No proof nets for MLL with units

Proof equivalence in MLL is PSPACE-complete

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and
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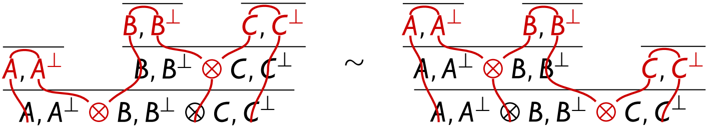
** Independent

Linear logic proof nets

Linear Logic:

- classical and computationally meaningful
- sequent calculus not natural deduction


A, A⊥
B, B⊥
C, C⊥
B, B⊥ ⊗ C, C⊥
A, A⊥ ⊗ B, B⊥ ⊗ C, C⊥
~
A, A⊥
B, B⊥
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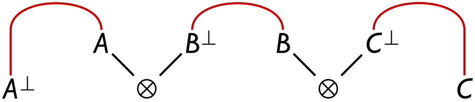
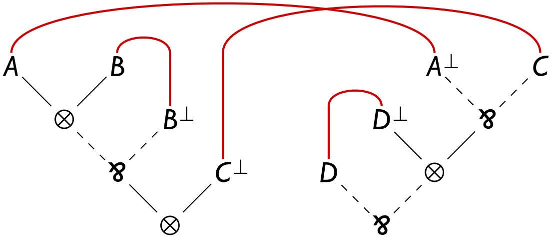

A⊥ A ⊗ B⊥ B ⊗ C⊥ C

Canonical proof nets

- canonical for proof equivalence
- independent of proofs by a correctness criterion



Canonical proof nets

- MLL^{-} [Girard 1987]
- ALL^{-} [Hu 1999; Hughes 2002]
- MALL^{-} [Hughes & Van Glabbeek 2005]
- ALL [Heijltjes 2011]

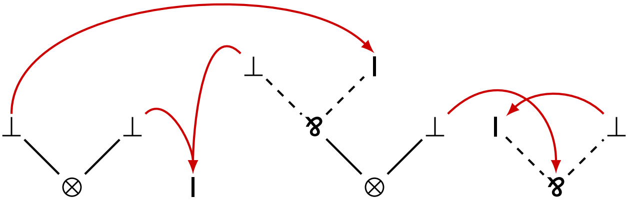
Main result

- MLL No: proof equivalence is too hard (PSPACE-complete)

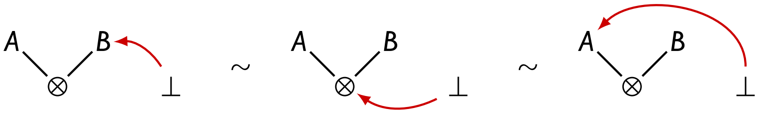
MLL proof equivalence

$$A, B, C \;:=\; \mathbf{I} \;\mid\; \bot \;\mid\; A \otimes B \;\mid\; A \parr B$$

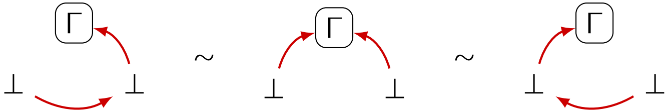
$$\frac{}{\mathbf{I}} \qquad\qquad \frac{\Gamma}{\Gamma, \bot} \qquad\qquad \frac{\Gamma, A \qquad B, \Delta}{\Gamma, A \otimes B, \Delta} \qquad\qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}$$



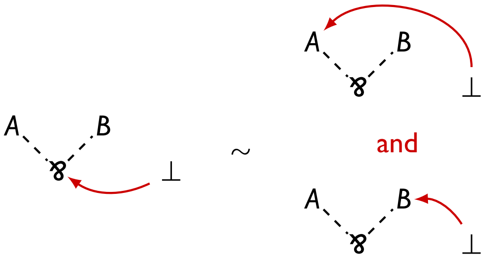
$$\dfrac{\Gamma, A \qquad \dfrac{B, \Delta}{B, \Delta, \bot}}{\Gamma, A \otimes B, \Delta, \bot} \quad \sim \quad \dfrac{\dfrac{\Gamma, A \qquad B, \Delta}{\Gamma, A \otimes B, \Delta}}{\Gamma, A \otimes B, \Delta, \bot} \quad \sim \quad \dfrac{\dfrac{\Gamma, A}{\bot, \Gamma, A} \qquad B, \Delta}{\bot, \Gamma, A \otimes B, \Delta}$$

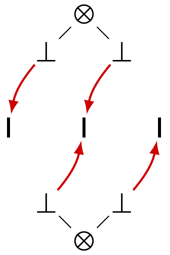


$$\dfrac{\dfrac{\Gamma}{\Gamma, \bot}}{\bot, \Gamma, \bot} \quad \sim \quad \dfrac{\dfrac{\Gamma}{\bot, \Gamma}}{\bot, \Gamma, \bot}$$

