Efficient approximation of solutions of parametric linear transport equations by ReLU DNNs

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Parametric Linear Transport Equation

The Cauchy problem for the linear parametric transport equation is given by

$$\begin{cases} \partial_t u(t,x,\eta) + V(t,x,\eta)\cdot\nabla_x u(t,x,\eta) = f(t,x,\eta), \\ u(0,x,\eta) = u_0(x), \end{cases}$$

where $x \in \mathbb{R}^n, \eta \in [0,1]^D$ and $t \in [0,T]$ for some $n, D \in \mathbb{N}$ and $T > 0$. The vector field $V \in C^k([0,T]\times\mathbb{R}^n\times\mathbb{R}^D;\mathbb{R}^n)$, the source term $f \in C^k([0,T]\times\mathbb{R}^n\times\mathbb{R}^D)$ and the initial condition $u_0 \in C^s(\mathbb{R}^n;\mathbb{R})$ are given with $s,k \in \mathbb{N}$.

We are especially interested in the case:

- $n \ll D$, (high-dimensional parameter space)
- $s \ll k$, (smooth velocity and rhs, but unregular initial condition)

$\Rightarrow$ Approximation of **high-dimensional functions of low regularity**

Regularity theory: NN complexity $\sim \varepsilon^{-d/s}, d := 1 + D + n$,
$\Rightarrow$ Approximation suffers from **curse of dimensionality** since $s \ll d$

We utilise special form of solution given by $u_0 \circ X$, X being the solution of the characteristic system of ODEs to prove bounds for ReLU NNs with

$$\text{complexity} \sim \varepsilon^{-n/s} + \varepsilon^{-d/k}$$

$\Rightarrow$ d and s have been decoupled

- Low-reg of u_0 is compensated by low-dimensionality of domain
- High regularity of X equalises high dimension d

Theory of transport equations

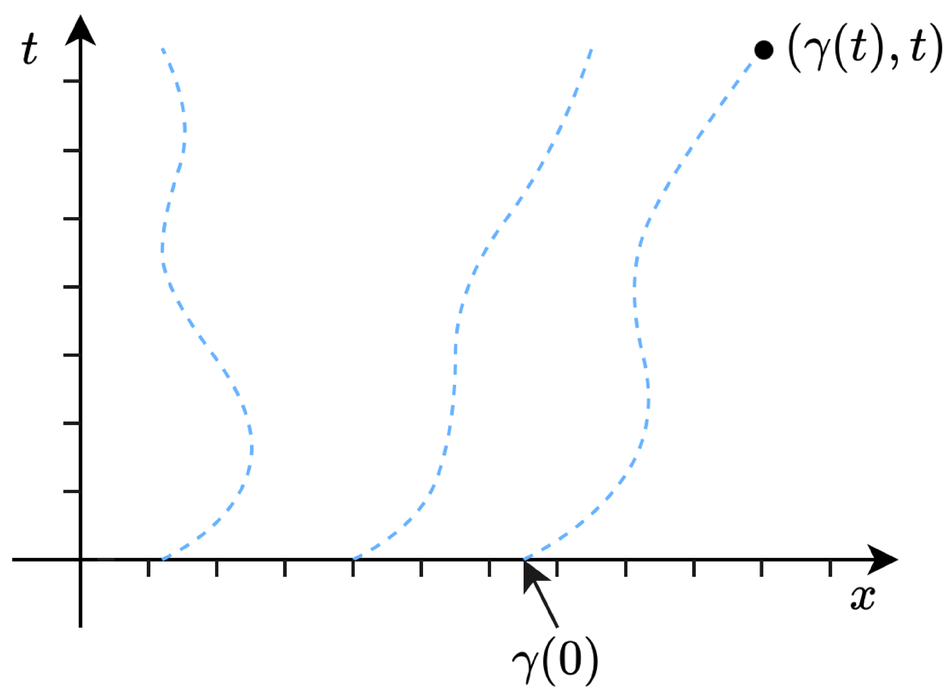

Figure 1. Characteristic curves

Characteristic curves

The characteristic curve of the transport operator $\partial_t + V(x,t,\eta)\cdot\nabla_x$ passing trough x at time $s = t$ is given by the set

$$\{(s,\gamma(s)) \mid s \in [0,T]\},$$

where γ is the solution of the characteristic system of ODEs

$$\begin{cases} \dot\gamma(s) = V(s,\gamma(s),\eta), \\ \gamma(t) = x. \end{cases}$$

u is constant along characteristic curves:
Assume $V(t,x,\eta) = v, v \in \mathbb{R}^n$. Then (drop $\eta-$dependency)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} u(t,\gamma(t)) &= \partial_t u(t,\gamma(t)) + \nabla_x u(t,\gamma(t))\cdot\dot\gamma(t) \\ &= \partial_t u(t,\gamma(t)) + \nabla_x u(t,\gamma(t))\cdot v \\ &= (\partial_t + V(t,x)\cdot\nabla_x)u(t,\gamma(t)) = 0. \end{aligned}$$

