Jordan structures in symmetric manifolds

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Objective

$$\begin{array}{ccc} & \textit{Jordan algebras} & \\ \swarrow ? & & \searrow ? \\ \textit{Lie algebras} & \longleftrightarrow & \textit{Geometry} \end{array}$$

Smooth vector fields on differentiable manifold M

= Lie algebra

$$[X, Y] = XY - YX$$

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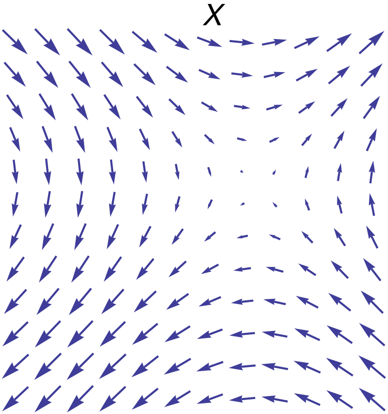

For a symmetric manifold M ($\dim M \leq \infty$)

Tangent space $T_x M =$ Jordan algebra (or Jordan triple).

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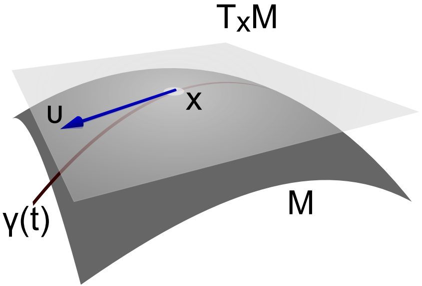

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$$\uparrow$$

Jordan structure

Origin of Jordan algebras

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Annals of Math. 1934

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S, T Hermitian $\Rightarrow$ $S \circ T := ST + TS$ Hermitian

(Hermitian operators, $\circ$) —> Jordan algebra

Jordan algebras

A (non-associative) algebra $\mathcal{A}$ (over F)

is a *Jordan algebra* if

$$ab = ba$$

$$a^2(ba) = (a^2b)a\,.$$

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is a *Jordan algebra* if

$$a\,b = b\,a$$

$$a^2(b\,a) = (a^2 b)a\,.$$

$F = \mathbb{R}, \mathbb{C}$ $(\dim \mathcal{A} \leq \infty)$

Examples

Any associative algebra $\mathcal{A}$ is a Jordan algebra in the product

$$a \circ b = \frac{1}{2}(ab + ba) \qquad (a, b \in \mathcal{A})$$

$(\mathcal{A}, \circ)$ called a *special* Jordan algebra.

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Exceptional Jordan algebra : $H_3(\mathcal{O})$

All 3×3 Hermitian matrices (a_{ij}) with $a_{ij} \in$ *Cayley algebra* $\mathcal{O}$ and multiplication

$$(a_{ij}) \centerdot (b_{ij}) = \frac{1}{2}((a_{ij})(b_{ij}) + (b_{ij})(a_{ij}))$$

Lie algebra

Lie algebras

A non-associative algebra $\mathfrak{g}$ $(\dim \mathfrak{g} \leq \infty)$ satisfying

$$[x, y] = -[y, x]$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

Lie algebras

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Poincaré - Birkhoff - Witt : Any Lie algebra can be obtained from an associative algebra $\mathfrak{g}$ by :

$$[x, y] := xy - yx.$$

Connections to Lie algebras

Tits (1962), Kantor (1964), Koecher (1967):

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Jordan triples (Jordan triple systems) $\leftrightarrow$ TKK Lie algebras

TKK Lie algebras = 3-graded Lie algebras with involution

Jordan triples $\leftrightarrow$ TKK Lie algebras

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Triple product $\{a, b, c\}$	$\leftrightarrow$	Lie triple product $[[\tilde{a}, \tilde{b}], \tilde{c}]$
Triple identity	$\leftrightarrow$	Jacobi identity.