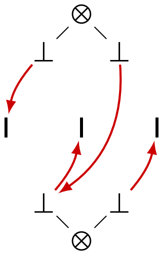


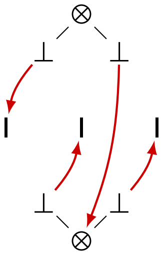
$$\dfrac{\dfrac{\Gamma, A, B}{\Gamma, A \parr B}}{\Gamma, A \parr B, \bot} \quad \sim \quad \dfrac{\dfrac{\Gamma, A, B}{\Gamma, A, B, \bot}}{\Gamma, A \parr B, \bot}$$

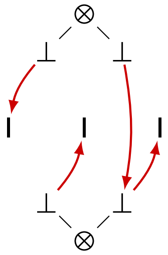


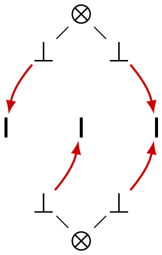


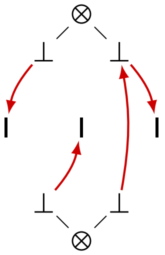
$$\bot \otimes \bot, \mathbf{I}, \mathbf{I}, \mathbf{I}, \bot \otimes \bot$$

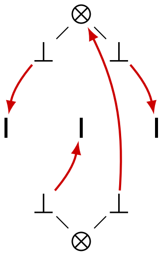


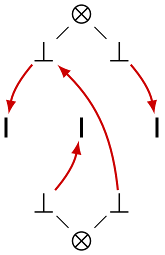


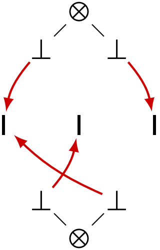


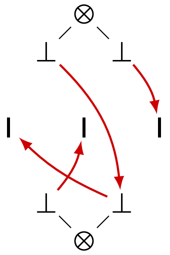


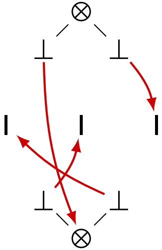








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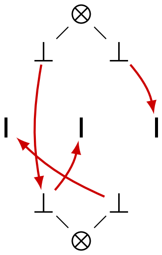
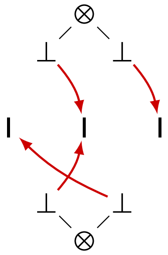




Proof equivalence

$\sim$

$*$-Autonomous categories

$\sim$

Proof net equivalence

(also generated by: rewire one jump preserving correctness)

[Seely 1989; Blute, Cockett, Seely & Trimble 1996; Hughes 2012]

Main result

MLL proof equivalence is PSPACE-complete

Corollary

Proof nets with

- canonicity
- tractable proof net equality
- tractable translation from proofs

would need P=PSPACE

PSPACE and Constraint Logic

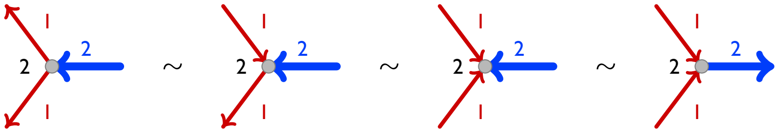
PSPACE

- Turing machines with polynomial space and unbounded time
- canonical problem: quantified Boolean formulae (QBF)

$$\text{NP, co-NP} \subseteq \text{PSPACE} \subseteq \text{EXPTIME}$$

Constraint Logic

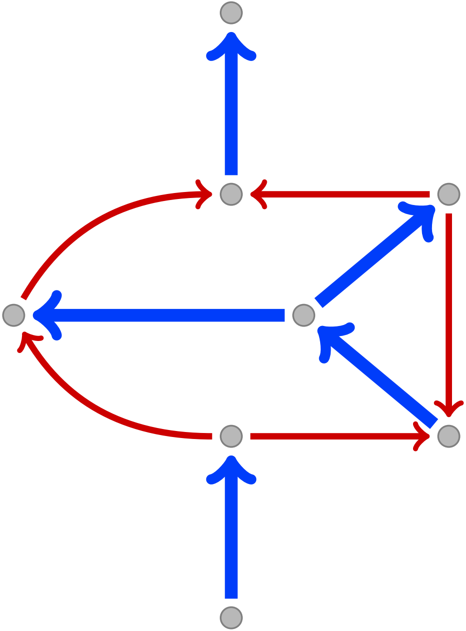
[Hearn & Demaine 2005, 2008]

