Proof nets for bi-intuitionistic linear logic

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Abstract

Bi-Intuitionistic Linear Logic (BILL) is an extension of Intuitionistic Linear Logic with a par, dual to the tensor, and subtraction, dual to linear implication. It is the logic of categories with a monoidal closed and a monoidal co-closed structure that are related by linear distributivity, a strength of the tensor over the par. It conservatively extends Full Intuitionistic Linear Logic (FILL), which includes only the par.

We give proof nets for the multiplicative, unit-free fragment MBILL-. Correctness is by local rewriting in the style of Danos contractibility, which yields sequentialization into a relational sequent calculus extending the existing one for FILL. We give a second, geometric correctness condition combining Danos-Regnier switching and Lamarche's Essential Net criterion, and demonstrate composition both inductively and as a one-off global operation.

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1 Introduction

Obtaining good proof-theoretic characterizations of FILL [17], intuitionistic linear logic with a "par" connective dual to the tensor, and BILL, which further adds "subtract" dual to linear implication, has proved difficult. The main challenge is in combining par, whose natural home is a multi-conclusion calculus, and linear implication, which is most naturally expressed by a single-conclusion calculus. The dual situation holds for tensor and subtraction (below on the right), where tensor naturally prefers multiple assumptions, but subtraction a single assumption. These are the natural sequent rules:

$$\frac{\Gamma \vdash \Delta\ \ C\ \ D}{\Gamma \vdash \Delta\ \ C \wp D} \qquad \frac{\Gamma\ \ A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{A\ \ B\ \ \Gamma \vdash \Delta}{A \otimes B\ \ \Gamma \vdash \Delta} \qquad \frac{D \vdash C\ \ \Delta}{D - C \vdash \Delta}$$

A system with the above rules, however, does not satisfy cut-elimination [22, 3]: the single-conclusion and single-assumption rules for linear implication and subtraction are too restrictive. But their multi-conclusion and multi-assumption variants,

$$\frac{\Gamma\ \ A \vdash B\ \ \Delta}{\Gamma \vdash A \multimap B\ \ \Delta} \qquad \frac{\Gamma\ \ D \vdash C\ \ \Delta}{\Gamma\ \ D - C \vdash \Delta}$$

are unsound: they collapse the logic into MLL, since mapping linear implication $A \multimap B$ onto $A^\perp \wp B$ and subtraction $D - C$ onto $D \otimes C^\perp$ preserves provability (in both directions) [6]. Intermediate ground between these variants is found by annotating the rules with a *relation* between the antecedent and the consequent, and requiring that the discharged assumption A in a rule introducing $A \multimap B$ is not related to any additional conclusions Δ (and dually for $D - C$). With this side-condition, and without

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describing the development of the relation R into S, the rules are as below. The sequent calculus (for FILL) with relational annotation enjoys cut-elimination [4, 11].

$$\frac{\Gamma\ A \vdash_R B\ \Delta}{\Gamma \vdash_S A \multimap B\ \Delta}\ (A \not R \Delta) \qquad\qquad \frac{\Gamma\ D \vdash_R C\ \Delta}{\Gamma\ D - C \vdash_S \Delta}\ (\Gamma \not R C)$$

Traditionally, the sequent calculus is a meta-calculus, describing the construction of natural deduction proofs. For linear logic, naturally described in sequent style, the question of what underlying proof objects were constructed led to the development of proof nets [12]. In this paper we ask the same question for BILL: what are the underlying, canonical proof objects of BILL?

Our answer is a notion of proof nets, presented as a graph-like natural deduction calculus, that embodies the perfect duality between tensor and par, and between implication and subtraction. It exposes the relational annotation of the sequent calculus as recording the directed paths through the proof net constructed by the sequent proof. We give two correctness conditions: one by local rewriting in the style of Danos *contractibility* [8] and the *parsing* approach of Lafont, Guerrini and Masini [18, 14]; and a global, geometric criterion that combines Danos–Regnier switching [9] and Lamarche's *essential net* condition [19]. We introduce our proof nets with an example in Section 1.2.

We have aimed for *canonical* proof nets: those that factor out all sequent calculus permutations. To this end we have restricted ourselves to the fragment MBILL−, multiplicative bi-intuitionistic linear logic without units. MBILL with units, even though it omits negation, includes unit-only MLL, where canonical proof nets are unavailable: the proof equivalence problem, which canonical proof nets would solve efficiently, is PSPACE-complete [15].

1.1 Background and related work

In the late 1960s Lambek initiated the study of substructural logics, which restrict contraction and weakening, through category theory and with a particular focus on non-commutative variants [20]. The central point of FILL, the relation between par and linear implication, was investigated in the early 1980s by Grishin [13]. The advent of linear logic in the late 1980s [12] created an interest also in intuitionistic variants. Schellinx observed that for a multi-conclusion sequent calculus with single-conclusion $\multimap R$ rule, cut-elimination fails [22, p.555].

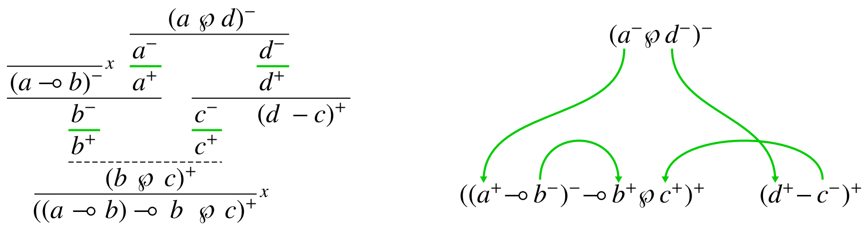
To obtain cut-elimination, Hyland and De Paiva formalize FILL through a sequent calculus annotated by a term calculus [17]. The terms describe natural deduction derivations whose open assumptions, identified by free variables in the terms, give a side-condition to a multi-conclusion $\multimap R$-rule similar to that of the current relational calculus. Unfortunately, as pointed out by Bierman, the term assignment introduces spurious dependencies that break cut-elimination. Three solutions to this problem were proposed: a modification of the term assignment by the first author, in private communication to Hyland and Bierman (cfr. [1]); a different term assignment using pattern matching by Bierman, [3]; and a sequent calculus with relational annotation by Braüner and De Paiva [4]. This is the calculus we adopt here, extended with subtraction. Eades and De Paiva [11] later revisited the term-annotated calculus, with the first author's correction, to prove semantic correctness. In the late 90s the first author developed proof nets for FILL (including the MIX rule) that sequentialize into the term-annotated sequent calculus [1]. Around the same time Cockett and Seely gave a graph-like natural deduction calculus for FILL, and for the variant of BILL corresponding to the plain, un-annotated multi-conclusion sequent calculus, which collapses onto MLL [6].

Recently, Clouston, Dawson, Goré and Tiu gave annotation-free alternatives to sequent calculi, in the form of deep-inference and display calculi for BILL that enjoy cut-elimination [5].

1.2 Proof nets for MBILL- via contractibility

We will introduce our proof nets through an example. It is shown below, in two modes of representation. On the left, it is viewed as a dag-like natural deduction proof. It is built from *links*, the equivalent of a natural deduction inference, shown as solid or dashed horizontal lines connecting *premises* above to *conclusions* below. The bottom link in the example, labelled x, introduces a linear implication, and as in natural deduction, closes the corresponding assumption by a matching link also labelled x. The (green) links from negative to positive atomic formulas, a^- to a^+, are *axiom links*.

In a multiplicative linear logic such as MBILL−, each connective in the conclusion of a sequent proof is introduced once, by exactly one proof rule; that is, connectives in the conclusion are 1–1 related to inferences in the sequent proof. Proof nets are similar: connectives in *open* assumptions and conclusions correspond 1–1 to (non-axiom) links. Via this correspondence, proof nets can be represented by only the *sequent* of open assumptions and conclusions, plus the *axiom links*, connected to the atomic subformulas in the sequent. This gives the second representation below.



