Regularising Inverse Imaging Problems using Generative Machine Learning Models

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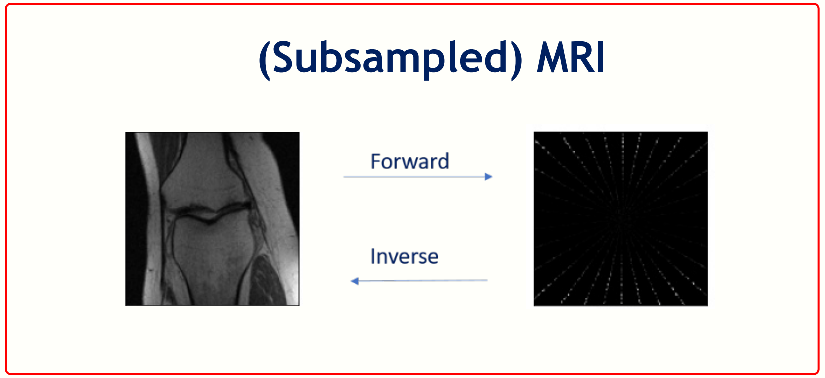
Overview

- Inverse problem

$$y \approx Ax$$

where $x \in \mathscr{X}$, $y \in \mathscr{Y}$.



- Variational approach: solve

$$\arg\min_{x \in \mathcal{X}} \|y - Ax\|_2^2 + \lambda \mathcal{R}_G(x)$$

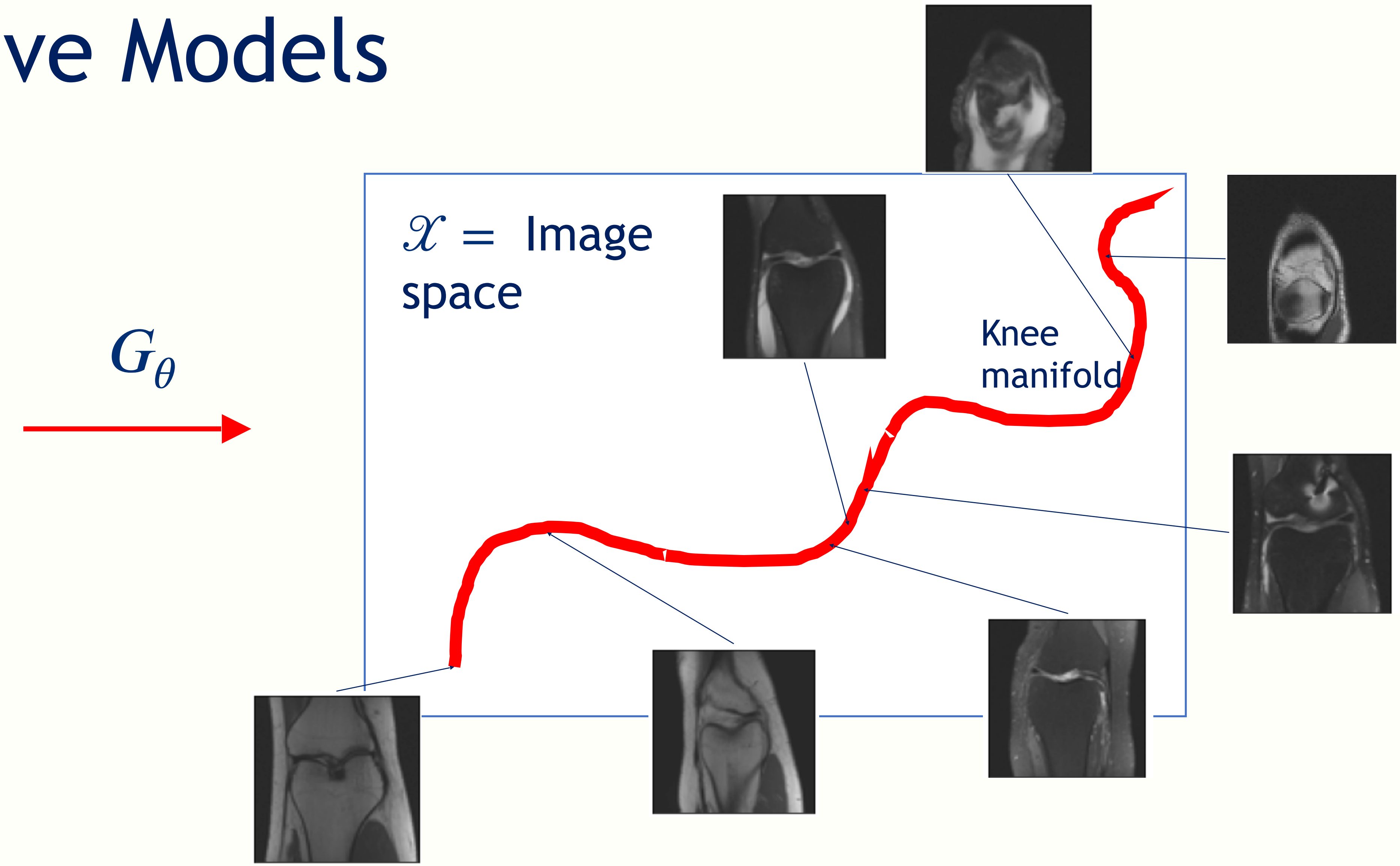
where $G: \mathscr{Z} \to \mathscr{X}$, a generative model.

- Penalise images far from the range of the generative model.

