MA10230 Homework Hints

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Abstract

Hints for homework problem sheet questions for the MA10230 Methods and Applications course at the University of Bath, during the 2020/21 academic year.

Problem Sheet 10

This problem sheet focuses on how to solve both free and forced second order linear differential equations.

This week you have learnt how to solve a *free* second-order ODE

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

but I shall now summarise how to solve a *forced* second-order ODE

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x), \quad f \neq 0$$

as that is what Question 5 is all about (you will go into more detail next week as to how you solve them).

Essentially, we aim to write the general solution in the form

$$y(x) = q(x) + p(x)$$

where

- $q(x)$ is called the *complementary function*, and deals with the fact that we want a *general* solution - i.e. some (two to be exact - see lecture notes as to why) arbitrary constants floating about in our answer.
- $p(x)$ is called the *particular integral* and deals with the fact that we have a *forced* ODE - when we plug y into the LHS of the ODE we want the term(s) $f(x)$ to survive/not cancel out. Because we already have two arbitrary constants floating about in q, all constants/coefficients in p will be determined/known (even if we don't have any initial/boundary conditions!).

Complementary function: We consider a *free* version of the ODE

$$a\frac{\mathrm{d}^2q}{\mathrm{d}x^2} + b\frac{\mathrm{d}q}{\mathrm{d}x} + cq = 0$$

and solve for the complementary function $q(x)$. Use the auxiliary equation approach as outlined in lectures to determine what q should look like (it will have two arbitrary constants - they are here to stay).

Particular integral: We now solve

$$a\frac{\mathrm{d}^2p}{\mathrm{d}x^2} + b\frac{\mathrm{d}p}{\mathrm{d}x} + cp = f(x)$$

for $p(x)$. In later modules you will be taught a more satisfying method as to how you go about solving this but, due to the scope of this course, we solve this by staring at $f(x)$ and giving a good long think.

- We want to think of a function $p(x)$ that has terms that could potentially look like $f(x)$, but also whose first and second derivatives either has terms that could potentially look like $f(x)$, or whose terms could potentially cancel out the non-f looking terms that crop up. When we initially guess p, we will include some unknown coefficients/constants that we will determine later. Here are some examples:
  - If $f = 10\cos(4x)$, try $p(x) = \ell\cos(4x) + m\sin(4x)$.
  - If $f = x^2$, try $p(x) = r_0 + r_1 x + r_2 x^2$
  - If $f = e^{3x}$, try $p(x) = C\,e^{3x}$ (things can potentially get a bit complicated in this sort of situation - the lecture notes will go into more detail but for Question 6 (non-homework) this sort of guess will suffice)
- Plug $p(x)$ (and it's first and second derivative) into the ODE. By comparing (the coefficients of) this answer to $f(x)$, determine *all* the unknown constants that you initially introduced in $p(x)$.

  Why $y = q+p$ is a general solution to the forced ODE: We went to all this trouble of calculating the complementary function q and particular integral p, but how do we know $q+p$ is a general solution of the ODE? Firstly, to see why it is a solution at all, lets plug it into the ODE:

$$\begin{aligned} ay'' + by' + cy &= a(q+p)'' + b(p+q)' + c(p+q) \\ &= (aq'' + bq' + cq) + (ap'' + bp' + cp) \\ &= 0 + f(x) \\ &= f(x), \end{aligned}$$

  so it is certainly *a* solution of the ODE. Recall from lectures that a general solution of a second order ODE has two arbitrary constants. Indeed, because q is a general solution of a free ODE - so has two arbitrary constants - and p has no arbitrary constants, $y = p + q$ is in fact the *general* solution to the ODE.

Question 1

Part a)

We want to find the general solution of the second-order linear ODE

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 2\frac{\mathrm{d}u}{\mathrm{d}x} + 2u = 0$$

- Calculate the auxiliary equation and use the general solution that corresponds to the type of roots the A.E. has (see lecture notes).