We stress that these are two different representations of one and the same graphical object, and thus the same proof net. Because the former is more explicit on logical inference, we choose it as our main representation, and as the basis of our definitions (we could have chosen either). We make axiom links explicit to emphasize the connection with the second presentation.

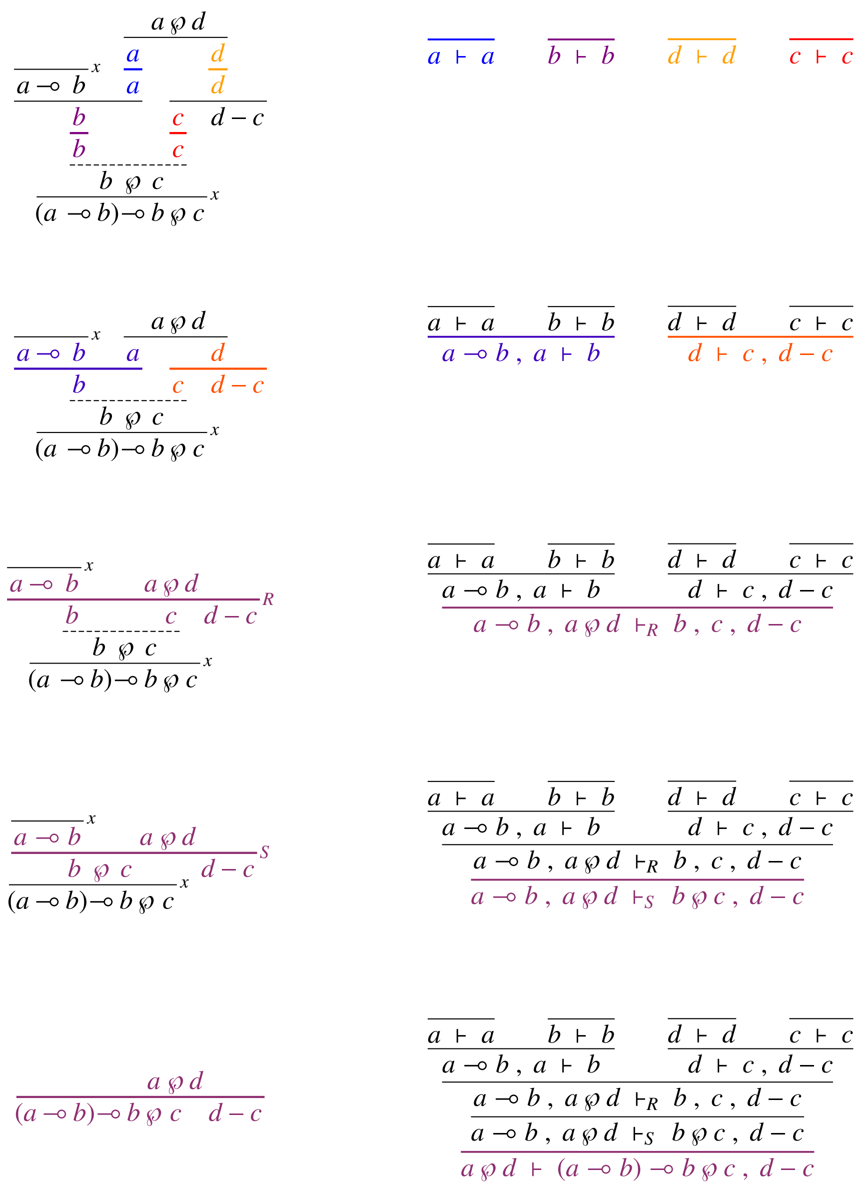
We may explicitly annotate formulae with their *polarity*, in the standard notion that reverses on the left of an implication. In BILL, it also reverses on the right of a subtraction. In a proof net, polarity is positive for conclusions and negative for assumptions, and indicates whether a formula is being *introduced* (+) or *eliminated* (−). An axiom link indicates a change from an elimination phase (above) to an introduction phase (below). In a sequent calculus, the negative formulae would be those in the antecedent Γ of a sequent $\Gamma \vdash \Delta$, and the positive those in the consequent Δ.

Figure 1 sequentializes the above example by contraction. It is initiated by giving an axiom for each axiom link (matched by colouring). Contraction is driven by the coloured links; in the second row, the links on a and b have contracted the $\multimap$-elimination link between them, and the links on c and d have contracted the −-introduction link. The corresponding sequent rules are added on the right.

The next step contracts both active links with the $\wp$-elimination link, and introduces an explicit relation R between the premises and the conclusions of the resulting link. Its purpose is to maintain the connectedness by directed (top-down) paths through the proof net. In this case, there was no directed path from $a \multimap b$ to c or to $d - c$, and to reflect this in the link created by the contraction, the relation R connects $a \multimap b$ only to b. In the third step, the $\wp$-introduction link is contracted. It uses a dashed line because it is *switched*, and may only contract if both premises connect to the same link.

Preserving top-down connectedness is the key to showing the correctness of $\multimap$-introduction links, in the last step, which must (at least) fulfil the standard intuitionistic condition: all directed paths from the discharged assumption to an (open) conclusion must pass through the discharging $\multimap$-introduction link (see [19]). The contraction step comes with the following side-condition, analogous to that of the sequent rule: the assumption $a \multimap b$ may only be related by S to the premise of the $\multimap$-introduction link, $b \wp c$, and not to other conclusions, here $d - c$. For simplicity we omit the annotation for the final link again, as it is the full relation between premises and conclusions.

This concludes the example: the net contracts to a single link, and is thus correct.



$$R = \{ (a \multimap b\,,\, b)\,,\ (a \wp d\,,\, b)\,,\ (a \wp d\,,\, c)\,,\ (a \wp d\,,\, d - c) \}$$
$$S = \{ (a \multimap b\,,\, b \wp c)\,,\ (a \wp d\,,\, b \wp c)\,,\ (a \wp d\,,\, d - c) \}$$

Figure 1 An example contraction and sequentialization sequence

$$\frac{}{A \vdash_T A} \qquad T = \frac{A}{A} \qquad\qquad \frac{\Gamma \vdash_R \Delta\ A \qquad A\ \Gamma' \vdash_S \Delta'}{\Gamma\ \Gamma' \vdash_T \Delta\ \Delta'} \qquad T = R \star \frac{A}{A} \star S$$

$$\frac{A\ B\ \Gamma \vdash_R \Delta}{A \otimes B\ \Gamma \vdash_T \Delta} \qquad T = \frac{A \otimes B}{A \quad B} \star R \qquad\qquad \frac{\Gamma \vdash_R \Delta\ A \qquad \Gamma' \vdash_S \Delta'\ B}{\Gamma\ \Gamma' \vdash_T \Delta\ \Delta'\ A \otimes B} \qquad T = (R \cup S) \star \frac{A \quad B}{A \otimes B}$$

$$\frac{C\ \Gamma \vdash_R \Delta \qquad D\ \Gamma' \vdash_S \Delta'}{C \wp D\ \Gamma\ \Gamma' \vdash_T \Delta\ \Delta'} \qquad T = \frac{C \wp D}{C \quad D} \star (R \cup S) \qquad\qquad \frac{\Gamma \vdash_R \Delta\ C\ D}{\Gamma \vdash_T \Delta\ C \wp D} \qquad T = R \star \frac{C \quad D}{C \wp D}$$

$$\frac{\Gamma \vdash_R \Delta\ A \qquad B\ \Gamma' \vdash_S \Delta'}{\Gamma\ A \multimap B\ \Gamma' \vdash_T \Delta\ \Delta'} \qquad T = R \star \frac{A \multimap B \quad A}{B} \star S \qquad\qquad \frac{\Gamma\ A \vdash_R B\ \Delta}{\Gamma \vdash_T A \multimap B\ \Delta}\, A \not{R} \Delta \qquad T = \frac{}{A} \star R \star \frac{B}{A \multimap B}$$

$$\frac{\Gamma\ D \vdash_R C\ \Delta}{\Gamma\ D - C \vdash_T \Delta}\, \Gamma \not{R} C \qquad T = \frac{D - C}{D} \star R \star \frac{C}{} \qquad\qquad \frac{\Gamma \vdash_R \Delta\ D \qquad C\ \Gamma' \vdash_S \Delta'}{\Gamma\ \Gamma' \vdash_T \Delta\ D - C\ \Delta'} \qquad T = R \star \frac{D}{C \quad D - C} \star S$$

