Adaptive Refinement for hp–Version Trefftz Discontinuous Galerkin Methods for the Homogeneous Helmholtz Problem

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We consider a Trefftz discontinuous Galerkin finite element (TDG) approximation of the solution to the homogeneous Helmholtz problem $-\nabla u - k^2 u = 0$ in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, for a given fixed wavenumber k. The TDG method for Helmholtz is a discontinuous finite element methods, where instead of using standard polynomial basis functions on each element we instead use (local) solutions to the Helmholtz equation. Here, we consider use of plane wave functions $\mathrm{e}^{ikd_\ell \cdot (\boldsymbol{x} - \boldsymbol{x}_K)}$, where $\boldsymbol{d}_\ell$, $\ell = 1, \ldots, p_K$, are distinct unit propagation directions on an element K with element centre $\boldsymbol{x}_K$. As the chosen direction are distinct, this leads to a finite element space on each element with dimension p_K. We also consider the so-called *effective polynomial degree* q_K of an element, from which we can derive the number of plane waves, such that the convergence rate of the TDG formulation is the same as the convergence rate of the corresponding polynomial basis function finite element approximation with polynomial degree q_K. It has been shown that p_K is of order q_K^{d-1}, compared to the number of degrees of freedom for the polynomial-based approximation which is of order q_K^d.

Existing work has already studied adaptive h-refinement of the finite element mesh (refinement of mesh element size), and we present a basic, numerically derived, extension to this work to handle p-refinement (increasing the number of plane wave basis functions per element). We also present a modified version of an existing algorithm for deciding when to perform h- or p-refinement.

We also consider the fact that wave propagation problems often have a primary propagation direction and aligning one of the directions of the plane wave basis functions with this primary direction can reduce the error. We show how we can use the eigenvalues and eigenvectors of the Hessian of the numerical solution to the Trefftz discontinuous Galerkin finite element method to approximate this primary propagation direction and, hence, adaptively adjust the selected plane wave directions for the plane wave basis functions.