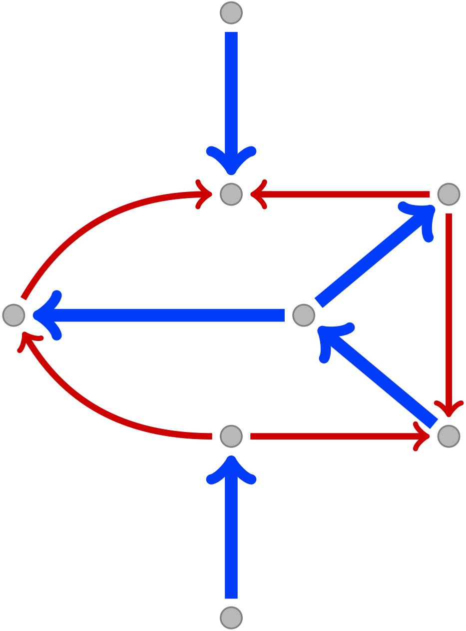


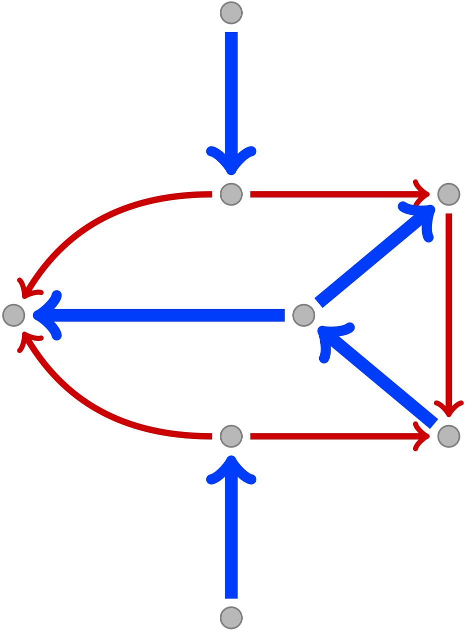
Constraint Graphs:

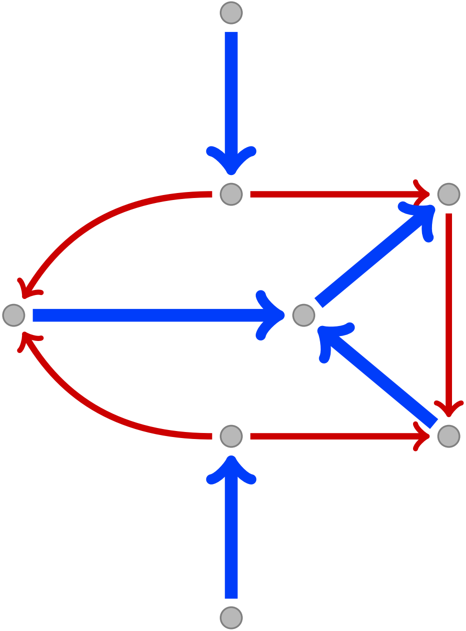
- weighted edges
- sum weight of incoming edges $\geq$ vertex inflow constraint
- step: reverse one edge

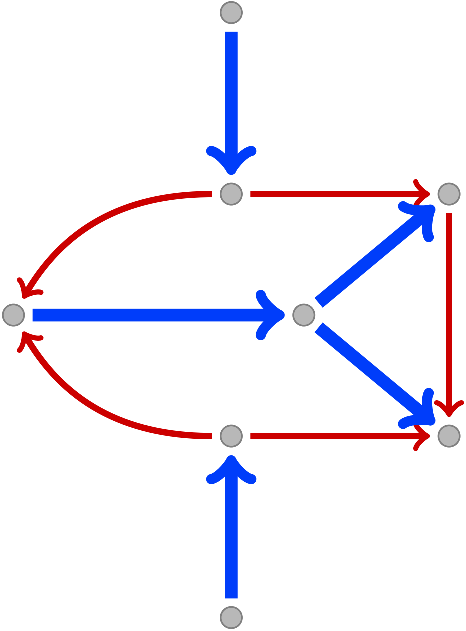
Equivalence of constraint graphs is PSPACE-complete