Figure 2 Relational sequent calculus for MBILL−

2 MBILL−

The language of MBILL− is given by the following grammar.

$$A, B, C \ ::= \ a \ | \ A \otimes B \ | \ A \multimap B \ | \ A \wp B \ | \ A - B$$

We use $a, b, c, \ldots$ to range over propositional atoms. The connectives are *tensor*, *(linear) implication*, *par*, and *subtraction*. The subformula occurrences of a formula have an implicit ***polarity*** $+$ or $-$, inherited from the parent formula but reversing to the left of an implication and to the right of a subtraction: $(A \multimap B)^+$ induces A^- and $(A - B)^+$ induces B^-, and similarly with $+$ and $-$ reversed.

Figure 2 gives the relational sequent calculus of Braüner and De Paiva [4], adapted for MBILL− by introducing rules for subtraction, dual to implication. A sequent is of the form $\Gamma \vdash_R \Delta$, where Γ and Δ are multisets of formulae and $R \subseteq \Gamma \times \Delta$ is a relation from Γ to Δ. (We assume that occurrences of the same formula can be distinguished, for instance by *naming* them.)

The relational annotation maintains a notion of *logical dependence* between the formulas of a sequent. Intuitively, it traces the *subformula* relation through a proof, and in addition connects across axioms. An introduction rule for a linear implication $A \multimap B$ then requires that no formula other than B depends on the assumption A. This is closely related to the correctness condition of Lamarche's *essential nets* [19] for intuitionistic linear logic: all paths from A must converge on $A \multimap B$. The subtraction rule has a corresponding side-condition.

We use the following standard notation: relational composition $R; S$ of $R \subseteq \Gamma \times \Delta$ with $S \subseteq \Delta \times \Lambda$, the identity relation ID_Γ on a sequent Γ, and ARB for $(A, B) \in R$. We extend the latter by writing $\Gamma R \Delta$ if ARB for some A in Γ and B in Δ, and $\Gamma \not{R} \Delta$ for the negation of this proposition. We further adopt a useful notion of relational composition of Braüner and De Paiva [4]. The ***star-composition*** $R \star S$ of two relations $R \subseteq \Gamma \times (\Delta \cup \Delta')$ and $S \subseteq (\Delta' \cup \Delta'') \times \Lambda$, where Δ, Δ', and Δ'' are pairwise disjoint, is

$$R \star S \quad = \quad (R \cup ID_{\Delta''})\,;\,(ID_\Delta \cup S) \quad \subseteq \quad (\Gamma \cup \Delta'') \times (\Delta \cup \Lambda)$$

The above composition consists of three parts: R restricted to $\Gamma \times \Delta$, S restricted to $\Delta'' \times \Lambda$, and $R; S$ restricted to $\Gamma \times \Lambda$. It is a relational equivalent of linear distributivity [7], and a generalization of both union (if Δ' is empty) and composition (if Δ and Δ'' are empty). For ease of presentation, we write $\frac{\Gamma}{\Delta}$ for the full relation $\Gamma \times \Delta$. Note that $\frac{}{A}$ stands for the *empty* relation from the empty sequent to A; it is used, with ($\star$)-composition, to restrict the domain of a relation by removing A.

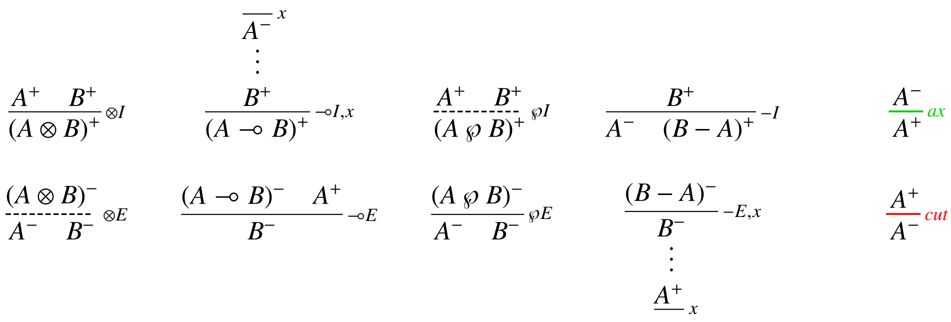


Figure 3 Links for the construction of MBILL− proof nets.

3 Proof nets

We shall define our proof nets for MBILL− as a graph-like natural deduction calculus. We make axioms and cuts explicit, as inference rules that only change the polarity of a formula. This gives a closer connection with sequent calculus and traditional proof nets, and simplifies the definition of contractibility. First we define the underlying graphs, or *pre-nets*; then we will introduce contractibility as a correctness condition, and define our proof nets as the pre-nets satisfying contractibility.

▶ **Definition 1** (Pre-nets). MBILL− pre-nets are built from the following notions.

- ***Link:*** a node with $n \geq 0$ ***premise*** ports and $m \geq 0$ ***conclusion*** ports labelled with formulas $A_1 \dots A_n$ and $B_1 \dots B_m$ and a possibly empty label ℓ. A ***relational link*** is labelled with a relation $R \subseteq \{A_1, \dots, A_n\} \times \{B_1 \dots B_m\}$. A link is drawn as follows.

$$\frac{A_1 \ \dots \ A_n}{B_1 \ \dots \ B_m}\,\ell$$

- ***Edge:*** a connection from a premise port to a conclusion port labelled with the same formula, of the same polarity.
- ***Pre-net:*** an acyclic directed graph $N = (V, E)$ with V a set of links as in Figure 3, and E a set of edges such that no two edges connect to the same port, satisfying the following conditions. A premise / conclusion port with no attached edge is an ***open assumption / conclusion***. The $\multimap I$ / $-E$ links are in bijection with the ***closed assumption / conclusion*** links, defined by the variable labels x in Figure 3. A ***relational pre-net*** may contain also relational links.

In Figure 3, note that the illustrations for $\multimap I$ and $-E$ links each show *two* links: the $\multimap I$ link itself, plus a closed assumption link; and the $-E$ link plus a closed conclusion link.

$$\frac{\Gamma}{\Delta}\,R$$

We abbreviate a pre-net with open assumptions Γ and open conclusions Δ by a double-lined link, as on the left. We may annotate it with a relation R that relates A in Γ to B in Δ if (and only if) there is a directed downward path from A to B.

3.1 Contractibility

Our correctness condition is in the style of Danos *contractibility* [8].[1] Contractibility for MLL proof nets is, in essence, top-down sequentialization [18, 14], starting from the axioms rather than the

[1] The second author has also used the term *coalescence* for the generalization of contractibility that includes the additives—but as these are not currently present, we feel it is more appropriate to use the terminology that was established earlier.