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Triple product

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$$\{a, b, c\} = (ab)c + a(bc) - b(ac)$$

We have

$$\{a, b, c\} = \{c, b, a\}$$

$\{\cdot, \cdot, \cdot\}$ is trilinear.

Triple identity

$$\{a, b, \{x, y, z\}\}$$

$$= \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

Jordan triples

A real vector space V is a *Jordan triple* if there is a trilinear map

$$\{\cdot,\cdot,\cdot\} : V^3 \longrightarrow V$$

satisfying

(i) $\{a, b, c\} = \{c, b, a\}$
(ii) Triple identity.

Jordan triples

A complex vector space V is a *Jordan triple* if there is a map

$$\{\cdot,\cdot,\cdot\} : V^3 \longrightarrow V,$$

linear in the 1st and 3rd variables, conjugate linear in the 2nd variable,
satisfying

(i) $\{a, b, c\} = \{c, b, a\}$
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(ii) complex Jordan algebras with involution * and triple product

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(iii) Tangent spaces of symmetric manifolds.

Tits-Kantor-Koecher construction

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Box operator (*left multiplication*)

$$a \square b : x \in V \mapsto \{a, b, x\} \in V$$

$$V_0 := V \square V := \left\{ \sum_k a_k \square b_k : a_k, b_k \in V, k = 1, \ldots, n \right\}$$

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Tits-Kantor-Koecher (TKK) Lie algebra

$$\mathfrak{L}(V) := V_{-1} \oplus V_0 \oplus V_1$$

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$$\mathfrak{L}(V) = \text{exceptional Lie algebra of type } E_7$$

3-graded Lie algebras

A Lie algebra $\mathfrak{g}$ is called 3-graded if

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satisfying

$$[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}.$$

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A TKK Lie algebra $\mathfrak{g}$ is a 3-graded Lie algebra with involution

$$\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying

$$\theta(\mathfrak{g}_n) = \mathfrak{g}_{-n}.$$

Jordan triple $\longleftarrow\!\!-\!-\!-\!\!\longrightarrow$ TKK Lie algebras

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- - → Jordan triple V, with $\mathfrak{g} = \mathfrak{L}(V)$:

Define $V := \mathfrak{g}_{-1}$ and

$$\{x, y, z\} := [\,[\,x, \theta(y)\,], z\,].$$

Geometry

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Compact (connected) Lie groups : $\quad s_e(x) = x^{-1}$

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Symmetry at $\mathbf{1} \in \Omega$: $s_{\mathbf{1}}(\omega) = \omega^{-1} \quad (\omega \in \Omega)$

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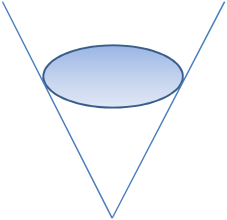

(n = 3)

future light cone

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biholomorphic : s_p is a holomorphic bijection,
s_p^{-1} is holomorphic.

Example

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Riemann Mapping Theorem

all 1-dim bounded symmetric domains $\approx \mathbb{D}$.

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$(\mathbb{C}^d, \| \cdot \|)$ is a Jordan triple !

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$D \approx \{z \in \mathbb{C}^d : \|z\| < 1\} \subset (\mathbb{C}^d, \|\cdot\|) \quad (d \geq 3)$

$$\|z\|^2 = \langle z, z\rangle + \sqrt{\langle z, z\rangle^2 - \langle z, z^*\rangle}$$

$$z^* = (\overline{z}_1, \ldots, \overline{z}_d)$$

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holomorphic maps

$V, W =$ Banach spaces
(Open set $D \subset V$)

$f : D \longrightarrow W$ is holomorphic if it has a Fréchet derivative $f'(a)$ at each $a \in D$, where

$$f'(a) : V \longrightarrow W$$

is a continuous linear map such that

$$\lim_{\|v\| \to 0} \frac{\|f(a+v) - f(a) - f'(a)(v)\|}{\|v\|} = 0.$$

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1. V is a complex Banach space;
2. V is a Jordan triple;
3. $a \square a : x \in V \mapsto \{a, a, x\} \in V$ is hermitian;
4. $\sigma(a \square a) \subset [0, \infty)$;
5. $\|\{a, b, c\}\| \leq \|a\|\|b\|\|c\| \qquad (a, b, c \in V)$.