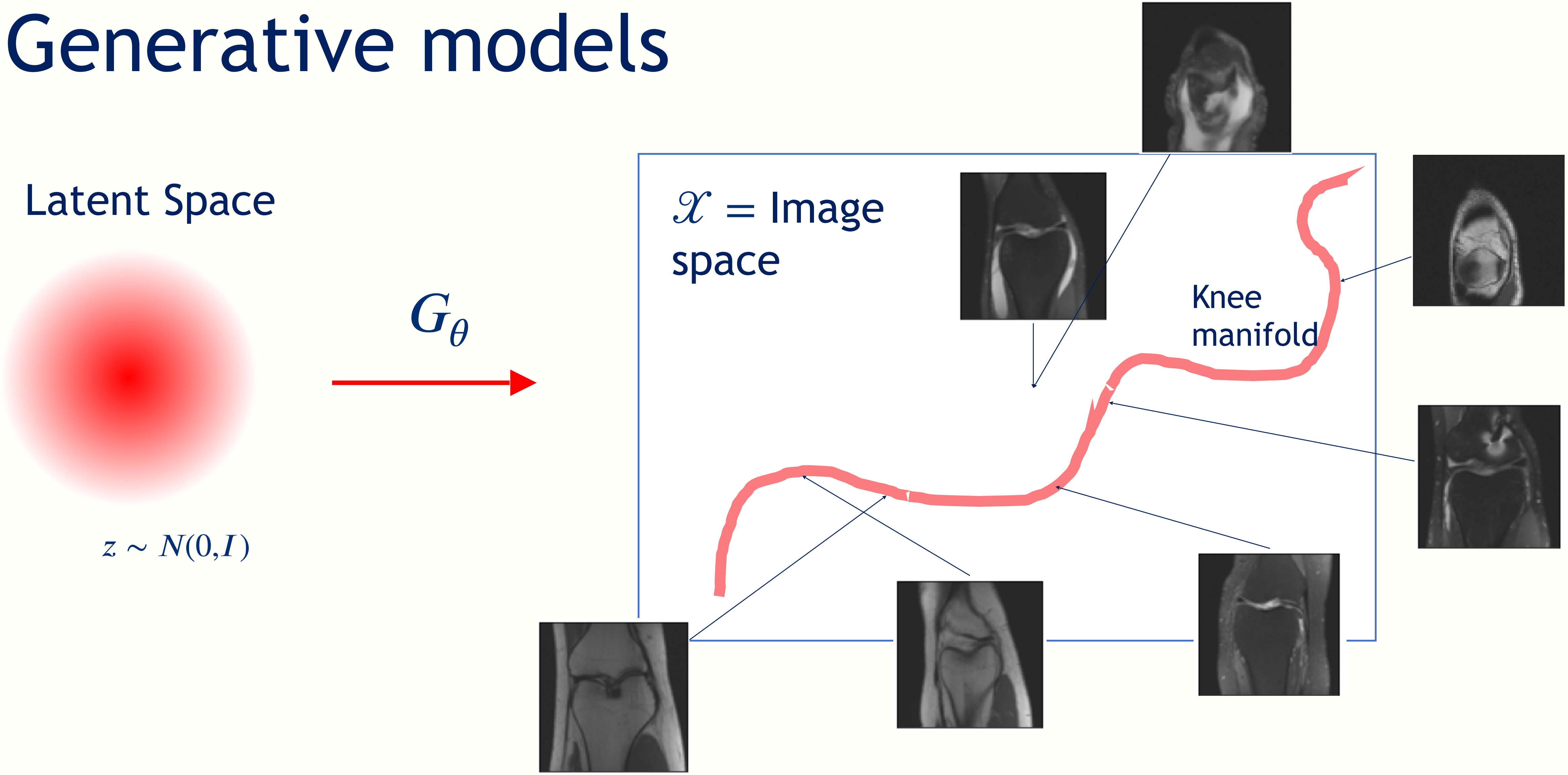
Generative Models

Latent Space

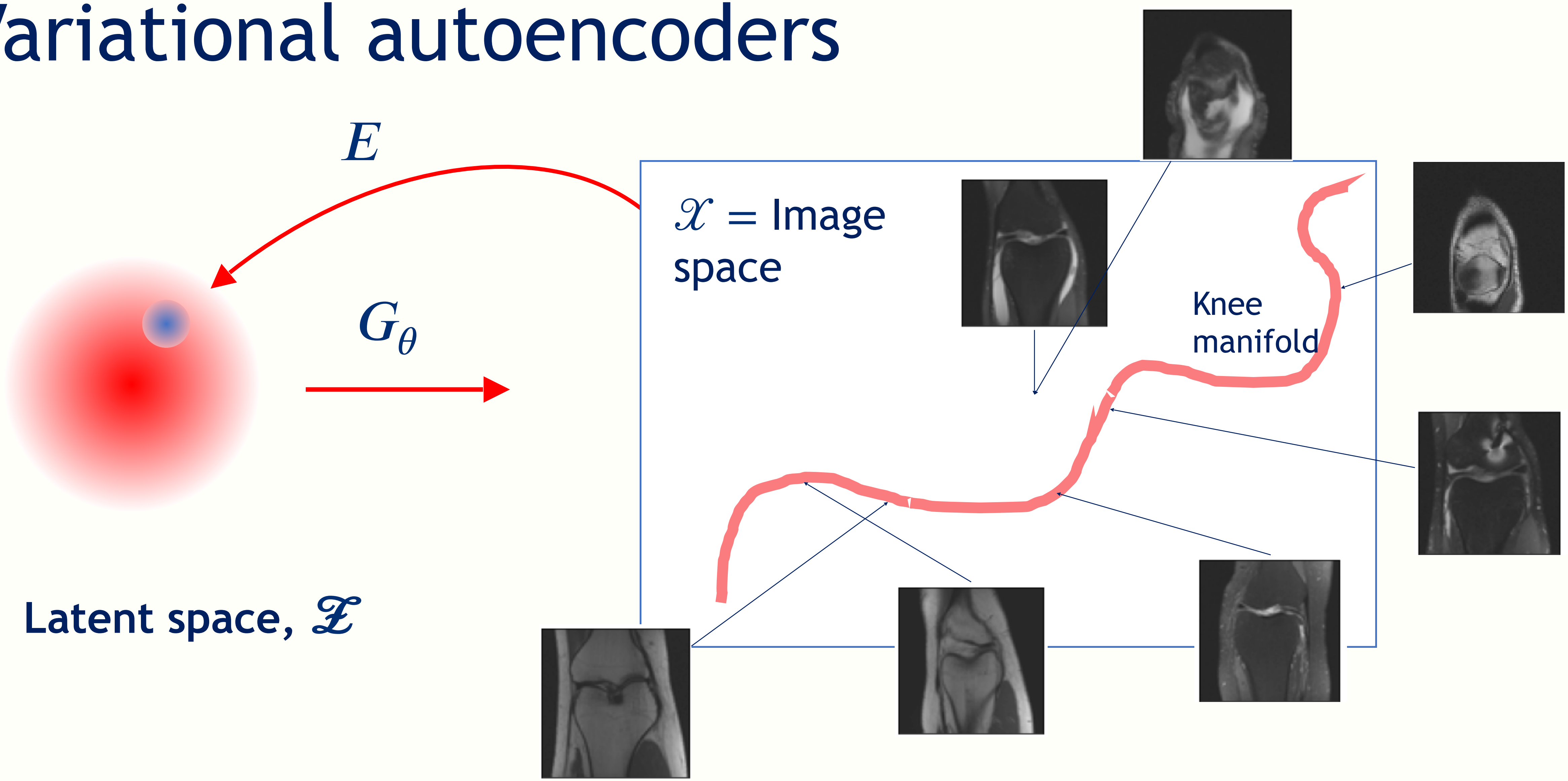
$$z \in \mathcal{Z}$$



Generative models



Variational autoencoders



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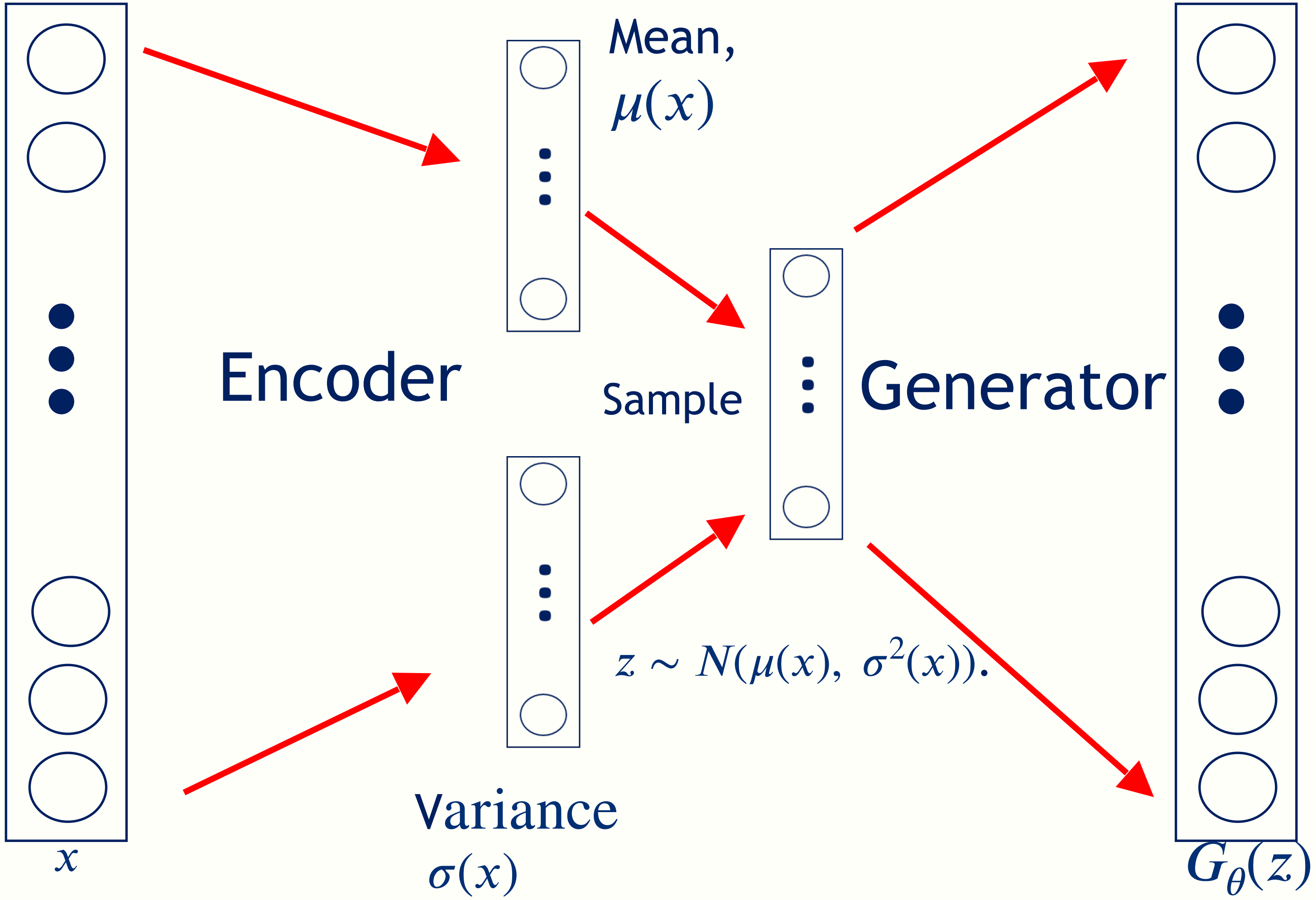


Objective function:

$$\min_{\mu,\,\sigma,\,\theta} \mathbb{E}_x[\mathbb{E}_{z|x}\left\|x - G_\theta(z)\right\|^2 + KL(p_z \,|\, p_{N(0,1)})]$$

Where $z\,|\,x \sim N(\mu(x),\ \sigma^2(x))$.

Tomography example: MNIST

$A: X \to Y$

Original Problem: Find x s.t.

$y = Ax + \epsilon$

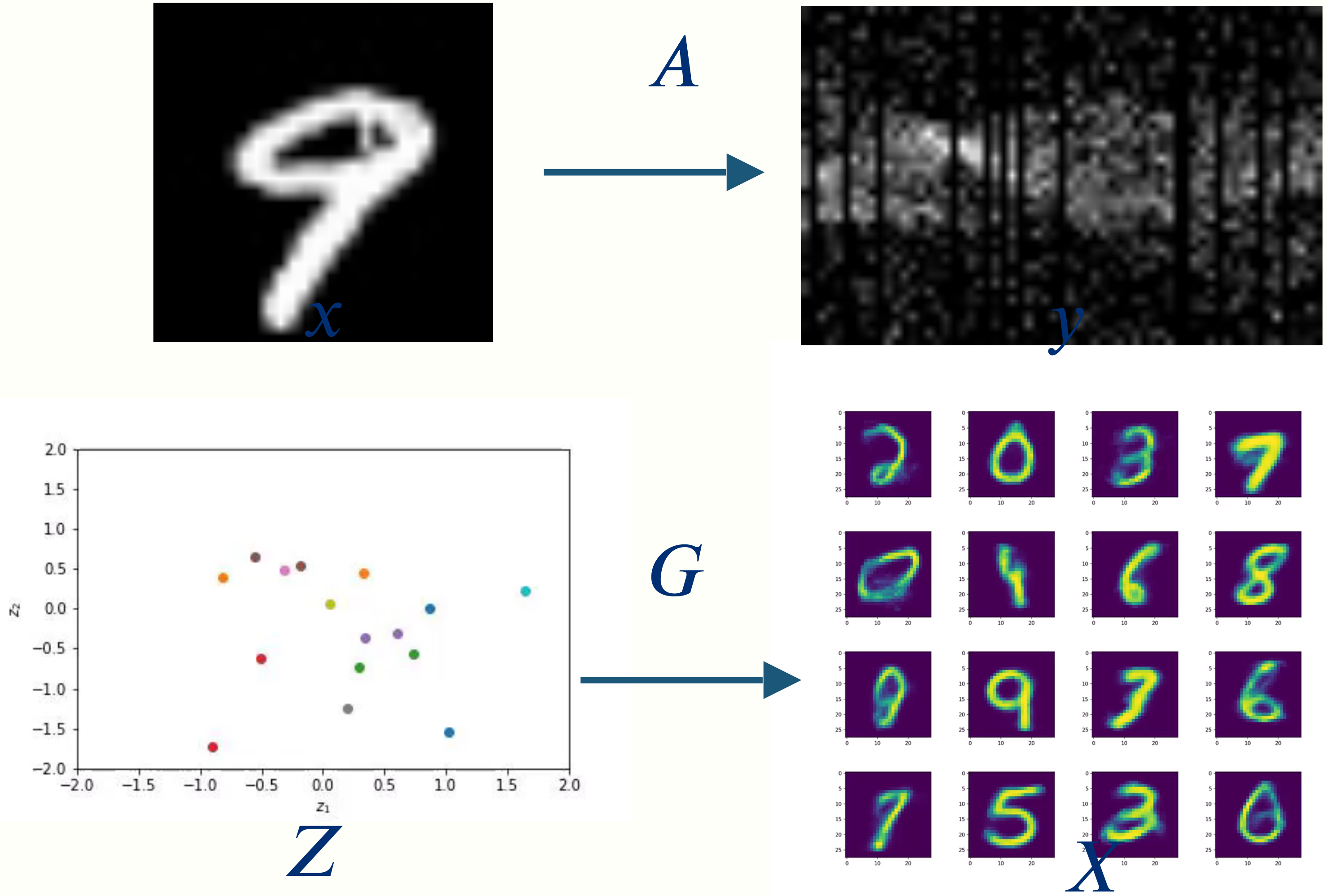
Generative model

$G: Z \to X$

New Problem: Find z s.t.