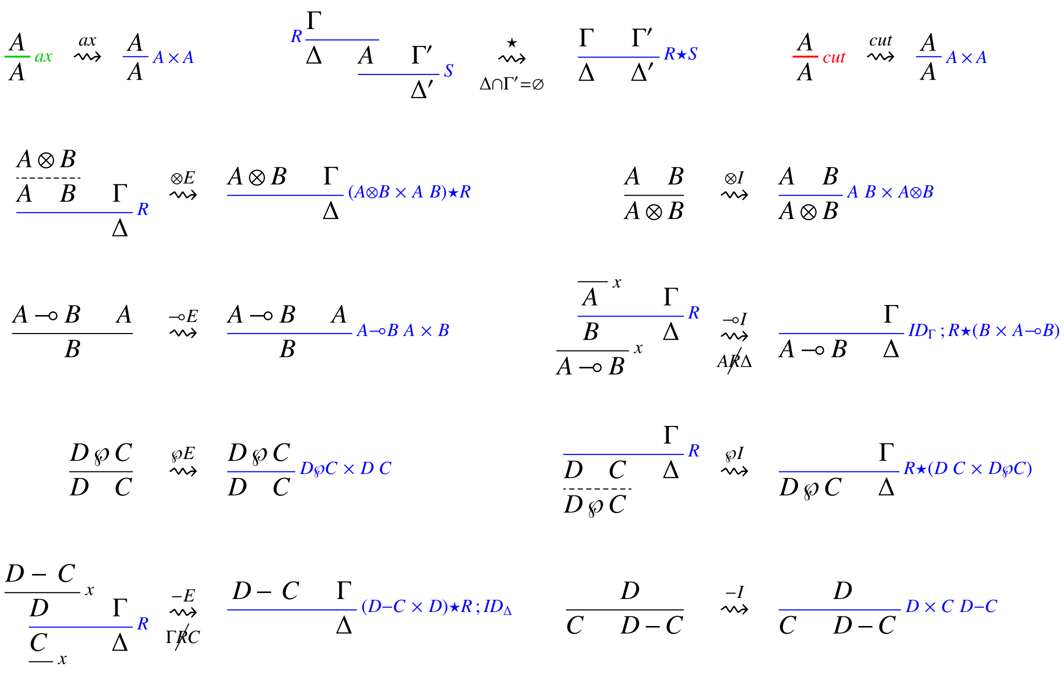


Figure 4 Contraction rules

conclusion of a proof net. In our current natural deduction style, contraction is *inside-out*, from axioms to assumptions and conclusions. Contracting a proof net corresponds to the construction of a sequent proof or other inductive proof object. This can be made explicit by carrying the constructed object as a label on the contracting links, which we will do in Section 4.

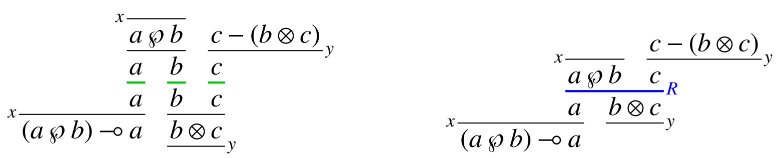
The links of a proof net being contracted correspond to sequents of the proof being constructed. As such, we will be contracting *relational links* (see Definition 1), corresponding to relational sequents.

▶ **Definition 2** (Contractibility)**.** ***Contraction*** is the rewrite relation on relational pre-nets given by the rewrite rules in Figure 4. Contraction is successful if it terminates with a single link. A pre-net ***contracts***, or is ***contractible***, if it has a successful contraction path. It ***strongly*** contracts if every contraction path is eventually successful.

▶ **Definition 3** (Proof nets)**.** A MBILL− ***proof net*** is a contractible MBILL− pre-net whose open assumptions and conclusions have negative respective positive polarity.

An example contraction sequence was given in Figure 1 in the introduction. An example of how contraction excludes incorrect nets is the following.

▶ **Example 4.** Below left is an incorrect pre-net. After several ax, $\wp E$, $\otimes I$ and $\star$ steps, we obtain the pre-net below right, where $R = \{\, (a \wp b\,,\, a)\,,\ (a \wp b\,,\, b \otimes c)\,,\ (c\,,\, b \otimes c)\,\}$. Because of the relation $(a \wp b\,,\, b \otimes c)$ this prevents further contraction: there are two potential steps, a $\multimap I$-step and a $-E$-step, and for both the side-condition is not met.



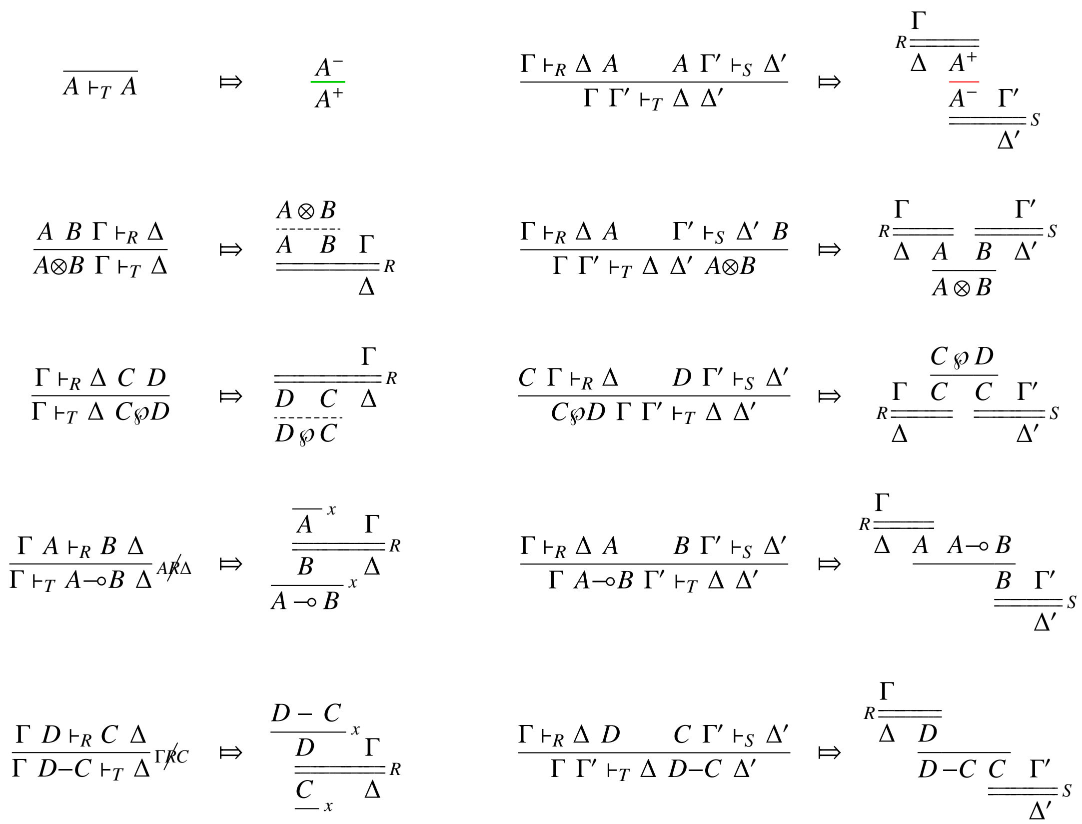


Figure 5 De-sequentialization

4 Sequentialization and de-sequentialization

To de-sequentialize a sequent proof to a proof net, intuitively, is to take each sequent rule, and separate the logical inference (e.g. from $A \multimap B$ and A to B) from the context (Γ and Δ). We visualize this in Figure 5, where the premises of each rule de-sequentialize to the given (double-lined) pre-nets.

