DURHAM UNIVERSITY

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PROJECT IV REPORT

Curve Shortening Flow

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Plagiarism Declaration

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

I have produced all the images contained in this piece of work myself using Inkscape, MathMod, and Python.

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This report is dedicated to my two dogs, Betty and Ralph, and my cat Cat, for helping to keep my spirits up during the long sessions spent typing away at my computer.

Abstract

In this report we give an overview of curve shortening flow and some of its uses. We begin by looking at curve shortening flow in the plane, then move on to study curve shortening flow on surfaces. We discuss the main results concerning curve shortening flow on closed embedded curves and the conditions under which they apply; in particular, we look at the connection between curve shortening flow and geodesics. We also look at applications and further generalisations of curve shortening flow.

This report is aimed at a 4th year MMath student with an interest in geometry; in particular, it assumes knowledge of the Differential Geometry III and Riemannian Geometry IV modules. However, we include some definitions and reminders to make it more accessible to students without such a geometry-heavy background.

Contents

Introduction

Originally, differential geometry arose as the study of 'curved' or 'bent' spaces fixed in time. Over the last four decades, however, geometers have overseen huge developments in the study of spaces which are not fixed but instead change, or 'flow', over time. Out of all the different flows which have been studied, the most well-known is *Ricci flow*, which Perelman famously used to prove the Poincaré conjecture, so far the only Millennium Prize Problem which has been solved. This work was made even more famous when Perelman declined the Fields Medal and $1,000,000 prize he was offered as a reward.

Although there are many ways in which a curve can evolve over time, one of the most intuitive and best understood ways is by *curve shortening flow*. The idea behind curve shortening flow is deceptively simple: a curve evolves by moving in the direction of its normal vector, with speed proportional to its curvature. Effectively, where it bends the most it moves the fastest. The study of curve shortening flow originated in materials science [1], but the main strides came in the mid-to-late 1980s when mathematicians such as Michael Gage, Richard Hamilton and Matthew Grayson were able to characterise the behaviour of the flow in detail. This report examines how the curve shortening flow behaves, why it behaves in such a manner, and how it can be utilised. It also includes plots and diagrams to showcase the visual interpretation of curve shortening flow.

This report is structured as follows: In Chapter 1, we introduce some basic material from Riemannian geometry which we will need to fully understand the rest of the report, and we introduce geodesics and examine their behaviour and properties. In Chapter 2, we look at curve shortening flow in the plane, a process by which we can deform a smooth curve in what is in some sense a 'natural' way. In Chapter 3, we generalise curve shortening flow to a wider class of 2-dimensional Riemannian manifolds and see an interesting connection to geodesics. In Chapter 4, we look at some applications of curve shortening flow in both pure and applied mathematics.

Chapter 1

Background

In this chapter, we recall some basic definitions and results from Riemannian geometry including material on curves, Riemannian manifolds as metric and topological spaces, curvature, and geodesics. This material will be used frequently in the later chapters of this report. For a more thorough and detailed introduction to Riemannian geometry, the reader may consult for example [2] or [3]; for facts about smooth manifolds the reader can consult [4].

Throughout this report we will let (M, g) denote a Riemannian manifold M equipped with metric g. For notational simplicity we will often drop the word "Riemannian" and the metric g, and we will write about "a manifold M" when really we mean "a Riemannian manifold (M, g)"; the Riemannian metric is still there.

1.1 Curves

Formally, we define a curve in the following way:

Definition 1.1.1. A *curve* is a continuous function from an interval $I \subset \mathbb{R}$ to a manifold M:

$$\boldsymbol{\gamma} \colon I \to M$$
$$u \mapsto \boldsymbol{\gamma}(u).$$

We frequently make use of the following terminology when talking about curves:

Definition 1.1.2. A curve $\boldsymbol{\gamma}$ defined on a closed interval $I = [a, b]$ is *closed* if its endpoints join up: that is, if $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$. In this case we can consider the curve as a function from the circle S^1 instead of I. One way to visualise this, for example, is by taking $[a, b]$ and 'joining' the two endpoints together to form a loop.

Definition 1.1.3. A curve is *smooth* if it is C^∞. For the purposes of this report it is usually sufficient to consider curves belonging to C^2, but for simplicity we will assume that all curves we consider are smooth.

Definition 1.1.4. Let M and N be smooth manifolds. A smooth function $f\colon N \to M$ is an *embedding* if f is a one-to-one diffeomorphism. Slightly abusing notation, we will say that a curve $\gamma\colon I \to M$ is *embedded* if γ is an embedding.