$y = A\big(G(z)\big) + \epsilon$

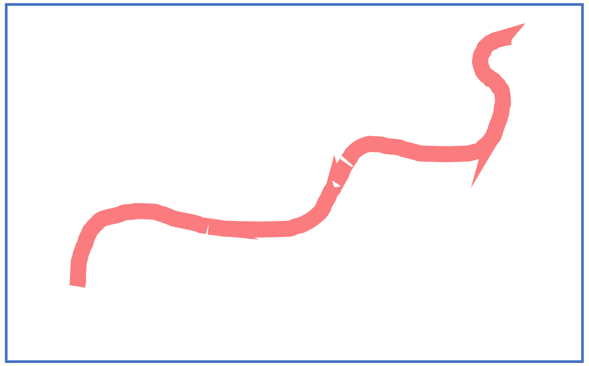
$x = G(z)$



Incorporating the generator

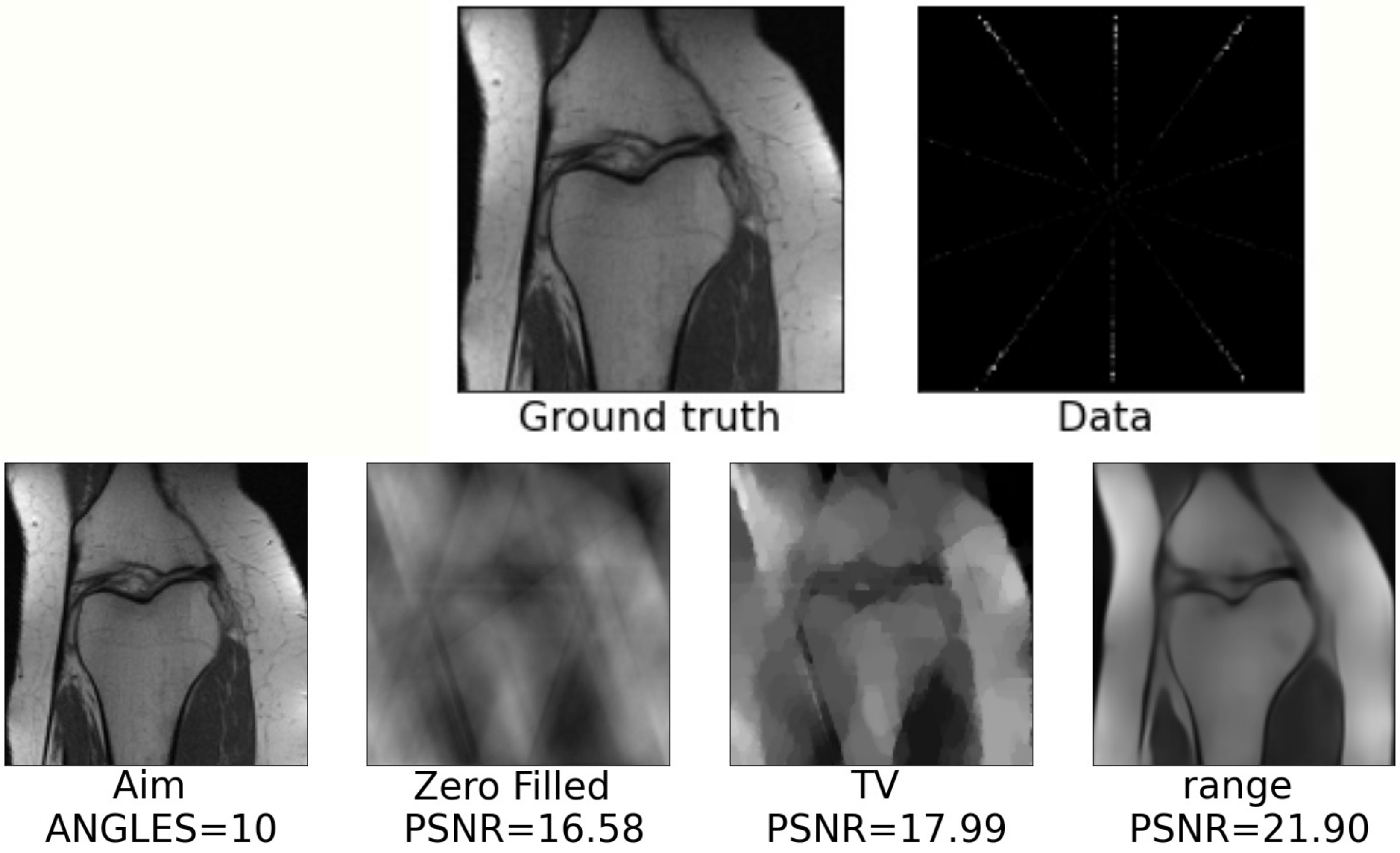
Image in the range of the generator

$$\mathcal{R}_G(x) = \min_{z \in \mathcal{Z}} \iota_{\{0\}}(G(z) - x) + \|z\|_2^2$$



Bora et al. “Compressed sensing using generative models". ICML 2017.

NYU FastMRI dataset



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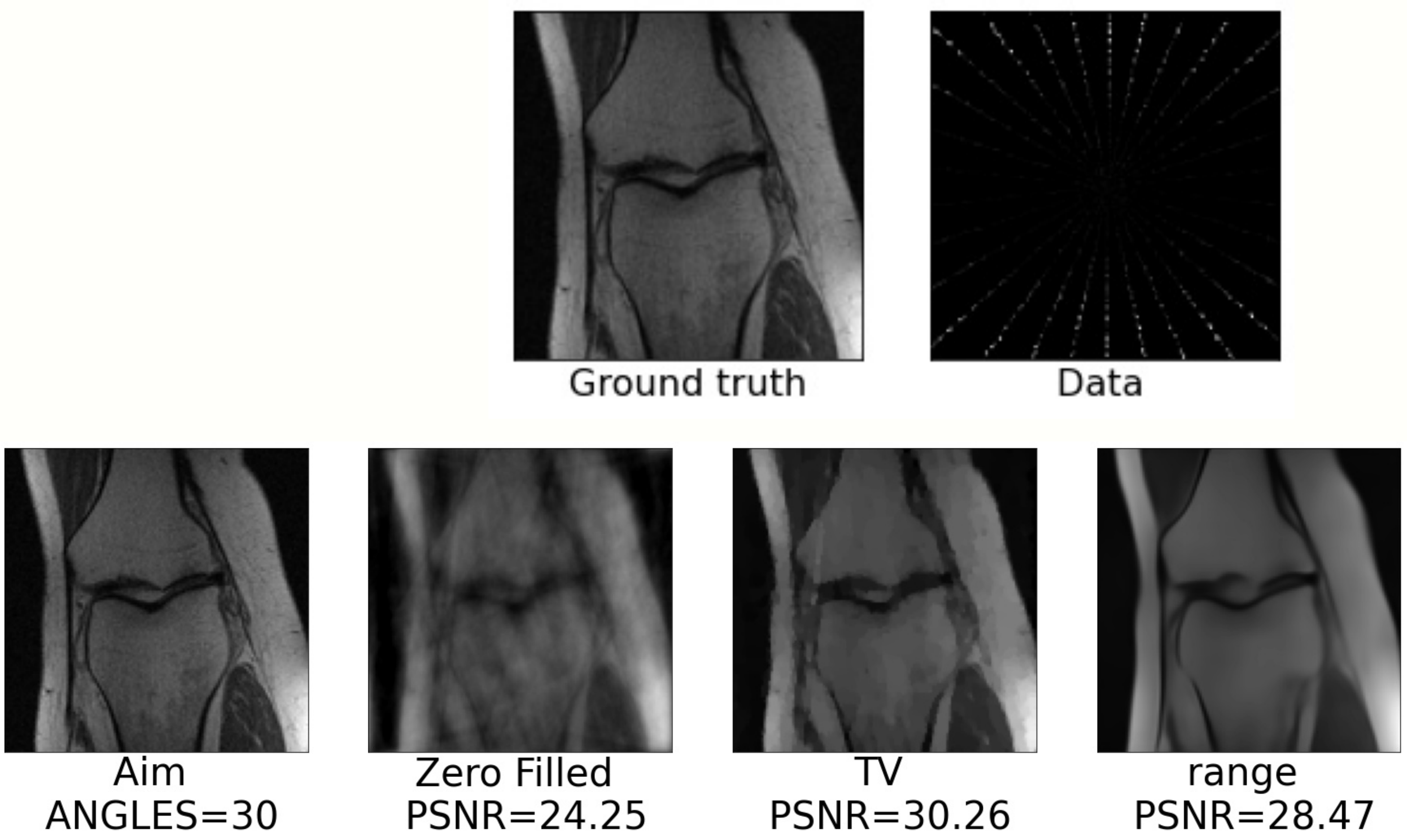

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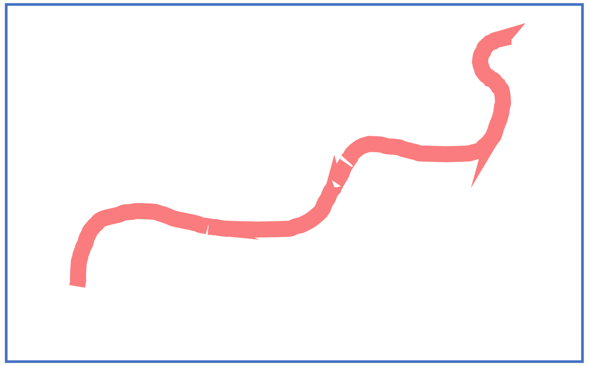
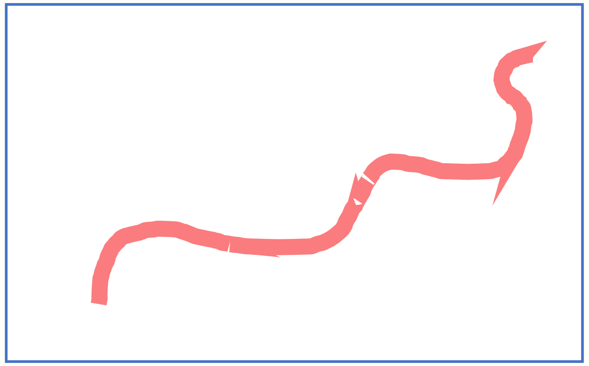