▶ **Definition 5.** A sequent proof ***de-sequentializes*** ($\Rightarrow$) to a proof net as illustrated in Figure 5.

▶ **Proposition 6.** *The de-sequentialization of a sequent proof contracts.*

Proof. By induction on the sequent proof. Following Figure 5, a de-sequentialization $\frac{\Gamma}{\Delta}R$ (double-lined) contracts to the relational link $\frac{\Gamma}{\Delta}R$. ◀

Sequentialization is by contraction. First, we introduce a notion of ***open proof***, a sequent proof from (open) premise sequents $\vdash A$ and $B \vdash$. We abbreviate an open proof by a double line, as below left. The given open proof will result from contracting a pre-net with negative assumptions Γ^- and positive conclusions Δ^+, plus *positive* assumptions $A_1^+ \dots A_n^+$ and *negative* conclusions $B_1^- \dots B_m^-$, below right. The domain and range of the annotating relation of a sequent are extended to include the open permises: $R \subseteq (\Gamma\, A_1\ \dots\ A_n) \times (\Delta\, B_1\ \dots\ B_m)$. The relation is otherwise constructed as before.

$$\frac{\vdash A_1 \dots \ \vdash A_n \quad B_1 \vdash \ \dots \ B_m \vdash}{\Gamma \vdash_R \Delta} \qquad\qquad \frac{\Gamma^-\ A_1^+ \dots A_n^+}{\Delta^+\ B_1^- \dots B_m^-}R$$

For sequentialization, we define a mapping from the contracting links of a proof net to sequent proofs. For a star-composition,

$$\dfrac{\dfrac{\Gamma^- \; A_1^+ \ldots A_n^+}{\Delta^+ \; B_1^- \ldots B_m^- \quad C^+}{\scriptstyle R} \quad \Gamma'^- \; A_{n+1}^+ \ldots A_p^+}{\Delta'^+ \; B_{m+1}^- \ldots B_q^-}{\scriptstyle S} \quad \overset{\star}{\rightsquigarrow} \quad \dfrac{\Gamma^- \quad \Gamma'^- \quad A_1^+ \ldots A_p^+}{\Delta^+ \quad \Delta'^+ \quad B_1^- \ldots B_q^-}{\scriptstyle R\star S}$$

if the links in the redex map onto the open proofs

$$\Pi = \dfrac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash \ldots B_m \vdash}{\Gamma \vdash_R \Delta \; C} \qquad \Phi = \dfrac{\vdash C \quad \vdash A_{n+1} \ldots \vdash A_p \quad B_{m+1} \vdash \ldots B_q \vdash}{\Gamma' \vdash_S \Delta'}$$

then the contractum is mapped onto the open proof

$$\dfrac{\vdash A_1 \ldots \vdash A_p \quad B_1 \vdash \ldots B_q \vdash}{\Gamma \; \Gamma' \vdash_{R\star S} \Delta \; \Delta'}$$

obtained by replacing the open premise $\vdash C$ of Φ with the open proof Π, and adding the conclusions Γ and Δ to each inference from $\vdash C$ down to the conclusion of Φ.

To the contractum of the steps *ax*, *cut*, $\otimes I$, $\multimap E$, $\wp E$, $-I$ we assign the respective proofs:

$$\dfrac{}{A \vdash A} \qquad \dfrac{\vdash C \quad C \vdash}{\vdash} \qquad \dfrac{\vdash A \quad \vdash B}{\vdash A \otimes B} \qquad \dfrac{\vdash A \quad B \vdash}{A \multimap B \vdash} \qquad \dfrac{C \vdash \quad D \vdash}{C \wp D \vdash} \qquad \dfrac{\vdash C \quad D \vdash}{\vdash C - D}$$

To the remaining steps we assign proofs as follows, where $\Gamma = \Gamma^- \; A_1^+ \ldots A_n^+$ and $\Delta = \Delta' \; B_1^- \ldots B_m^-$.

$$\dfrac{\dfrac{A \otimes B}{A \quad B} \quad \Gamma}{\Delta}{\scriptstyle R} \quad \overset{\otimes E}{\rightsquigarrow} \quad \dfrac{A \otimes B \quad \Gamma}{\Delta}{\scriptstyle (A\otimes B \times A\; B)\star R} \qquad \dfrac{\dfrac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash \ldots B_m \vdash}{A \;\; B \;\; \Gamma' \vdash_R \Delta'}}{A \otimes B \;\; \Gamma' \vdash_T \Delta'}$$

$$\dfrac{\dfrac{\dfrac{}{A}{\scriptstyle x} \quad \Gamma}{B \quad \Delta}{\scriptstyle R}}{A \multimap B}{\scriptstyle x} \quad \underset{A \not R \Delta}{\overset{\multimap I}{\rightsquigarrow}} \quad \dfrac{\Gamma}{A \multimap B \quad \Delta}{\scriptstyle ID_\Gamma \,;\, R\star(B \times A\multimap B)} \qquad \dfrac{\dfrac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash \ldots B_m \vdash}{\Gamma' \;\; A \vdash_R B \;\; \Delta'}}{\Gamma' \vdash_T A\multimap B \;\; \Delta'}{\scriptstyle A \not R \Delta'}$$

$$\dfrac{\Gamma}{\dfrac{D \quad C}{D \wp C} \quad \Delta}{\scriptstyle R} \quad \overset{\wp I}{\rightsquigarrow} \quad \dfrac{\Gamma}{D \wp C \quad \Delta}{\scriptstyle R\star(D\; C \times D\wp C)} \qquad \dfrac{\dfrac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash \ldots B_m \vdash}{\Gamma' \vdash_R C \;\; D \;\; \Delta'}}{\Gamma' \vdash_T C \wp D \;\; \Delta'}$$

$$\dfrac{\dfrac{D - C}{D}{\scriptstyle x} \quad \Gamma}{\dfrac{C}{}{\scriptstyle x} \quad \Delta}{\scriptstyle R} \quad \underset{\Gamma \not R C}{\overset{-E}{\rightsquigarrow}} \quad \dfrac{D - C \quad \Gamma}{\Delta}{\scriptstyle (D-C \times D)\star R \,;\, ID_\Delta} \qquad \dfrac{\dfrac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash \ldots B_m \vdash}{\Gamma' \;\; D \vdash_R C \;\; \Delta'}}{\Gamma' \;\; D-C \vdash_T \Delta'}{\scriptstyle \Gamma' \not R C}$$

Finally, recall that a proof net has only negative assumptions and positive conclusions. If it contracts to a single link, this link maps to a regular (relational) sequent proof, without open premises.

▶ **Definition 7** (Sequentialization)**.** A proof net ***sequentializes*** to a proof Π if it contracts to a single link that maps onto Π.

▶ **Proposition 8.** *The de-sequentialization of a sequent proof* Π *sequentializes to* Π.

Proof. By induction on the sequent proof. Following Figure 5, a de-sequentialization $\dfrac{\Gamma}{\Delta}{\scriptstyle R}$ of Π contracts to the relational link $\dfrac{\Gamma}{\Delta}{\scriptstyle R}$ mapping to Π. ◀

5 A geometric characterization

In this section we give a geometric correctness condition for MBILL− proof nets, and demonstrate that a pre-net contracts if and only if it is correct. The condition has two components: a ***switching*** condition in the style of Danos and Regnier [9] that integrates the condition on Lamarche's *essential nets* [19], and a ***bi-functionality*** condition that further refines the essential net condition. We begin by giving the necessary definitions.

▶ **Definition 9** (Switching). In a pre-net N:

Switched / solid the ***switched*** links are $\wp I$, $\otimes E$, $\multimap I$, and $-E$; other links are ***solid***. A ***switched*** edge is one connecting to an auxiliary port of a switched link or to a closed assumption or conclusion link; other edges are ***solid***.