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For $\mathbb{D} \subset \mathbb{C} = V$,

$$\{x, y, z\} = x\overline{y}z \qquad (x, y, z \in \mathbb{C})$$

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type IV (classical) : Lie ball

$$D \approx \{z \in \mathbb{C}^d : \|z\| < 1\} \subset (\mathbb{C}^d, \|\cdot\|) \quad (d \geq 3)$$

$$\{x, y, z\} = \langle x, y \rangle z + \langle z, y \rangle x - \langle x, z^* \rangle y^*.$$

Open unit balls of JB*-triples

are

bounded symmetric domains

More examples of JB*-triples

- Hilbert spaces H : $\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$

$$(\mathbb{C} \ : \ \{x, y, z\} = x\bar{y}z)$$

- C*-algebras : $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$

- $L(H, K)$: $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$

$L(H, K)$

= all bounded linear operators between Hilbert spaces H and K

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$D \hookrightarrow$ JB*-triple V

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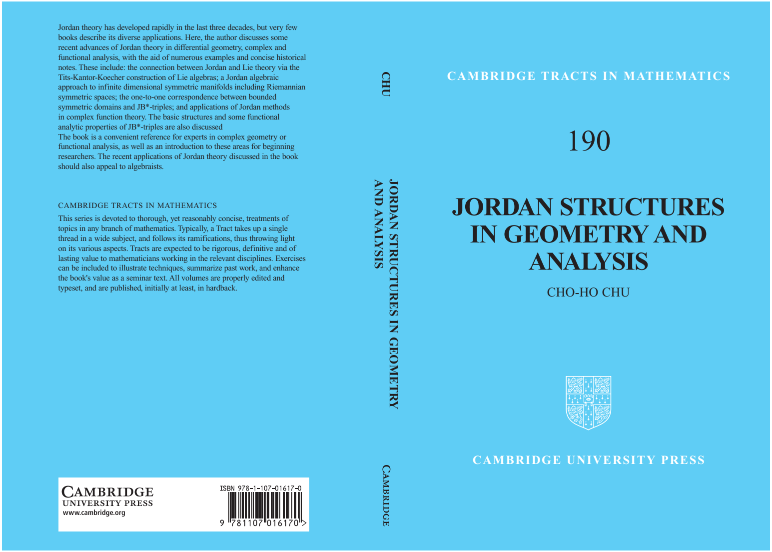
$V := \mathfrak{g}_{-1}$ with Jordan triple product

$$\{x, y, z\} := [[x, \theta(y)], z]$$


Jordan theory has developed rapidly in the last three decades, but very few books describe its diverse applications. Here, the author discusses some recent advances of Jordan theory in differential geometry, complex and functional analysis, with the aid of numerous examples and concise historical notes. These include: the connection between Jordan and Lie theory via the Tits-Kantor-Koecher construction of Lie algebras; a Jordan algebraic approach to infinite dimensional symmetric manifolds including Riemannian symmetric spaces; the one-to-one correspondence between bounded symmetric domains and JB*-triples; and applications of Jordan methods in complex function theory. The basic structures and some functional analytic properties of JB*-triples are also discussed
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Geometric Analysis

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Bounded symmetric domain D

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Special features : dim $\leq \infty$; ambient Jordan structures