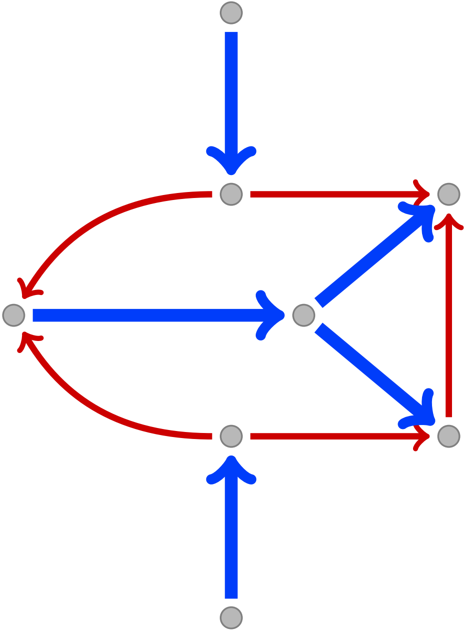


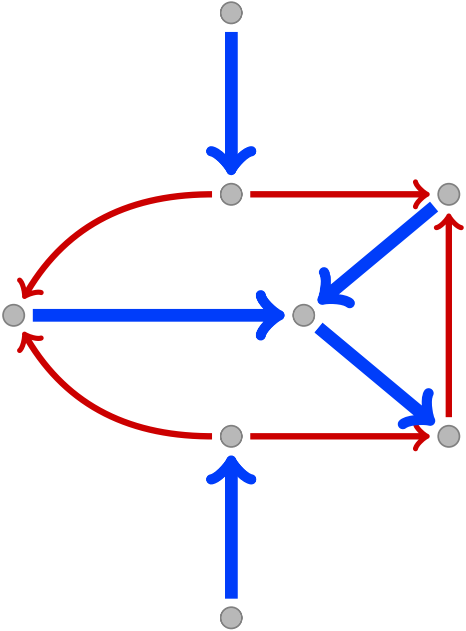


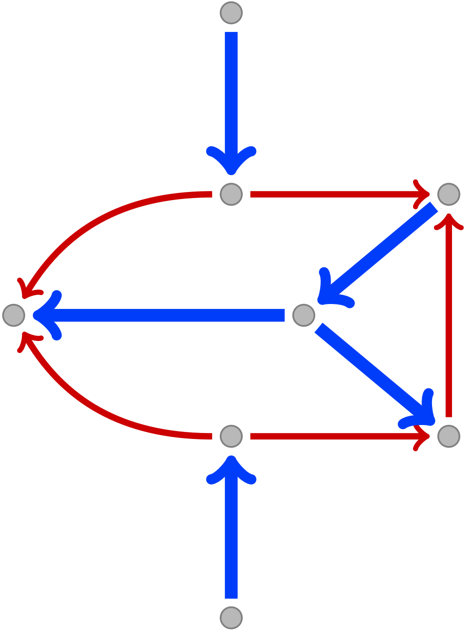


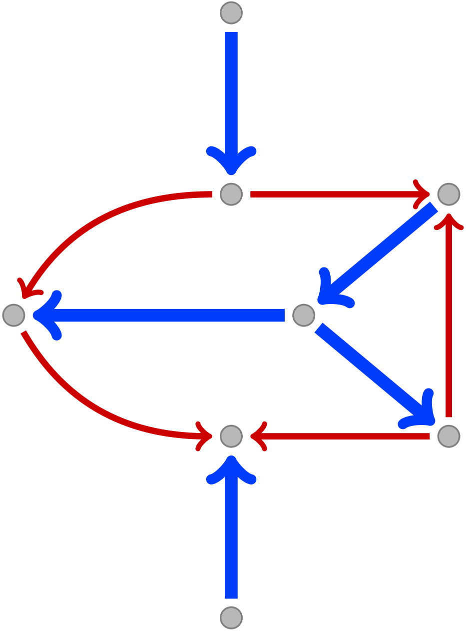


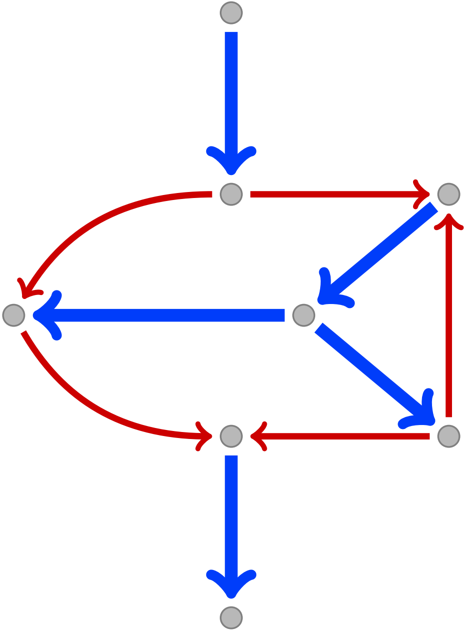


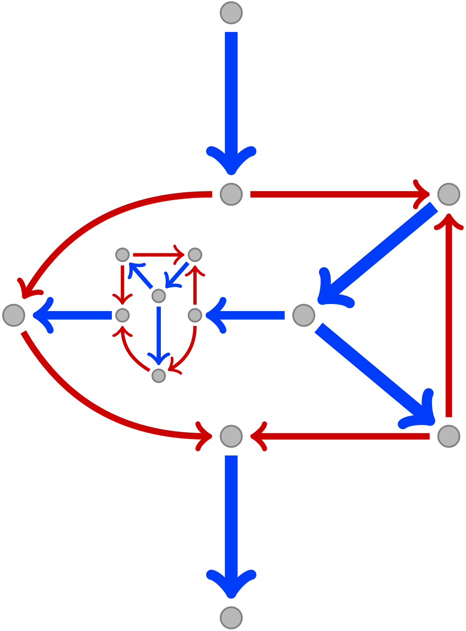








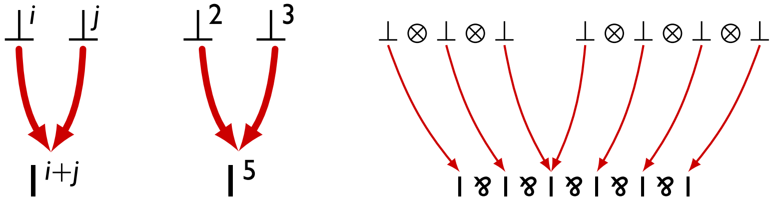


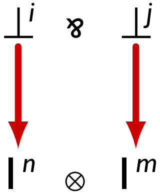


Encoding Constraint Logic

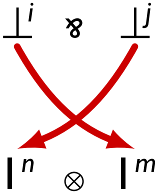
$$\bot^n \qquad \underbrace{\bot \otimes \ldots \otimes \bot}_{n+1}$$

$$1^n \qquad \underbrace{1 \parr \ldots \parr 1}_{n+1}$$





Provable iff $i = n$ and $j = m$



Provable iff $i = n$ and $j = m$ (or $i = m$ and $j = n$)