This means that a curve is embedded if its tangent field $\boldsymbol{\gamma}'$ is non-zero everywhere and it has no self-intersections: $\boldsymbol{\gamma}(u_1) = \boldsymbol{\gamma}(u_2)$ if and only if $u_1 = u_2$.

Definition 1.1.5. A curve is *regular*, or *immersed*, if its derivative $\boldsymbol{\gamma}'$ is non–zero everywhere.

Definition 1.1.6. The *length* of a curve is defined to be

$$L(\boldsymbol{\gamma}) = \int_I \left\|\boldsymbol{\gamma}'(u)\right\| \, \mathrm{d}u.$$

The length of a curve does not depend on how it is parametrised.

Definition 1.1.7. A curve is *arclength*, or *parametrised by arclength*, if its derivative has unit magnitude everywhere: $\|\boldsymbol{\gamma}'\| \equiv 1$. A curve is *parametrised proportionally to arclength* if its derivative has constant non-zero magnitude everywhere.

Arclength measures the length along a curve from its start point. When a curve is parametrised by arclength we denote the space parameter by s instead of u.

Remark 1.1.8. Any regular curve admits an arclength reparametrisation.

Let I_1 and I_2 be intervals. We can consider a *family of curves*, a function which takes two arguments:

$$\begin{aligned} \boldsymbol{\gamma}\colon I_1 \times I_2 &\to M \\ (u, t) &\mapsto \boldsymbol{\gamma}(u, t) =: \boldsymbol{\gamma}_t(u). \end{aligned}$$

For each (fixed) t in I_2, we can let u vary and consider the curve $\boldsymbol{\gamma}_t$, with u as its argument. We could do the same with the variables switched, and look at $\boldsymbol{\gamma}_u$ for a fixed u in I_1, but this turns out not to be useful for the purposes of this report and so we will ignore it. We will treat the variable u as a space parameter and the variable t as a time parameter, so as we vary t the curve $\boldsymbol{\gamma}_t$ 'evolves' or varies with time. Note that $\boldsymbol{\gamma}_t$ does not represent differentiation with respect to t.

Sometimes we will slightly abuse notation and refer to "a curve" when technically we are talking about "the image (or trace) of a curve". In this way, we are effectively looking at curves modulo an equivalence relation, where two curves are considered equivalent if they have the same image. In this way, for example, we will call a curve embedded if there exists a parametrisation of it that makes it embedded. Where we want to consider the specific map defining a curve and not just its image, we will explicitly make reference to the parametrisation of the curve.

1.2 Topological Properties of Manifolds

Definition 1.2.1. A manifold is *connected* if every pair of points p, q in M has a path between them.

When we have a connected manifold M with metric g, we can define a distance function on it by

$$d(p,q) = \inf\{L(\boldsymbol{\gamma}) : \boldsymbol{\gamma} \text{ is piecewise smooth}, \boldsymbol{\gamma}\colon [a,b] \to M, \boldsymbol{\gamma}(a) = p, \boldsymbol{\gamma}(b) = q\}.$$

It can be shown that this distance function turns M into a metric space (M, d) (note that d depends on the metric g here). The topology of this metric space is precisely the 'original' topology of M as a smooth manifold; since the metric g induces d, this connects the geometry and the topology of a Riemannian manifold.

Definition 1.2.2. A subset $V \subseteq M$ is *compact* if it is compact in the metric space (M, d).

Remark 1.2.3. Closed curves (precisely, the images of closed curves) are always compact: this is because S^1 is compact, and a closed curve is by definition a continuous map from S^1 to a manifold M. Because the map is continuous, it preserves compactness, so its image is compact.

Definition 1.2.4. A manifold M is *complete* if every geodesic can be extended indefinitely in both directions (that is, its domain I can be extended to $\mathbb{R}$).

For example, the unit sphere $S^n \subset \mathbb{R}^{n+1}$ with the standard round metric induced from the Euclidean metric of $\mathbb{R}^{n+1}$ is complete: since the geodesics are great circles, which are closed, we can loop around them indefinitely.

Remark 1.2.5. As a consequence of the Hopf-Rinow theorem [2, Chapter 7], if M is connected and complete then any two points p, q in M can be joined together by a length-minimising geodesic.

Definition 1.2.6. Let M be a Riemannian manifold which is connected and complete. A subset $K \subseteq M$ is *convex* if for any two points p, q in K, the shortest geodesic between the points is entirely contained in K.