Image close to the range of the generator

$$\mathcal{R}_G(x) = \min_{z \in \mathcal{Z}} \|G(z) - x\|_2^2 + \mu\|z\|_2^2$$



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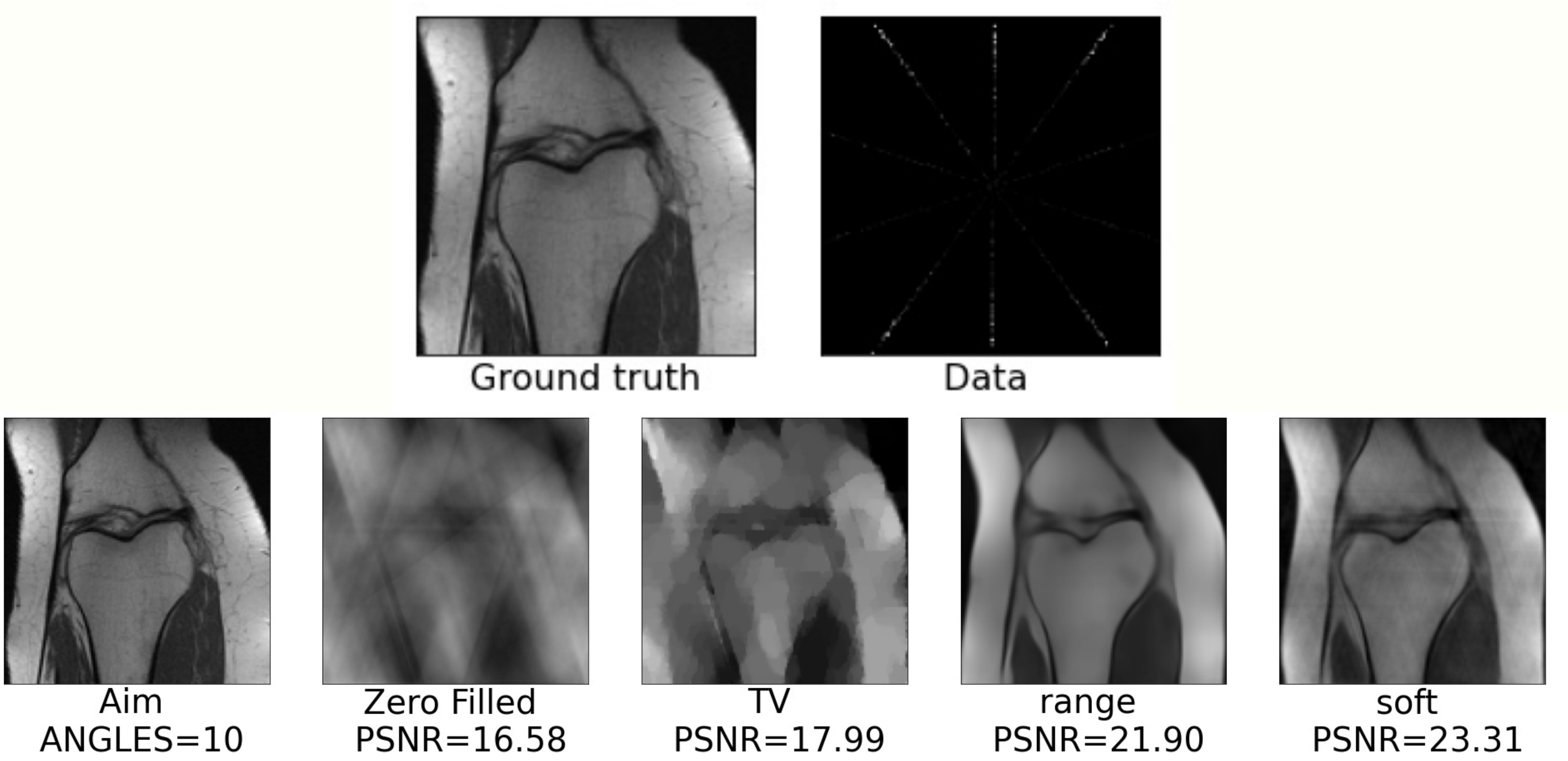

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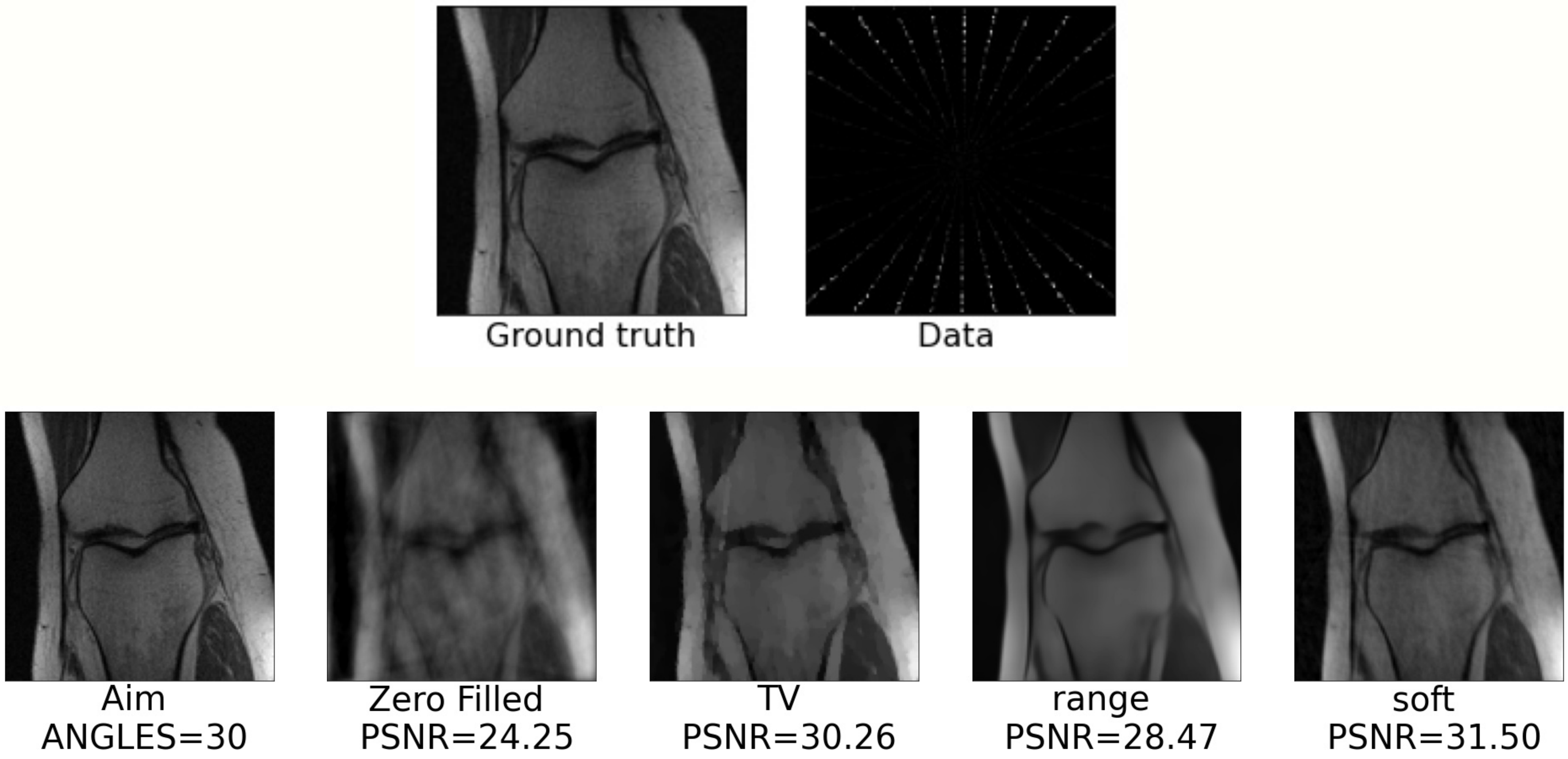
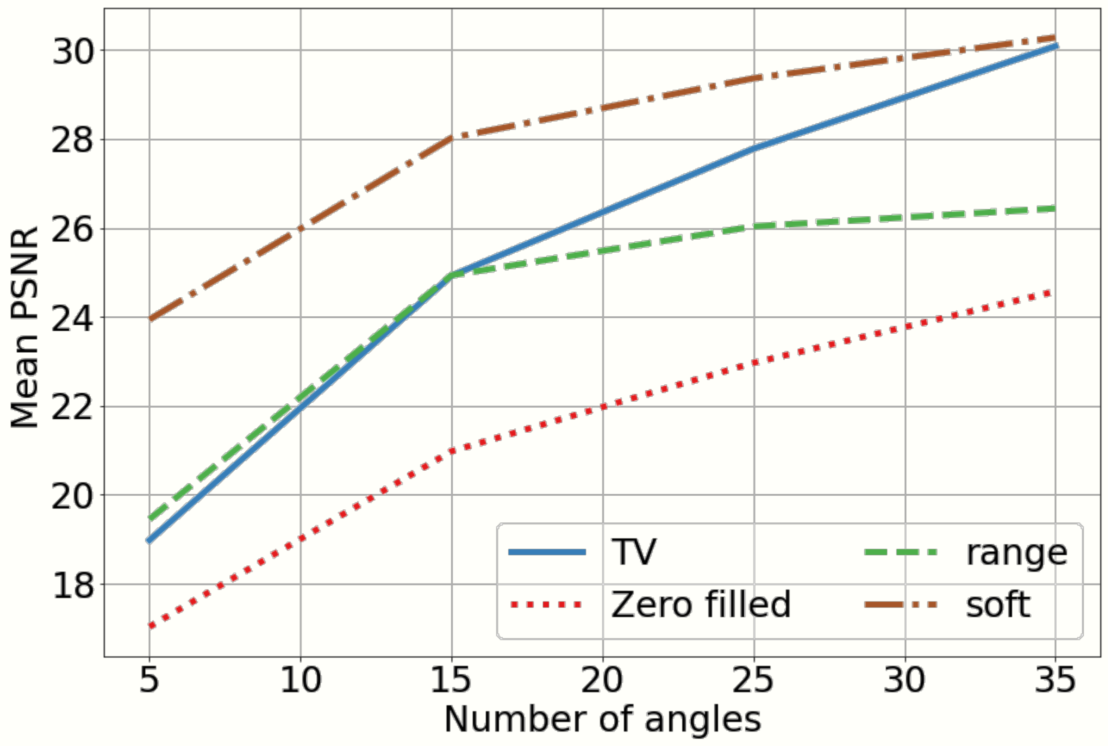
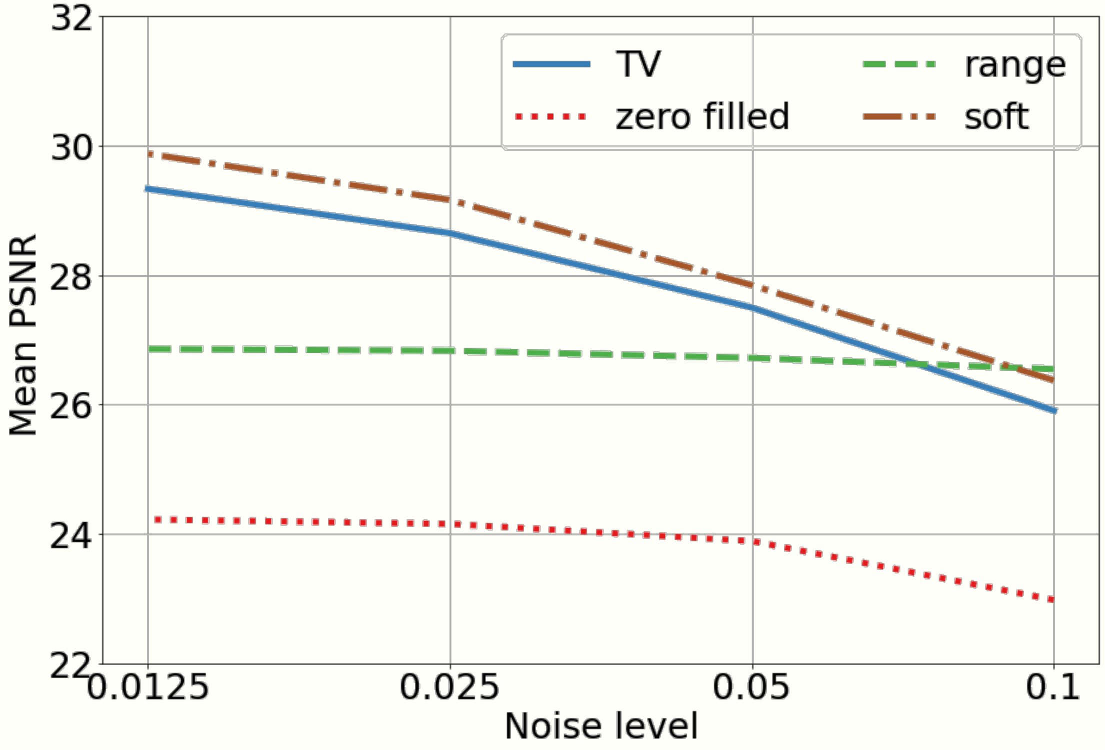