3-Partition (NP-complete) [Garey & Johnson 1975]

- multiset $\{i_1, \ldots, i_{3n}\}$ with sum $n \times k$
- partition into n triples $\{i_a, i_b, i_c\}$ with sum k
- $k/4 < i < k/2 \quad \Rightarrow \quad$ any subset with sum k is a triple

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$$\bot^{i_1} \quad \bot^{i_2} \quad \bot^{i_3} \quad \bot^{i_4} \quad \bot^{i_5} \quad \bot^{i_6} \quad \ldots \quad \bot \quad \bot \quad \bot^{i_n}$$

$$|^{k} \quad \otimes \quad |^{k} \quad \otimes \quad \ldots \quad \otimes \quad |^{k}$$

3-Partition (NP-complete) [Garey & Johnson 1975]

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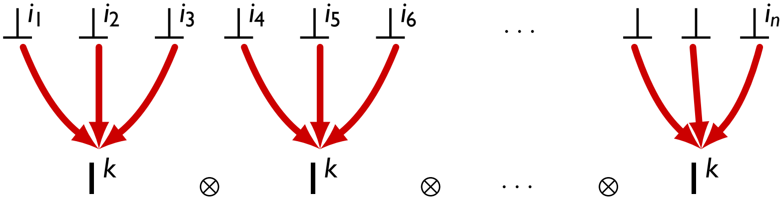


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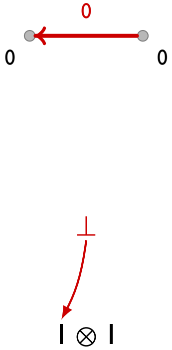
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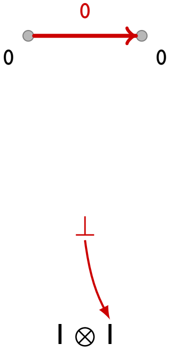
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- vertices connected by $\otimes$
- note: edges may connect to every vertex


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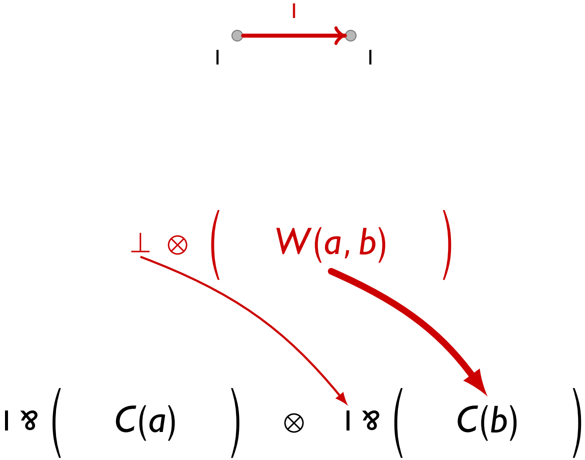
$$\bot \otimes \Big(\quad W(a, b) \quad \Big)$$

$$1 \parr \Big(\quad C(a) \quad \Big) \otimes 1 \parr \Big(\quad C(b) \quad \Big)$$



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$$\bot \otimes \left(\bot^{i} \parr \bot^{j} \parr \bot^{k} \right)$$

$$\mathbf{1} \parr \left(\mathbf{1}^{i} \otimes \mathbf{1}^{j+k} \right) \otimes \mathbf{1} \parr \left(\mathbf{1}^{i+j} \otimes \mathbf{1}^{k} \right)$$


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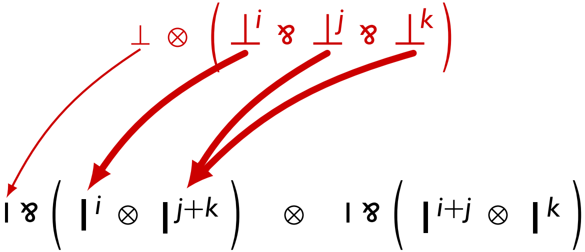
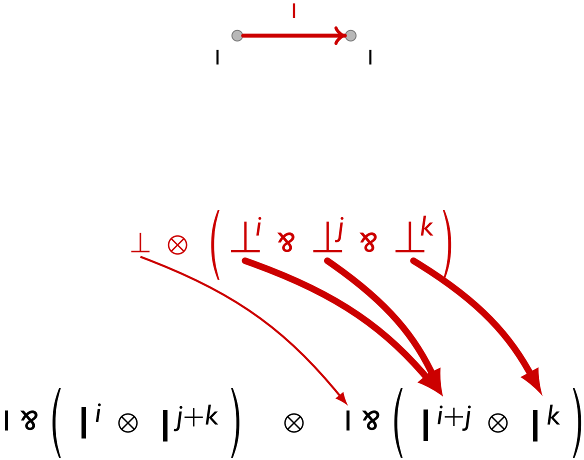