Targets The ***targets*** of a switched link are as follows:

- the targets of a $\wp I$ or $\otimes E$ link are the two links connected by a switched edge;
- the targets of a $\multimap I$ link $A \multimap B$ are the link connected to the auxiliary port B plus all links on a directed downward path starting from the associated closed assumption link A, but not passing through $A \multimap B$;
- the targets of a $-E$ link $D - C$ are the link connected to the auxiliary port D plus all links on a directed downward path ending at the associated closed conclusion link C, but not passing through $D - C$.

Switching graph A ***switching graph*** G for N is an undirected graph (V, E) whose ***vertices*** V are the links of N, and whose edges E connect:

- any two links connected by a solid edge in N;
- any switched link to exactly one of its targets.

Switching condition a pre-net satisfies the ***switching condition*** if every switching graph is acyclic and connected.

▶ **Definition 10** (Bi-functionality). A pre-net satisfies the ***bi-functionality condition*** if

- a directed path from a closed assumption x to an open conclusion passes through $\multimap I, x$;
- a directed path from an open assumption to a closed conclusion y passes through $-E, y$;
- a directed path from a closed assumption x to a closed conclusion y passes through $\multimap I, x$ or $-E, y$.

▶ Remark. Closer observation will reveal that the first two components of the bi-functionality condition are equivalent to assuming an implicit $\wp I$-link connecting all open conclusions, and a $\otimes E$-link connecting open assumptions. The third component is equivalent to considering a closed assumption x and its implication introduction link $\multimap I, x$ to be one and the same link for the purpose of the switching graph (though not for downward reachability).

▶ **Definition 11** (Geometric correctness). A pre-net N is ***geometrically correct*** if it satisfies both the switching condition and the bi-functionality condition.

A ***switching path*** is an undirected path in a switching graph G, which we will indicate by ($\overset{G}{\text{——}}$). A single, switched edge will be written ($\overset{G}{\text{----}}$), and we may omit the superscript if G is understood. For simplicity, we will refer to a link by its principal formula when indicating switching paths. For a link A and switched link B in a switching graph G, write $A \ll_G B$ if A is on a switching path between two targets B_1 and B_2 of B, i.e. if there is a switching path $B_1 \overset{G}{\text{——}} A \overset{G}{\text{——}} B_2$.

▶ **Definition 12.** A link A is ***in scope of*** a switched link B, written $A \ll B$, if $A \ll_G B$ for some G. The ***scope*** of a link B is the set $\{A \mid A \ll B\}$.

We take the scope relation ($\ll$) as ranging over all links, though note that for a solid link B there is never any $A \ll B$.

▶ **Lemma 13.** *In a pre-net satisfying the switching condition, $(\ll)$ is a strict partial order.*

Proof. **Irreflexivity:** $A \not\ll A$. Immediate, since a switching path $A_1 \text{ —— } A \text{ ---- } A_2$ (with A switched to A_2) creates a cycle $A_1 \text{ —— } A \text{ ---- } A_1$ by switching A to A_1.

Transitivity: if $A \ll B \ll C$ then $A \ll C$. Let B be a switched link with jump targets B_1, B_2, and B_3, and C a switched link with targets C_1 and C_2. Let $A \ll B \ll C$ be witnessed by switching graphs G and H, so that $A \ll_G B \ll_H C$, via the following paths.

$$B_2 \overset{G}{\text{——}} A \overset{G}{\text{——}} B_3 \qquad C_1 \overset{H}{\text{——}} B_1 \overset{H}{\text{----}} B \overset{H}{\text{——}} C_2$$

We allow the possiblity that B_1 is the same as either of B_2 and B_3, as is necessarily the case for a binary switched link. First, we create a switching K which agrees with H everywhere except the links on the below path, where it agrees with those links.

$$B \text{ ---- } B_2 \overset{G}{\text{——}} A \overset{G}{\text{——}} B_3$$

Crucially, no other path in G from B may connect to the above path, and so any path in K not ending with a switched edge of B must agree with H. In particular this includes the path $B \text{ —— } C_2$. Moreover, in H no path from the principal port of B reaches C_1, since there is already a path $C_1 \text{ —— } B_1 \text{ ---- } B$. Then also in K no path from the principal port of B, which must all agree with H, can reach C_1. Instead, C_1 and B must then be connected as follows.

$$C_1 \overset{K}{\text{——}} B_2 \overset{K}{\text{----}} B$$

Let X be the link where this path first intersects the path $B_2 \overset{G}{\text{——}} A \overset{G}{\text{——}} B_3$, where K agrees with G; without loss of generality, assume thats X comes before A. This gives the following.

$$C_1 \overset{K}{\text{——}} X \overset{K}{\text{——}} B_2 \qquad B_2 \overset{K}{\text{——}} X \overset{K}{\text{——}} A \overset{K}{\text{——}} B_3$$

Switching B to B_3 we have the following path.

$$C_1 \overset{K}{\text{——}} X \overset{K}{\text{——}} A \overset{K}{\text{——}} B_3 \text{ ---- } B \overset{K}{\text{——}} C_2$$

Then $A \ll C$, as required. ◀

Our notion of scope is related to the first author's notion of *loop* for MLL nets with Mix [1]. It is further closely related to the De Naurois–Mogbil correctness condition [10]. This uses the relation $(\ll_G)$, over a fixed switching graph G. Unlike $(\ll)$ the relation $(\ll_G)$ is not necessarily transitive. We write $(\ll_G^*)$ for the transitive closure and $(\ll_G^n)$ for the n-fold relational composition,

$$A_0 \ll_G^n A_n \quad = \quad A_0 \ll_G A_1 \ll_G \cdots \ll_G A_n \,.$$

▶ **Proposition 14.** *In a pre-net satisfying the switching condition, $A \ll_G^* B$ if and only if $A \ll B$.*

Proof. From left to right, $A \ll_G B$ implies $A \ll B$, and $(\ll)$ is transitive. From right to left, we proceed by induction on the distance between A and B in $(\ll)$. First consider the case where A and B are immediate neighbours (distance 1), i.e. there is no C such that $A \ll C \ll B$. Then there is a path between the premises of B that does not contain any switched links. Whichever way G switches on B, we have $A \ll_G B$. In the case where there is a C such that $A \ll C \ll B$, by induction we have $A \ll_G^* C$ and $C \ll_G^* B$, and hence $A \ll_G^* B$. ◀

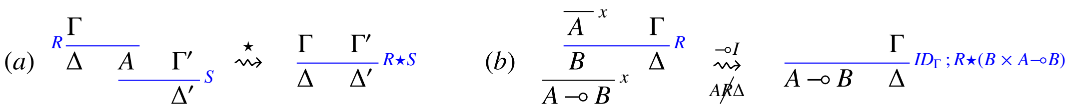
The scope of a link A includes exactly those links that must be contracted before A can be contracted itself. (We will use this to prove that a correct pre-net contracts, by demonstrating that any link that is minimal in $(\ll)$ may be contracted, as part of the proof of Theorem 16 below.) The scope of A then corresponds to the smallest open subproof of A in any sequentialization. In this way, the notion of scope is also closely related to the standard notion of ***kingdom*** [2]: the kingdom kA of a subformula A corresponds to the smallest *subproof* of A in any sequentialization.

For an MLL proof net, the kingdom kA is the smallest subgraph such that $A \in kA$ and:

1. if $B \in kA$ and B is in an axiom link with $B^\perp$, then $B^\perp \in kA$;
2. if $B \otimes C \in kA$ then $B \in kA$ and $C \in kA$;
3. If $B \wp C \in kA$ then kA includes the scope of $B \wp C$: if $D \ll B \wp C$ then $D \in kA$.