Method comparison





The Benefits of Generative Regularisers

- Don’t require supervised (paired) training data

- Flexible to changes in the forward problem

- Some degree of mathematical insight and control.

Generative Model Desired Properties

	Variational Autoencoder	Generative Adversarial Network
Generate all 'feasible' images	✓	Susceptible to mode collapse
Generate no 'unfeasible' images	Can produce blurry images	✓
Smoothness with respect to	Depends on the network Encoder distribution	Depends on the network
Known latent space distribution	Only the prior is known	Only the prior is known

Takeaway points

$A: X \to Y$

Original Problem: Find x s.t. $y \approx Ax$

Generative model

$G: Z \to X$

New Problem:

$$\arg\min_{x \in \mathcal{X}} \|y - Ax\|_2^2 + \lambda \mathcal{R}_G(x)$$

- Generative models can be used as priors for inverse problems
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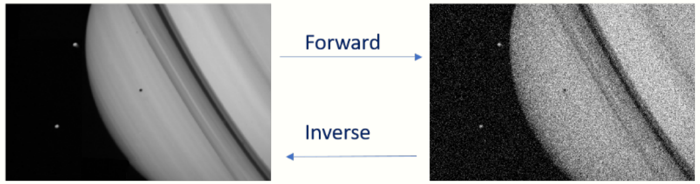
https://arxiv.org/abs/2107.11191

Inverse Problems are Everywhere

Matrix inversion

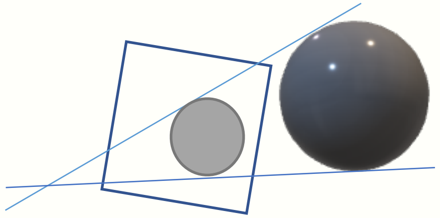
$$A = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1001}{1000} \end{pmatrix}$$

Denoising

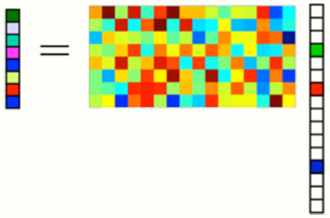


Original image, $x \in X$ Observed image, $y \in X$

3d representation



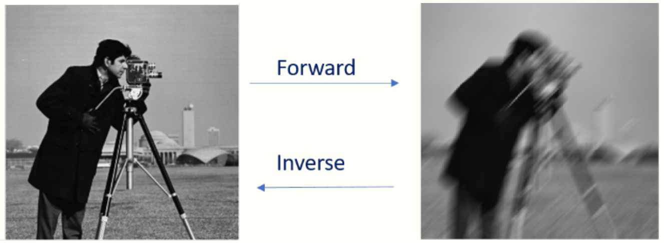
Compressed sensing



Differentiation/ integration

$$A(f) = \int_{t_0}^{t_1} f(t)\, dt$$

Deblurring



Original image, $x \in X$ Observed image, $y \in Y$

Image/video segmentation

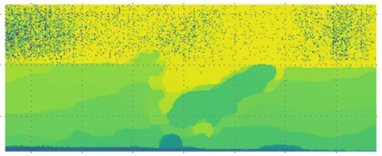
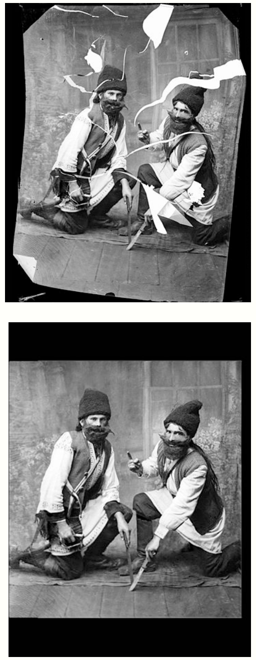
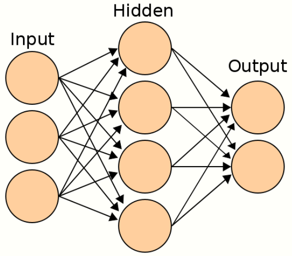


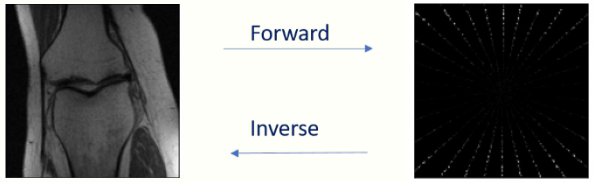
Image restoration



Neural network training

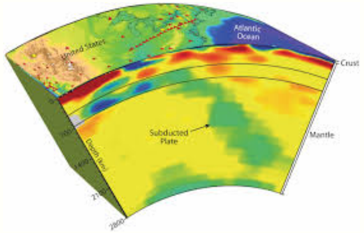


(Subsampled) MRI

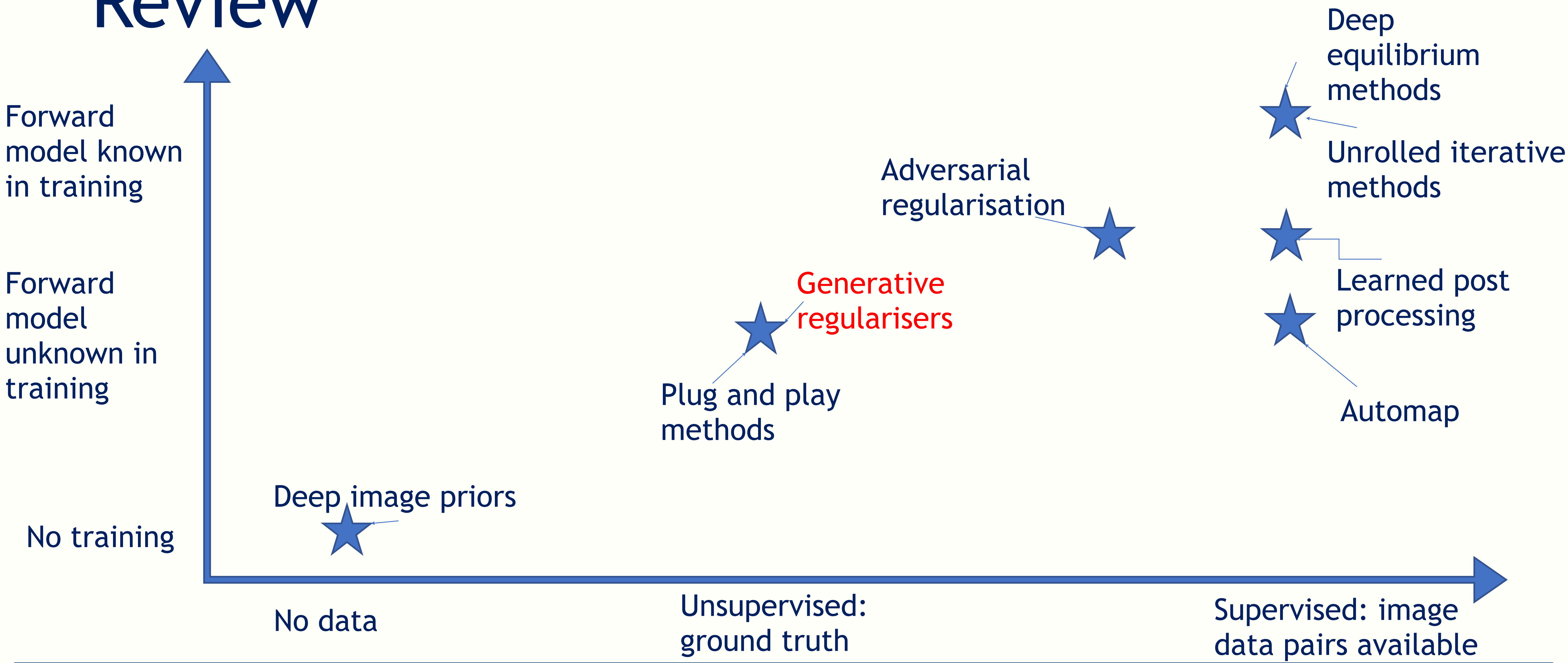


Original image, $x \in X$ Observed image, $y \in Y$

Seismic tomography



Deep Learning and Inverse Problems: Review



What properties do we need for the generator?