$\perp \otimes \left(\perp^i \parr \perp^j \parr \perp^k \right)$
$I \parr \left(I^i \otimes I^{j+k} \right) \otimes I \parr \left(I^{i+j} \otimes I^k \right)$



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I ⅋ (I^i ⊗ I^{j+k}) ⊗ I ⅋ (I^{i+j} ⊗ I^k)

- constraint units for vertex m: $\mathbf{1}^m \otimes \mathbf{1}^n$
- weight units for edge $i - j$: $\bot^i \parr \bot^{i-j} \parr \bot^k$
- $i \equiv j \equiv 1 \pmod 3$

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$$\bot \otimes \left(\bot^i \parr \bot^j \parr \bot^k \right)$$

$$I \otimes I$$

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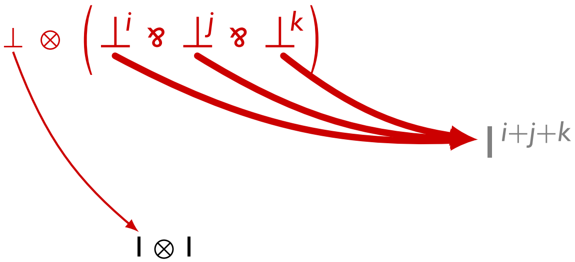
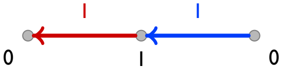


⊥ ⊗ (⊥^i ⅋ ⊥^j ⅋ ⊥^k)
I^{i+j+k}
I ⊗ I

- weight sink $\mathbf{1}^m$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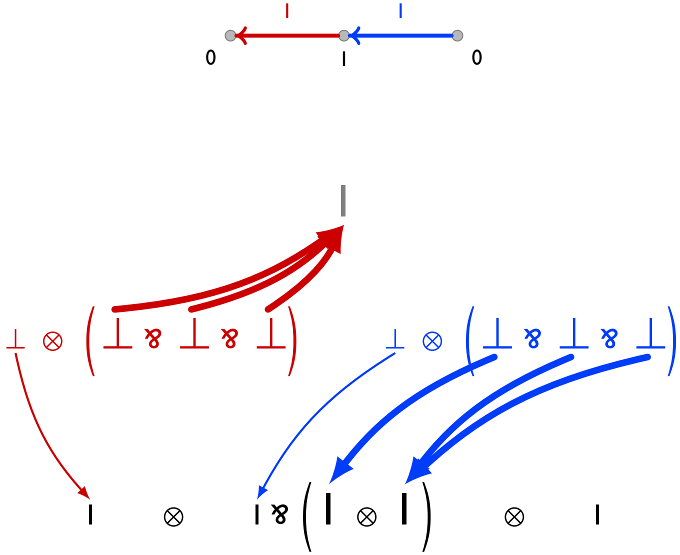