Solution at (t,x) = initial data evaluated at $\gamma(0)$

Assumptions

(H1) For some $k \in \mathbb{N}_{\geq 1}$ there holds

$$V \in C^k([0,T]\times\mathbb{R}^n\times[0,1]^D;\mathbb{R}^n).$$

(H2) There exists a $C > 0$ s.t.

$$|V(t,x,\eta)| \leq C(1+|x|) \text{ for all } (t,x,\eta) \in [0,T]\times\mathbb{R}^n\times[0,1]^D.$$

Theorem (Classical Solution)

Let V satisfy the assumptions (H1) and (H2) and $u_0 \in C^s(\mathbb{R}^n)$. Then the Cauchy problem for the parametric linear transport equation

$$\begin{cases} \partial_t u(t,x,\eta) + V(t,x,\eta)\cdot\nabla_x u(t,x,\eta) = 0, \\ u(0,x,\eta) = u_0(x), \end{cases}$$

has a unique solution $u \in C^{\min\{s,k\}}([0,T]\times\mathbb{R}^n\times[0,1]^D)$, which is given by

$$u(t,x,\eta) = u_0(X(0,t,x,\eta)).$$

The map X is given by

$$X(s,t,x,\eta) := \gamma(s)$$

with regularity $C^k([0,T]\times[0,T]\times\mathbb{R}^n\times[0,1]^D)$.

Our Approximation results

Main Result (Laakmann and Petersen 2021)

Let V satisfy assumptions (H1) and (H2) for $k, n, D \in \mathbb{N}$, and $T > 0$. Let $u_0 \in C^s(\mathbb{R}^n)$ and let $u \in C^{\min\{s,k\}}([0,T]\times\mathbb{R}^n\times[0,1]^D)$ denote the unique solution of the Cauchy problem for the parametric linear transport equation

$$\begin{cases} \partial_t u(t,x,\eta) + V(t,x,\eta)\cdot\nabla_x u(t,x,\eta) = 0, \\ u(0,x,\eta) = u_0(x). \end{cases}$$

Then, for every $\varepsilon \in (0,1)$ and every compact subset $K \subset \mathbb{R}^n$, there exists a NN $\Phi^{\overline{u},\varepsilon}$ with d-dimensional input, where $d := 1 + n + D$, such that for the restriction $\overline{u} := u\big|_{[0,T]\times K\times[0,1]^D}$ there holds that, for $c = c(n,s,d,k,K,T,\|V\|_{C^k},u_0) > 0$,

(i) $L(\Phi^{\overline{u},\varepsilon}) \leq c\cdot(\ln(1/\varepsilon)+1)$,
(ii) $W(\Phi^{\overline{u},\varepsilon}) \leq c\cdot\left(\varepsilon^{-n/s}+\varepsilon^{-d/k}\right)\cdot(\ln(1/\varepsilon)+1)$,
(iii) $\|\overline{u} - \mathrm{R}(\Phi^{\overline{u},\varepsilon})\|_{L^\infty([0,T]\times K\times[0,1]^D)} < \varepsilon$,

Generalisations and Extensions

1. **Weak solutions:** An analogous statement to the Main Result can be derived for weak solutions with initial data $u_0 \in W^{1,\infty}_{loc}$.
2. **Conservative Form:** A similar statement to the Main Result can be derived for classical solutions for the conservative formulation given by
$$\begin{cases} \partial_t u(t,x,\eta) + \mathrm{div}_x(V(t,x,\eta)u(t,x,\eta)) = 0, \\ u(0,x,\eta) = u_0(x). \end{cases}$$
3. **Source terms and amplification factors:** A similar statement to the Main Result can be derived for appropriate assumptions on the source term $f(t,x,\eta)$ and an additional amplification term $a(t,x,\eta)u(t,x,\eta)$.
4. **Bounded domains:** Results can extended to pure characteristic boundaries, periodic boundary conditions and inflow boundary conditions.

Example for Application (Radiative transfer models)

A core constituent of radiative transfer models is given by

$$\begin{cases} \partial_t u(t,x,\eta) + \eta\cdot\nabla_x u(t,x,\eta) + a(t,x,\eta)u(x,t,\eta) = f(t,x,\eta), \\ u(0,x,\eta) = u_0(x), \\ u(t,x,\eta) = u_b^-(t,x,\eta), \quad \text{for } (x,\eta)\in\partial\Omega^-, \end{cases}$$

with

$$\partial\Omega^- := \left\{(x,\eta)\in\partial\Omega\times[0,1]^D : \eta\cdot\nu < 0\right\}$$

- Describes the propagation of particles in a collisional medium
- η can describe a unit direction vector taken from the $(n-1)$-dimensional unit sphere
- For $V(t,x,\eta) = \eta$, obviously all assumptions are fulfilled with $k = \infty$

References

Laakmann, F. and P. Petersen (2021). "Efficient approximation of solutions of parametric linear transport equations by ReLU DNNs". In: *Advances in Computational Mathematics* 47.1.