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Generator properties

- Generator produces all ‘feasible’ images
- Generator produces no ‘unfeasible’ images
- The generated probability distribution matches the training data distribution

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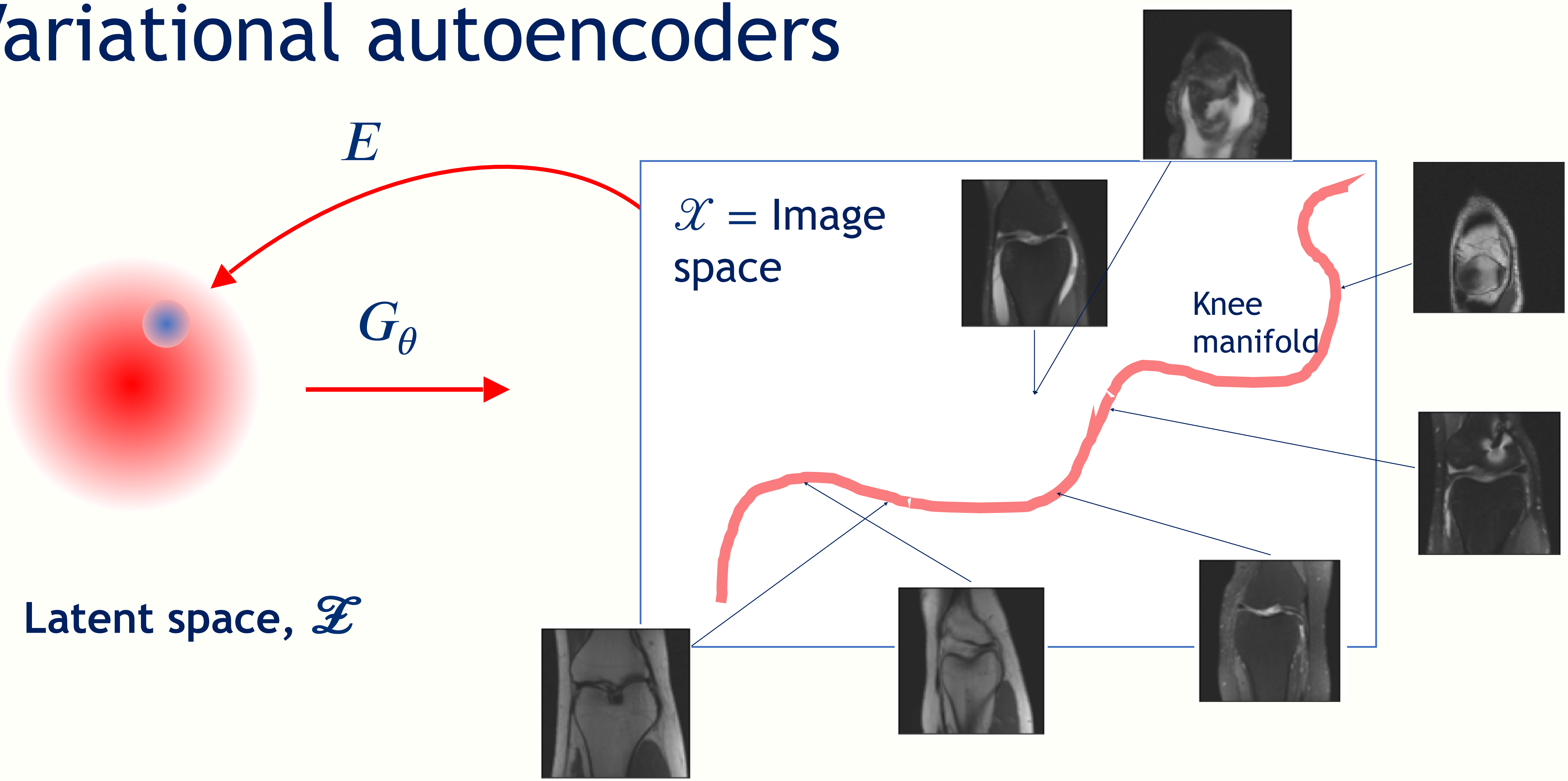
Latent space properties

- Smoothness of the generator with respect to z
- The area of the latent space that maps to feasible images is known

Variational autoencoders



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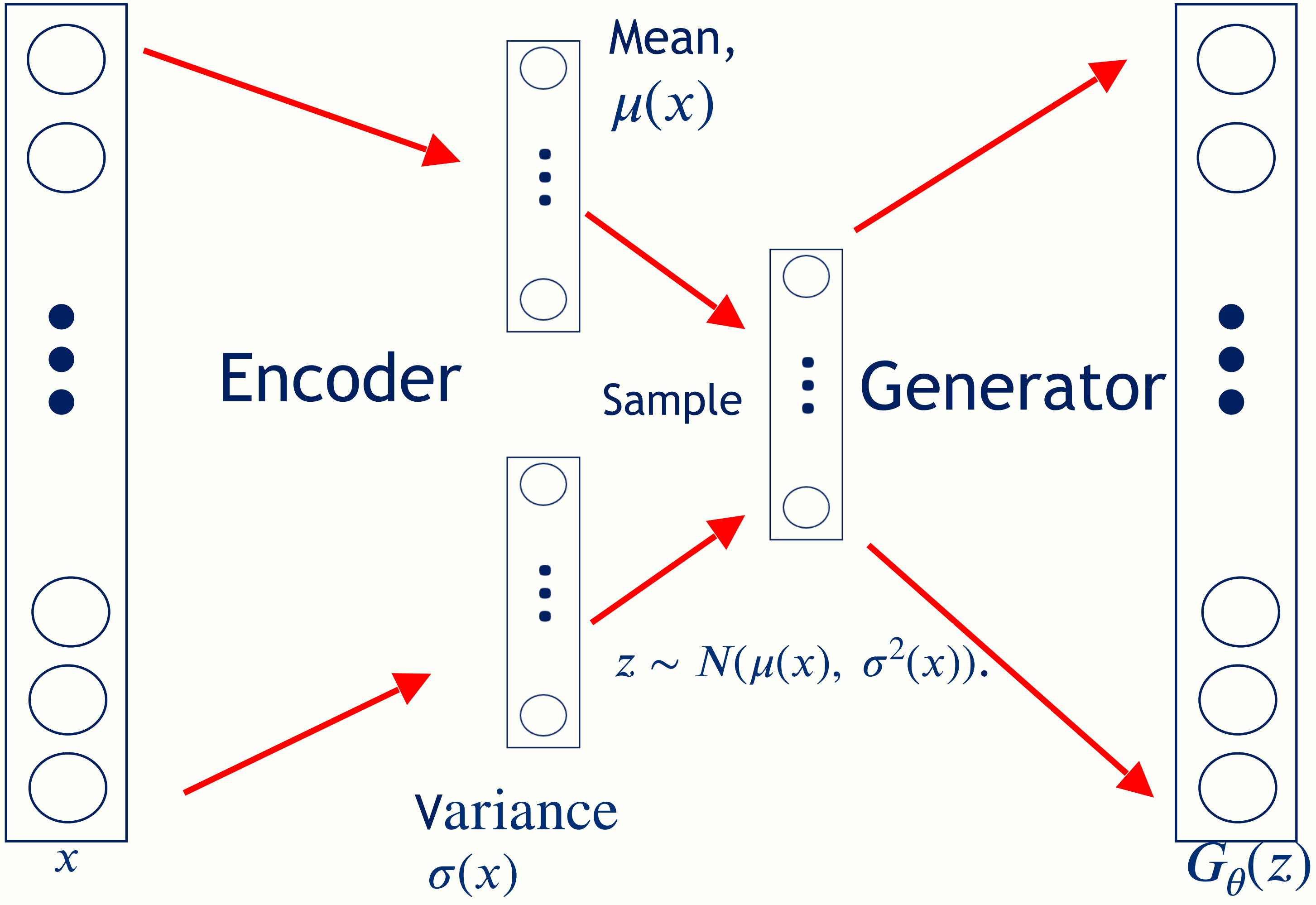

Objective function:

