

$$E \ll C \wp D \ , \quad C \wp D \ll A \wp B \quad \Rightarrow \quad E \ll A \wp B$$



Lemma: $A \ll B$ if and only if A must contract before B

Cut elimination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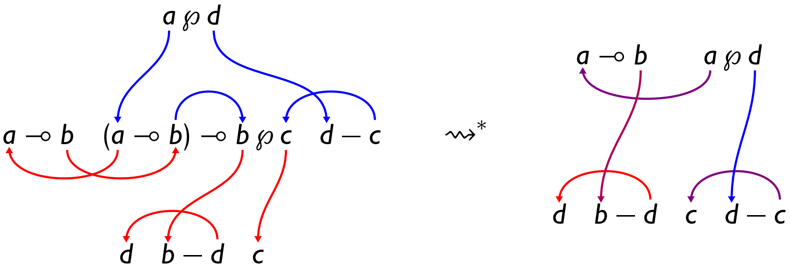

a ℘ d
a ⊸ b (a ⊸ b) ⊸ b ℘ c d — c
d b — d c
⇝*
a ⊸ b a ℘ d
d b — d c d — c

Thank you