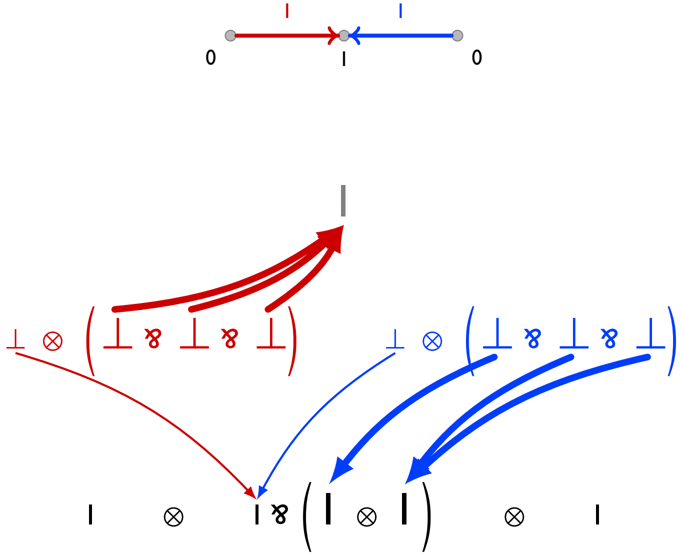

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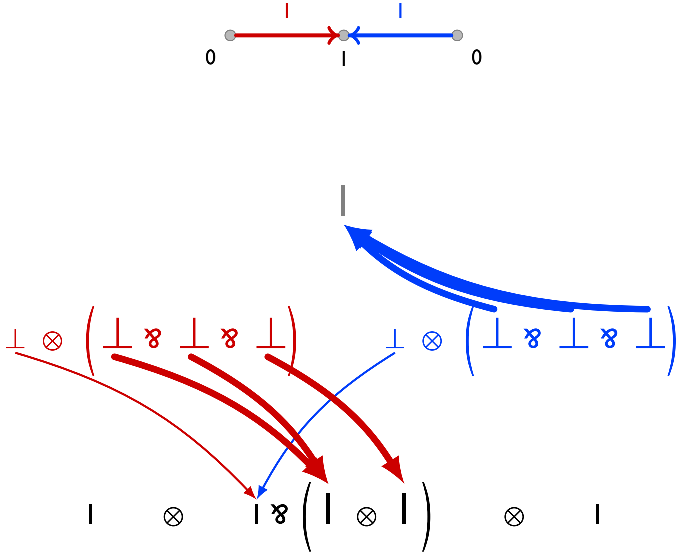
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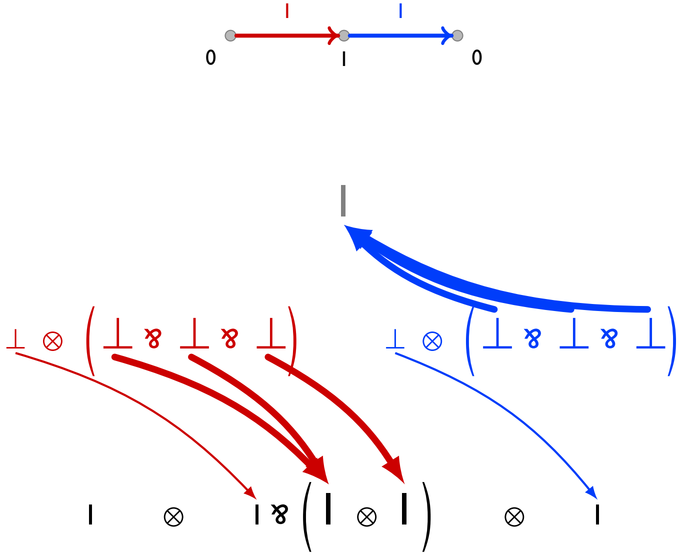

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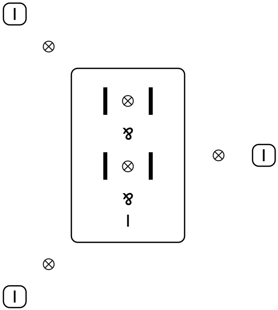