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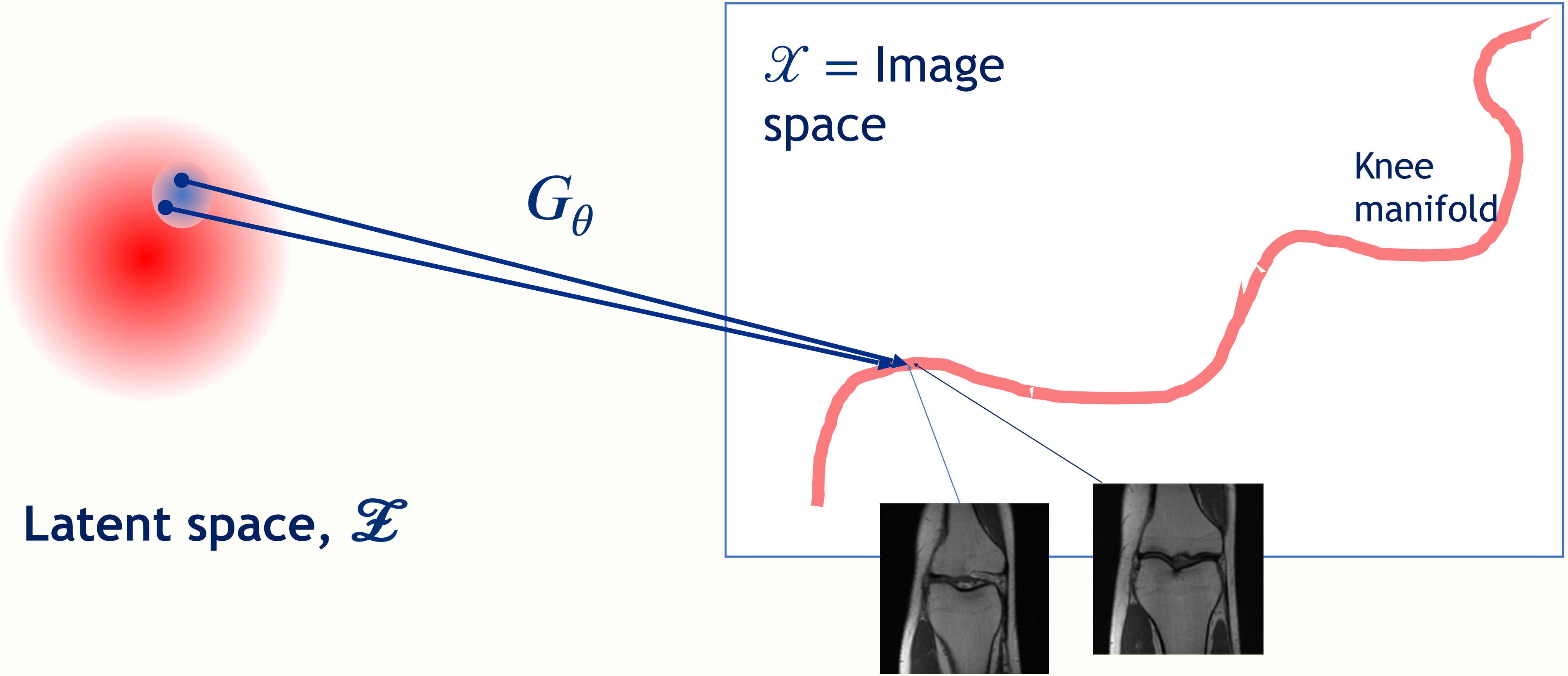
Where $z \,|\, x \sim N(\mu(x),\ \sigma^2(x)).$

Variational autoencoders



Generative Adversarial Networks (GANs)

$z \sim p_z$ $z \in Z$

Generator

Fake data $G(z) \in X$

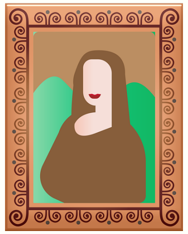
Discriminator

$D(x) \in (0,1)$

Probability the image is real

$x \sim p_{data}$ $x \in X$

Real data

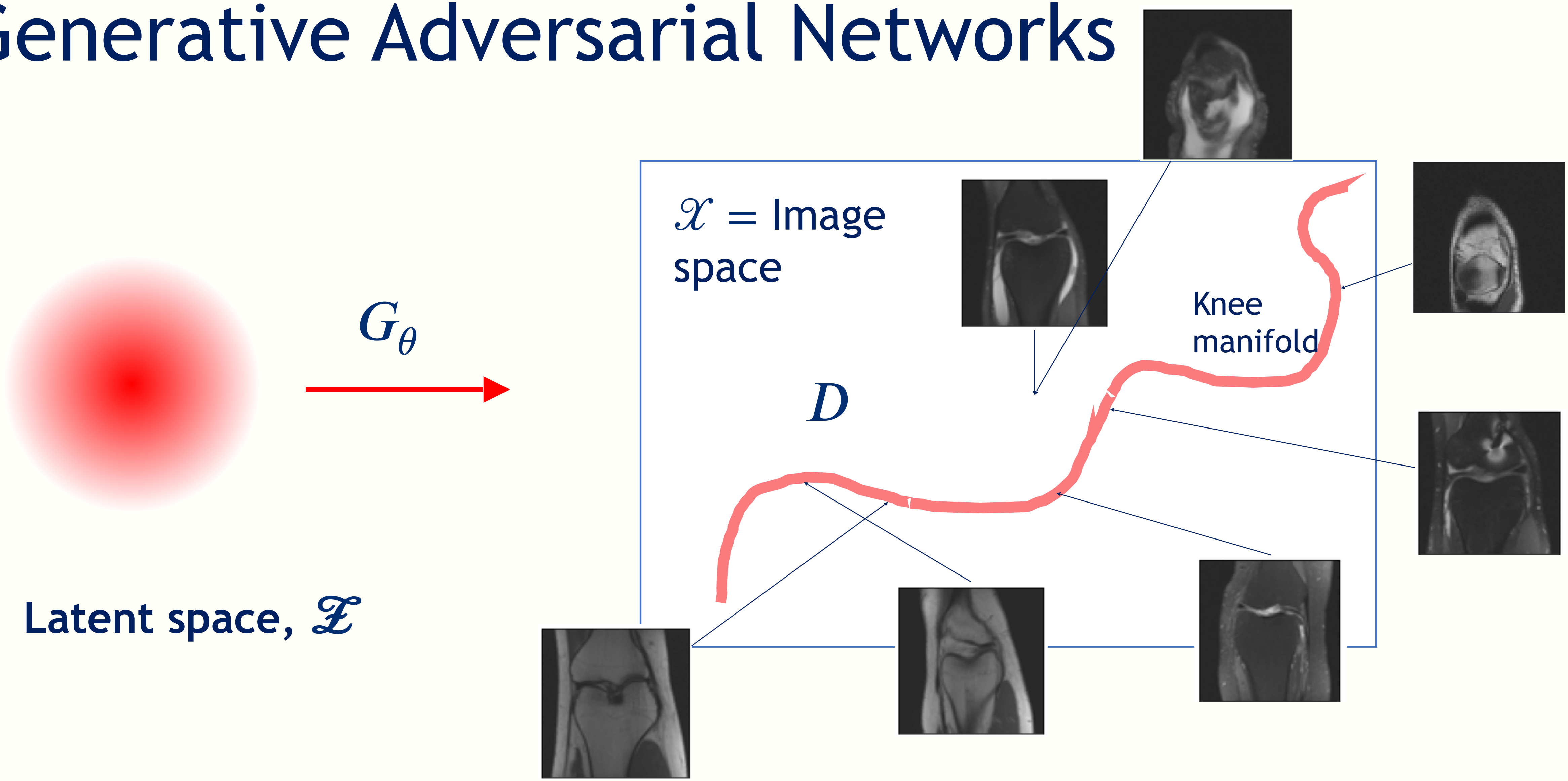


Objective function:

$$\min_G \max_D V(G, D)$$

$$V(G, D) = \mathbb{E}_{x \sim p_{data}} \log D(x) + \mathbb{E}_{z \sim p_z} \log\left(1 - D\big(G(z)\big)\right)$$

Generative Adversarial Networks



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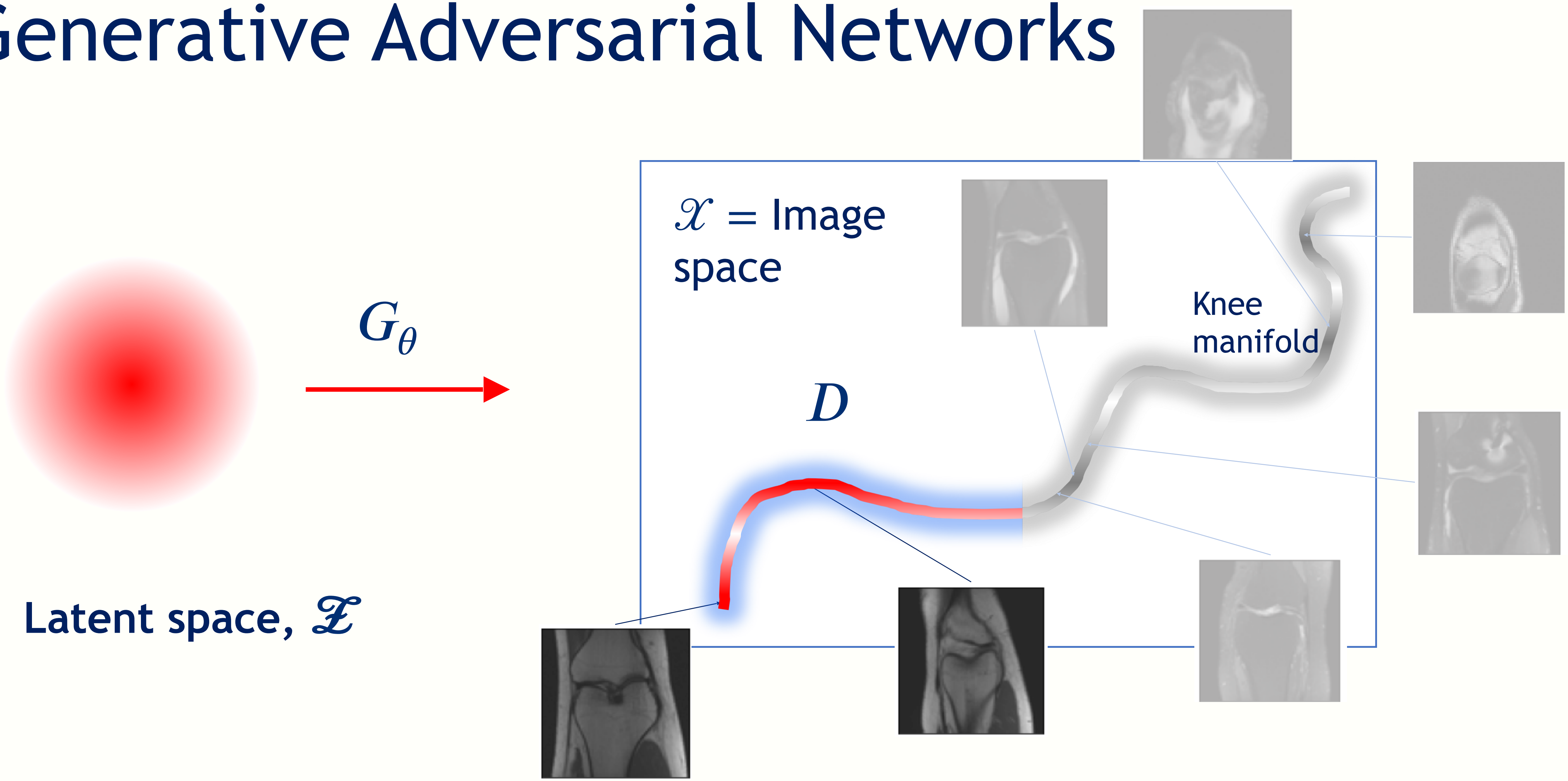
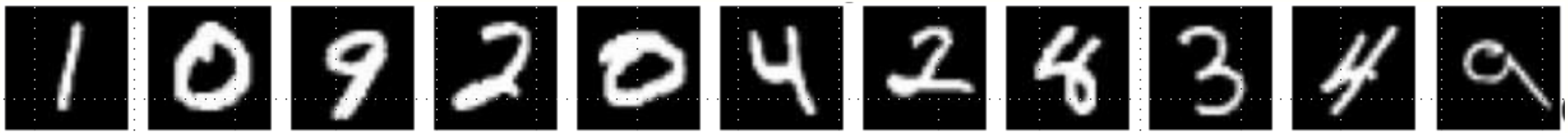
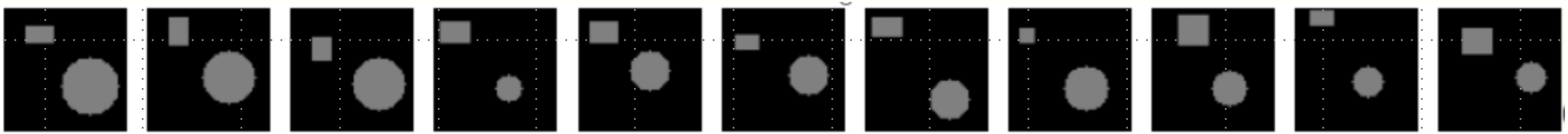