Observe that (2) corresponds to the fact that a *subproof* containing $B \otimes C$ must contain also subproofs for B and for C; however, an *open subproof* need not. Because scope is transitive, and because it does not need to be closed under (2) like kingdoms, we may avoid an inductive definition. Interestingly, this implies that (smallest) *open* subproofs are a *geometric* concept, not an inductive one.

We will now show that contractibility and geometric correctness are equivalent conditions. First, we establish that if N contracts to M, then if either of N and M is geometrically correct, both are. This is a straightforward induction on the contraction sequence.



▶ **Lemma 15.** *Contraction preserves and reflects geometric correctness.*

Proof. We will treat the star-contraction rule (a) and the contraction rule for linear implication (b); the other rules are similar, or trivial.

Let $N \rightsquigarrow M$ by a $\star$-step. The composition $R \star S$ ensures that directed paths are maintained through the contraction step. It follows that the targets of any $\multimap I$ or $-E$ link are the same in both N and M, save that if one of both contracted links in N is a target then the resulting link in M is a target, and vice versa. This leaves the geometry of the switching graphs in N and M unchanged.

Next, let $N \rightsquigarrow M$ by a $\multimap I$-step. Because of the side-condition $A \not R \Delta$, the only target of the link $A \multimap B$ is the contraction link R. It follows that there is a one-to-one correspondence between switching graphs in N and in M, preserving their geometry. ◀

▶ **Theorem 16.** *A pre-net N contracts if and only if it is geometrically correct.*

Proof. From left to right, assume that N contracts. The end result, a single contracted link, is geometrically correct. Since contraction reflects geometric correctness, by Lemma 15, by induction on the contraction sequence N is geometrically correct.

From right to left, it must be shown that if N is geometrically correct, a contraction step applies. As contraction preserves geometric correctness (Lemma 15), it then follows that N contracts, by induction on its size.

Contraction steps on solid links have no side conditions, and the star-contraction rule (a) applies to any adjacent relational links. Applying these steps first, we may assume that N consists solely of relational links separated by switched links. Consider a switched link that is minimal in $(\ll)$. We will treat the case of a $\multimap I$ link $A \multimap B$ and show that a $\multimap I$-step (b) applies; the other three cases are similar.

Let X be the link connected to the port A of the closed assumption of $A \multimap B$, and Y the link connected to the auxiliary port B of the link $A \multimap B$. In any switching graph G the links X and Y must be connected, and since both are targets of $A \multimap B$, they cannot be connected through its principal port, as this would violate irreflexivity of $(\ll)$. Because $A \multimap B$ is minimal in $(\ll)$ there can be no switched link on the switching path X —— Y, and since relational links are not adjacent (they would have been contracted), there can be only one. Then $X = Y$ is the unique relational link to which both ports A and B connect, as required by the $\multimap I$ contraction step (b).

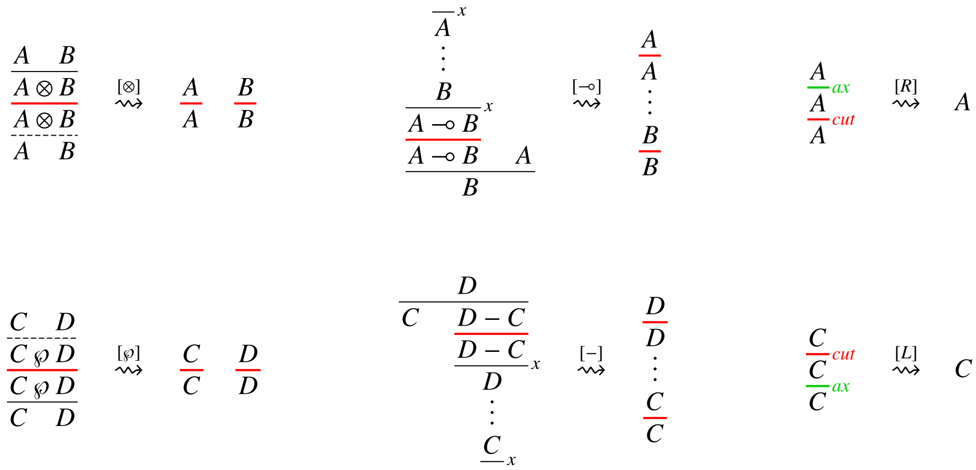


Figure 6 Proof net normalization rules

Finally, we show that the side condition $A \not R \Delta$ is satisfied. Suppose there is a port D in Δ such that ARD. By the bi-functionality condition D cannot be an open conclusion, and cannot connect to a closed one. The link L connected at D must then be a switched link (since adjacent relational links were assumed to have been contracted). Note that L is a target of $A \multimap B$. If L is a $\otimes E$, $\multimap I$, or $\wp I$ link, also at least one link connected at an auxiliary port of L (possibly X) is a target of $A \multimap B$. This would mean $L \ll (A \multimap B)$, contradicting the assumption that $A \multimap B$ was minimal. It follows that $A \not R \Delta$, and a $\multimap I$-contraction step applies to $A \multimap B$. ◀

To be effective, it is crucial to have *strong* contractibility, where *any* contraction path (eventually) terminates with a single link. If only some paths would eventually be successful, an algorithm for correctness would need to backtrack (or have a guaranteed strategy). Instead, we should be able to use any contraction sequence, without the chance of failure. This is established by the following theorem.

▶ **Theorem 17** (Strong contractibility). *MBILL– proof nets are strongly contractible.*

Proof. Since proof nets are correct (Theorem 16), and contraction preserves correctness (Lemma 15), any contraction step yields a correct proof net, which must then contract (Theorem 16). ◀

6 Normalization

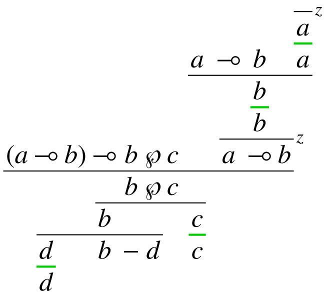
We give proof reduction as a graph-rewrite relation on pre-nets. There are six reduction steps, one for each connective and two for axioms, given in Figure 6. Since proof nets strictly reduce in size, termination is immediate. So is confluence: the only redexes that may overlap are $[L]$ and $[R]$, but this critical pair converges trivially. A pre-net is in ***normal form*** if it has no cuts, and in ***expanded normal form*** if in addition the formulas of axiom links are atomic. The unique expanded normal form of a net N is denoted $N{\downarrow}$. The example in the introduction is in expanded normal form.

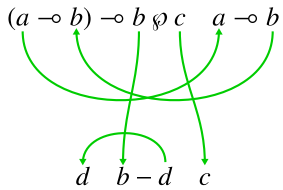
▶ **Theorem 18** (Normalization preserves correctness). *A proof net reduces to a proof net.*

Proof. By inspection of the normalization steps, geometric correctness is preserved. ◀

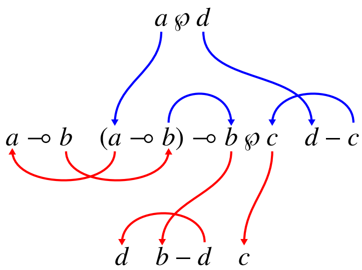
6.1 One-step composition

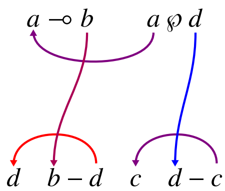
Proof nets in expanded normal form have a compact alternative representation. In a purely multiplicative logic such as MBILL–, a proof (or proof net) has exactly one rule (or link) for every connective in the conclusion sequent. Identifying links with connectives, we can display a proof net by drawing its open assumptions (above) and conclusions (below), and connecting these with the axiom links. An example was given in the introduction; here is another.