Useful tools

V = JB*-triple

$$a \square b : z \in V \mapsto \{a, b, z\} \in V$$

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Möbius transformation $g_a : D \longrightarrow D \quad (a \in D)$

$$g_a(z) = a + B(a, a)^{1/2}(I + z \square a)^{-1}(z)$$

$$Aut\ D = \{\ell \circ g_a : a \in D\}$$

$$\ell : V \longrightarrow V \qquad \text{(linear isometry)}$$

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Iteration of holomorphic maps

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \qquad \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$

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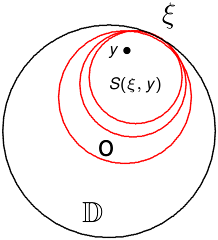

2. (Denjoy-Wolff Theorem)

$f^n \longrightarrow h(\cdot) = \xi$ as $n \to \infty$.
(uniformly on compact sets)

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Does Denjoy-Wolff Theorem hold for f?

Not always!

How to prove Wolff Theorem for D ?

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

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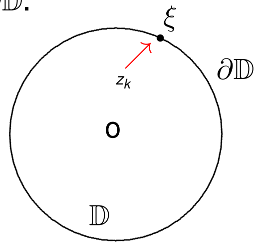

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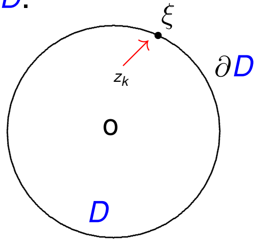

Construction of horodisc $S(\xi, y) \subset \mathbb{D}$

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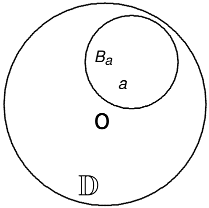
$\kappa(z, a)$: Poincaré distance



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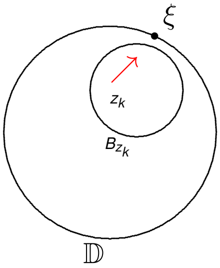

Horoball $S(\xi, y)$ in bounded symmetric domain

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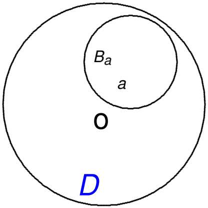

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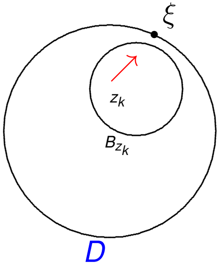
Not working well for ∞-dim !



Alternative construction of $S(\xi, y)$ for $\mathbb{D}$

$S(\xi, y)$

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$$S(\xi, y)$$

$$= \left\{ z \in \mathbb{D} : \lim_{k\to\infty} \frac{1-|z_k|^2}{1-|g_{-z_k}(z)|^2} < \lambda \right\} \quad (\textit{some } \lambda > 0)$$

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$(some\ \lambda_y > 0)$.

$$f(S(\xi, y)) \subset S(\xi, y) \text{ because } \kappa(f(a), f(b)) \leq \kappa(a, b).$$

Special cases

1. D = Hilbert ball Goebel (Nonlinear Anal. 1982)

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4. D = Lie ball Chu (Adv. Math. 2014)

Denjoy-Wolff theorem??

$$f^n \longrightarrow \xi \quad ?$$

Denjoy-Wolff type result (dim $< \infty$)

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D = *bounded symmetric domain*

Boundary of $D = \cup_\alpha K_\alpha$ *(disjoint union)*

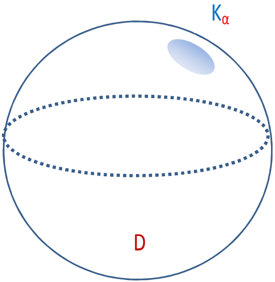
K_α = boundary component (convex domain)

$\exists$ K_α such that

$\forall$ $h = \lim_k f^{n_k}$ with $h(D)$ weakly closed

$\Rightarrow$ $h(D) \subset K_\alpha$.

e.g. *If* f = *Möbius transformation*, *then* $h(D) = K_\alpha$.



Thank you !