Generative model comparisons

- Datasets:
  - MNIST



  - Squares and circles



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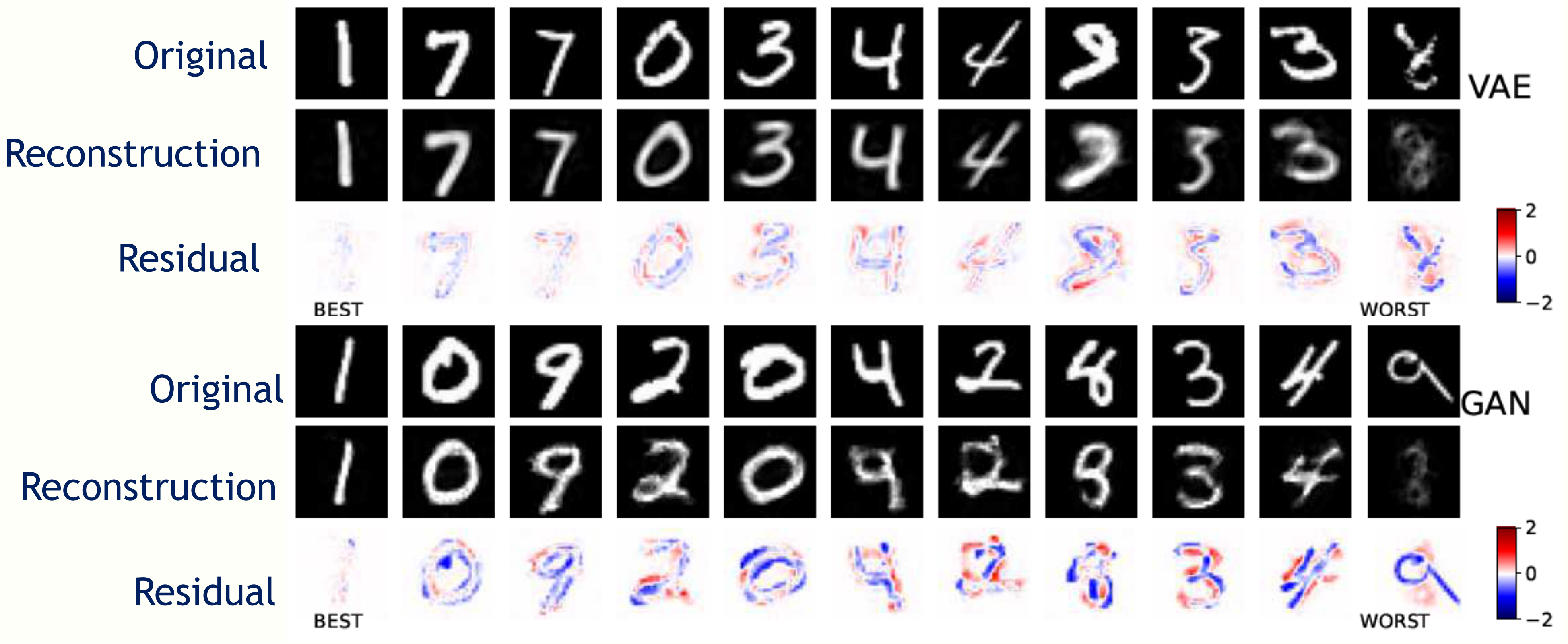
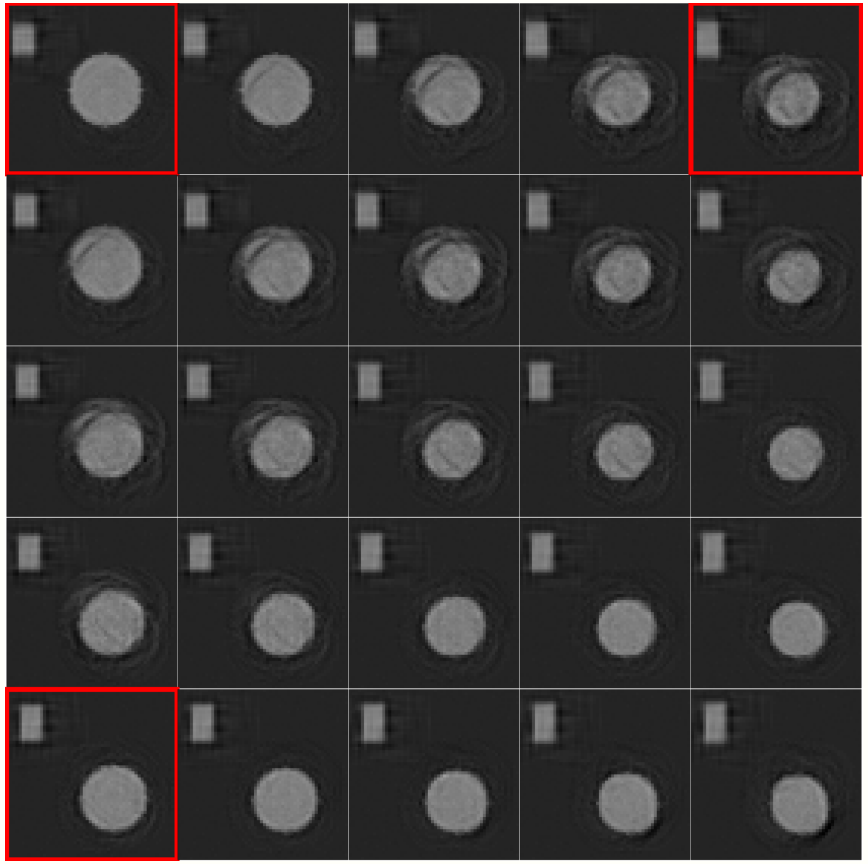
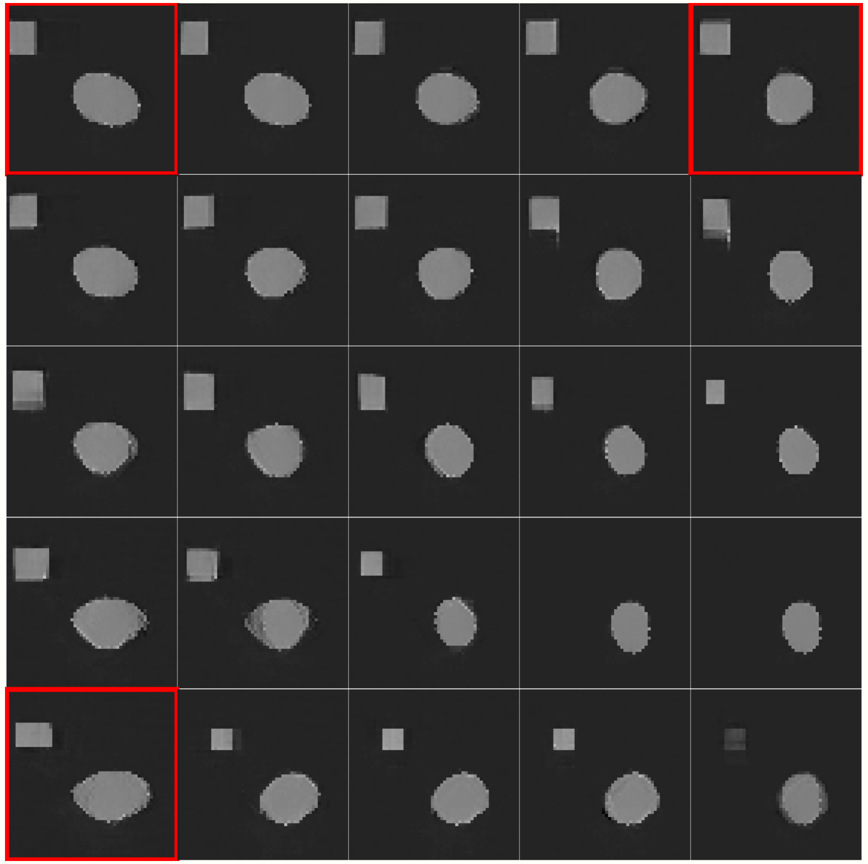


Generative model comparisons

- Smoothness of the generator with respect to z

VAE





GAN

VAE and GAN Comparison

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