Question 2

Part a)

We are asked to find the general solution to the ODE

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 6\frac{\mathrm{d}u}{\mathrm{d}x} + 5u = 0$$

- Calculate the auxiliary equation and use the general solution that corresponds to the type of roots the A.E. has (see lecture notes).

Part b)

We are asked to find the particular solution to the ODE in part a) satisfying the initial conditions

$$u(0) = 0, \qquad \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 1.$$

- Calculate the first derivative of u (at this stage it will still have some unknown constants floating about). Using the initial conditions, determine the unknown coefficients.

Part c)

We are asked to find the particular solution to the ODE in part a) satisfying the boundary conditions

$$u(0) = 0, \qquad u(1) = 1.$$

- When you apply the boundary conditions, the coefficients that you get won't be as nice-looking as those in part b).

Question 5

Instead of giving hints directly tailored to Q5, I shall instead go through how you should answer Q4, as it follows a very similar structure.

Solutions to Question 4

This question is essentially asking us to solve the forced ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9y = \cos(x) + \sin(x).$$

They break the general recipe of solving forced ODEs, as I described above, into several parts so it is good to see how each part of the question contributes to the general method.

Part a)

Find the general solution to the ODE

$$\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} + 9q = 0.$$

- As this is just the free version of the ODE, here we are calculating the *complementary function* - the part of our solution that makes it the *general* solution to the forced ODE.
- We have that the auxiliary equation is

  $$\lambda^2 + 9 = 0$$

  and so $\lambda = \pm 3i$. It follows (from the lecture notes) that the general solution to the free ODE (i.e. the complementary function) is

  $$q(x) = A\cos(3x) + B\sin(3x).$$

Part b)

Let $p(x) = \ell \sin(x) + m\cos(x)$, where ℓ and m are constants. Find

$$\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} + 9p$$

- Notice that the function p they prescribe is none other than an initial guess for the particular integral - the part of our solution that ensures we solve a *forced* ODE. If we peek at part d) we see that $f(x) = \cos(x) + \sin(x)$, and so the choice of $p(x)$ here is a generalisation of f suited to the fact that we are taking derivatives.
- To substitute p (our guess of the particular integral) into the ODE, we need to calculate its second derivative:

  $$\begin{aligned} p &= \ell\sin(x) + m\cos(x) \\ \implies p' &= -\ell\cos(x) + m\sin(x) \\ \implies p'' &= -\ell\sin(x) - m\cos(x) = -p. \end{aligned}$$

  It follows that

  $$\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} + 9p = 8p = 8\ell\sin(x) + 8m\cos(x).$$

Part c)

Let $p(x)$ be as in (b), and suppose that it satisfies

$$\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} + 9p = \cos(x) + \sin(x).$$

Using your answer to (b), find the values of ℓ and m by comparing coefficients of $\cos(x)$ and $\sin(x)$.

- This is precisely the property that we want the particular integral to have - that our general solution is that of a *forced* ODE. Since the general-ness of our end-goal solution is taken care of by the complementary function q, all the coefficients m and ℓ that we introduced in our guess for the particular integral p must be determined/known.

- From part (b), we have that

  $$p'' + 9p = 8\ell \sin(x) + 8m \cos(x) = \sin(x) + \cos(x),$$

  and so

  $$\ell = m = \frac{1}{8}.$$

Part d)

Using your answers to parts (a), (b), and (c), show that the general solution to the forced second-order linear ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9y = \cos(x) + \sin(x)$$

is given by $y = q(x) + p(x)$.

- Let

  $$y = q(x) + p(x) = A\cos(3x) + B\sin(3x) + \frac{1}{8}(\cos(x) + \sin(x)).$$

  By parts (a) and (b), this is certainly *a* solution to the given ODE. However, since it depends on two arbitrary constants, it is in fact the *general* solution to the ODE.