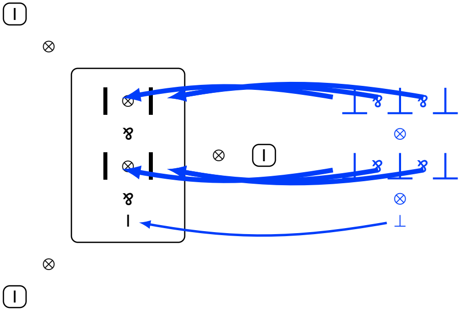

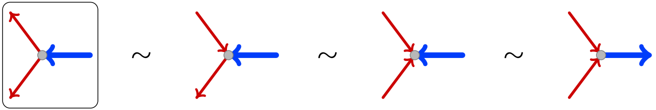

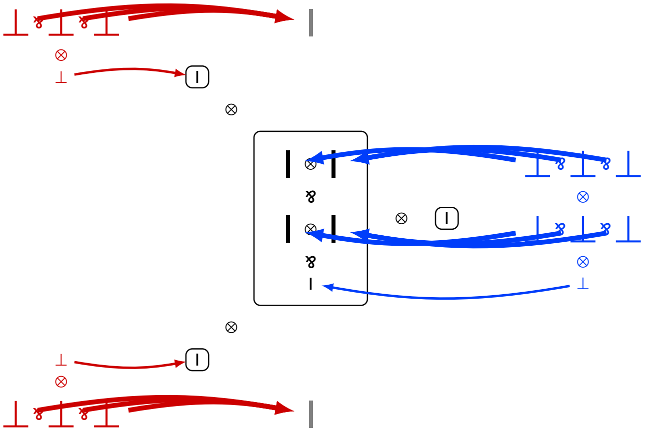

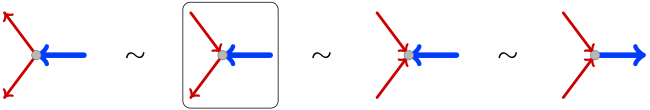

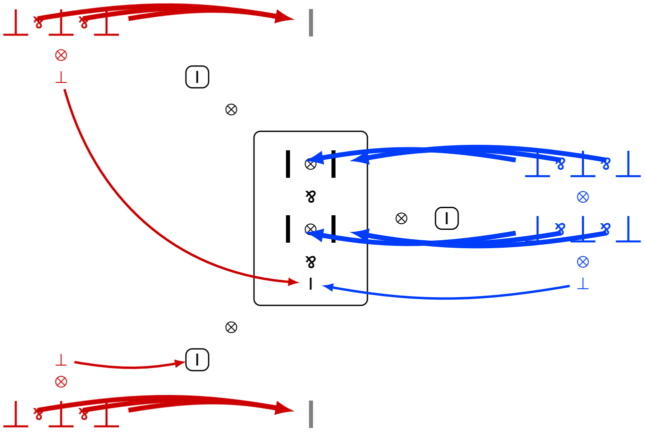

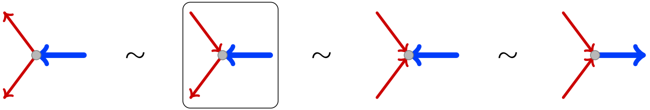

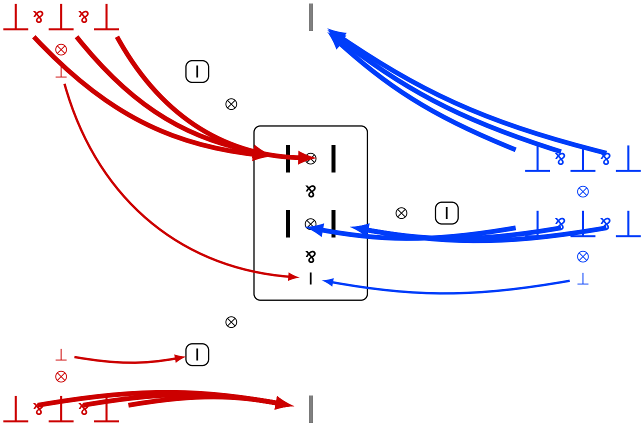

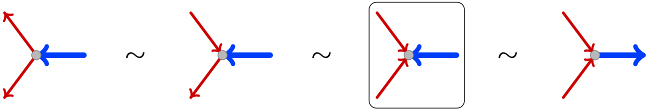

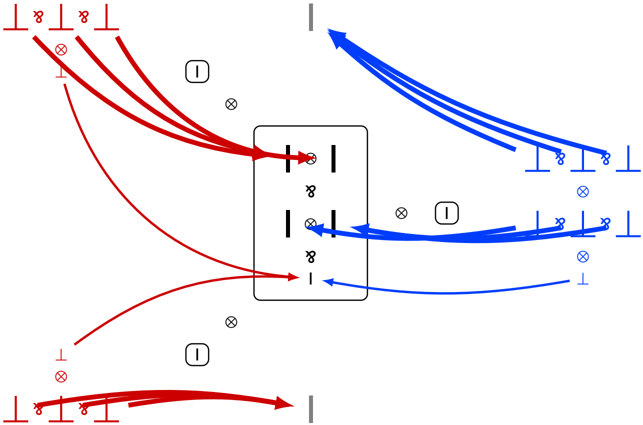

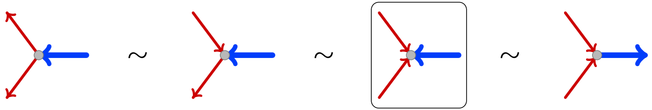

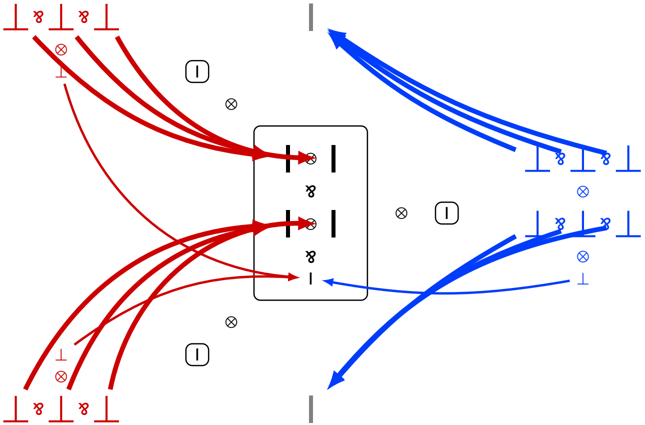

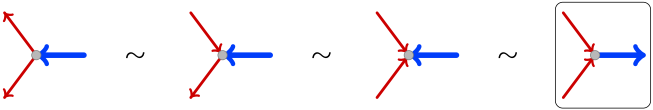

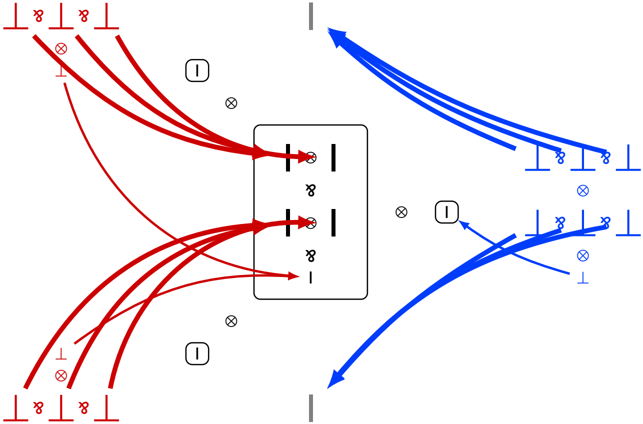

MLL proof equivalence is PSPACE-complete

- PSPACE-hard by the reduction from Constraint Logic
- in PSPACE by Savitch's Theorem (PSPACE = NPSPACE)

The End