We will formalize such proof nets as the ***compact form*** of a net in expanded normal form. As in classical and intuitionistic MLL [16], composition of compact forms in MBILL– is particularly nice: it is path-composition along the axiom links of both nets, as connected through the formula along which they are composed. This is demonstrated below. On the left are the net from the introduction, in blue, and that from above in red (with the assumption $a \multimap b$ re-positioned on the left), with their common open conclusion and assumption superimposed. Composing these nets along that common formula gives the net below right.





We will formalize this concisely, as follows.

▶ **Definition 19.** The ***compact form*** $\lfloor N \rfloor = \Lambda : \Gamma \vdash \Delta$ of a pre-net N in expanded normal form consists of the open assumptions Γ, the open conclusions Δ, and the axiom links Λ of N.

Given two compact forms $\lfloor M \rfloor = \Lambda_M : \Gamma_M \vdash \Delta_M\, A^+$ and $\lfloor K \rfloor = \Lambda_K : A^- \, \Gamma_K \vdash \Delta_K$, define their ***composition along*** A as $\Lambda : \Gamma_M\, \Gamma_K \vdash \Delta_M\, \Delta_K$ where Λ consists of all maximal paths in the undirected graph formed by Λ_M, Λ_K, and connecting corresponding atoms in A^+ and A^-. Correspondingly for (non-compact) pre-nets, the ***cut-composition along*** A of pre-nets M with open conclusion A^+ and K with open assumption A^-, is the (disjoint) union of both graphs together with a cut-link with premise A^+ and conclusion A^-.

▶ **Theorem 20.** *If N is the cut-composition along A of proof nets M and K in expanded normal form, then $\lfloor N{\downarrow} \rfloor$ is the composition along A of $\lfloor M \rfloor$ and $\lfloor K \rfloor$.*

Proof. By induction on the cut-formula. ◀

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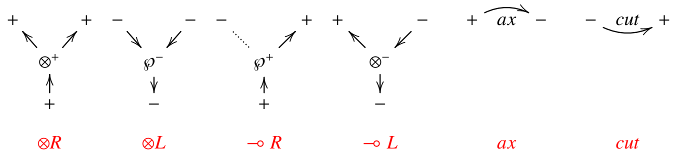
A Relations with existing syntax.

Lamarche [19] (see also Murawski and Ong [21]) developed a system of *essential nets* for ILL where nets are polarized, edges are directed and the polarization of links reflects the structure of ILL sequent calculus inferences. Notice that a $\wp^-$ link is *not switched* and $\wp^+$ links have a canonical *right* switch. The links of polarized classical MLL− formuas correspond to the intuitionistic ILL− inferences in red.

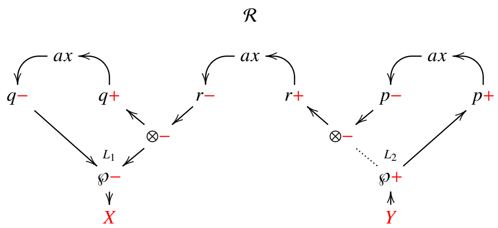


▸ **Definition 21.** An **essential net** $\mathcal{E}$ is a structure satisfying the following conditions:

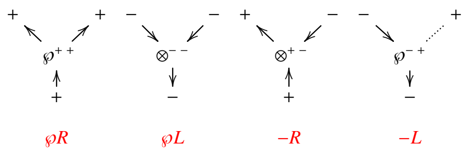
1. *(acyclicity)* there is no cycle of *directed edges* in $\mathcal{E}$;
2. *(functionality of implications) for every $\wp_+$ link with premises A^- and B^+, every directed path from (the only positive) conclusion of $\mathcal{E}$ to A^- passes through B^+.*

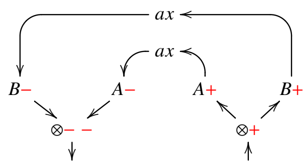
Lamarche proves that every correct proof net can be sequentialized into an ILL sequent derivation.

▸ **Example 22.** Essential net for $q \otimes (q \multimap r) \vdash (r \multimap p) \multimap p$, where $X = q \otimes (q \multimap r)$ and $Y = (r \multimap p) \multimap p$.



In order to extend the above representation to FILL− and BILL− we may add links for intuitionistic *par* and *subtraction*, below left. However, in this extension it is no longer possible to verify the *acyclicity* condition on *directed* paths. There is no directed cycle in the pre-net below right:





A solution is *first* test the MLL− acyclicity and connectedness condition of *undirected* DR-graphs with switchings on *par-like* links, namely, links representing MBILL− $\otimes L$, $\wp R$ (for $\multimap R$ and $-L$ the switching is canonical), and then test a specific correctness condition, the *bifunctionality condition on $\multimap R$ and $-L$.*

The first author [1] sequentializes proof nets for FILL into Hyland and De Paiva's labelled sequent calculus.

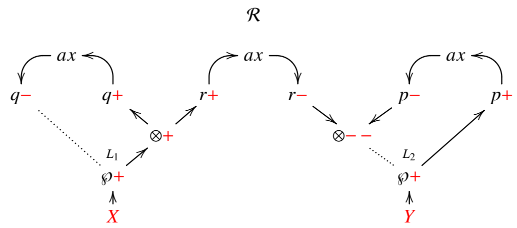
▸ **Definition 23.** A proof net $\mathcal{R}$ for FILL- is a polarized MLL- structure satisfying the following conditions:

1. (*DR condition*) for every switching s, $s\mathcal{R}$ is acyclic and connected;
2. (*functionality of implications*) *for every $\wp_+$ link with premises A^- and B^+, and conclusion $(A\wp B)^+$ every directed path from any positive conclusion X^+ of $\mathcal{R}$ to A^- passes through $(A\wp B)^+$.*

To prove sequentialization the following lemma is needed:

Lemma. *Let $\mathcal{D}$ be a labelled sequent calculus derivation of S and let $\mathcal{D}^-$ be the polarized proof net resulting from de-sequentializing $\mathcal{D}^-$. Then $x : A$ occurs in $t : B$ in some sequent of $\mathcal{D}$ iff there is a directed path from $(B')^+$ to $(A')^-$ in $\mathcal{D}^-$, where $(B')^+$ and $(A')^-$ are the translations of B and A in polarized MLL.*

▸ **Example 24.**



Here $X = q \multimap (q \otimes r)$, $Y = (r \wp p) \multimap p$ and there is a directed path from X to the premise $r \wp p$ of Y against the functionality of implication. In the following sequent derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{y : q \vdash y : q \qquad z : r \vdash z : r}{y : q, z : r \vdash y \otimes z : q \otimes r} \qquad x : p \vdash x : p}{v : r\wp p, y : q \vdash \texttt{let}\ v\ \texttt{be}\ z^r -\ \texttt{in}\ y \otimes z : q \otimes r, \texttt{let}\ v\ \texttt{be}\ - x\ \texttt{in}\ x : p}}{v : r\wp p \vdash \lambda y.\texttt{let}\ v\ \texttt{be}\ z^r -\ \texttt{in}\ y \otimes z : q \multimap q \otimes r, \texttt{let}\ v\ \texttt{be}\ - x^p\ \texttt{in}\ x : p}\multimap R}{\vdash \lambda y.\texttt{let}\ v\ \texttt{be}\ z^r -\ \texttt{in}\ y \otimes z : q \multimap q \otimes r, \lambda v.\texttt{let}\ v\ \texttt{be}\ - x^p\ \texttt{in}\ x : (r\wp p) \multimap p}\multimap R$$

the last inference $\multimap R$ is incorrect because v still occurs free in the succedent.