We now have the machinery to define the convex hull of a subset:

Definition 1.2.7. The *convex hull* H of a subset $U \subseteq M$ is the intersection of all convex sets K containing U.

The convex hull of U is the smallest convex set containing U, in the sense that if K is a convex set with $U \subseteq K$ then $H \subseteq K$.

Definition 1.2.8. M is *convex at infinity* if every compact subset of M has a compact convex hull.

1.3 Curvature of Curves

Definition 1.3.1. Let $\gamma\colon I \to M$ be a curve parametrised by arclength s. The *unsigned curvature* of γ is defined as

$$\varkappa(s) = \left| \frac{D}{ds}\gamma'(s) \right|,$$

where $\frac{D}{ds}$ denotes the covariant derivative along γ with respect to s.

Since by Remark 1.1.8 any regular curve can be reparametrised so that it is arclength, this gives a definition of curvature for any regular curve: by reparametrising by arclength, computing the curvature and then undoing our reparametrisation, we get that

$$\varkappa(u) = \frac{\left|\frac{D}{du}\gamma'(u)\right|}{|\gamma'(u)|^2} - \frac{g_{\gamma(u)}\left(\frac{D}{du}\gamma'(u), \gamma'(u)\right)}{|\gamma'(u)|^3}.$$

It is important to note that curvature depends on the manifold M. For example, let M_1 be $\mathbb{R}^3$ with the Euclidean metric, and M_2 the unit sphere embedded in $\mathbb{R}^3$ centred at the origin with metric induced from the Euclidean metric. Then the curve given by

$$\gamma(u) = (\cos(u), \sin(u), 0)$$

has constant curvature $\varkappa \equiv 1$ everywhere when considered as part of M_1, but zero curvature everywhere as part of M_2. Curvature can be viewed as how much a curve bends within its manifold [5, p. 118]; our definition of curvature coincides with the definition of *geodesic curvature* from the Differential Geometry III module.

Definition 1.3.2. Let M be an orientable manifold, and $\boldsymbol{\gamma}$ be a curve on M. Pick one of the two consistent choices of unit normal vector $\boldsymbol{n}$ along $\boldsymbol{\gamma}$: that is, the field such that at points where $\varkappa(s) \neq 0$ the equation

$$\boldsymbol{n}(s) = \frac{1}{\varkappa(s)} \frac{D}{ds} \boldsymbol{\gamma}'(s)$$

holds, or the field such that the equation

$$\boldsymbol{n}(s) = \frac{-1}{\varkappa(s)} \frac{D}{ds} \boldsymbol{\gamma}'(s)$$

holds.
Then the *signed curvature* of a curve is the quantity κ defined implicitly by the equation

$$\frac{D}{ds} \boldsymbol{\gamma}'(s) = \kappa(s) \boldsymbol{n}(s).$$

This definition assumes that there is at least one point at which $\varkappa(s) \neq 0$. If this is not the case, the signed curvature can be defined to be identically 0 and choosing a normal vector will not matter.

Note that this definition makes sense: as s parametrises $\boldsymbol{\gamma}$ by arclength, we have that

$$g_{\boldsymbol{\gamma}(s)}(\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s)) = 1.$$

Differentiating this with respect to s and applying the Riemannian property of the covariant derivative yields that

$$g_{\boldsymbol{\gamma}(s)} \left(\frac{D}{ds} \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s) \right) = 0,$$

so the normal vector is indeed orthogonal to the tangent vector.

The signed curvature is a scalar quantity with the same magnitude as the unsigned curvature, but it has the same sign as the quantity $g_{\boldsymbol{\gamma}(s)}(\frac{D}{ds}\boldsymbol{\gamma}'(s), \boldsymbol{n}(s))$. In particular, if we change the sign of $\boldsymbol{n}$ we induce a change in sign of κ, and so the quantity $\kappa \boldsymbol{n}$ does not depend on our choice of normal vector field. This proves to be a critical point later in the report when we define curve shortening flow, and so from this point onwards we will work exclusively with the signed curvature; when we refer to "curvature" without mentioning whether it is signed or unsigned, we will mean signed curvature.

1.4 Geodesics

Geodesics are essentially curves which play the role of straight lines on a manifold. Although the concept of a geodesic originates from differential geometry, they are used in many different areas. In general relativity, geodesics generalise the notion of straight lines to curved spacetime, and can be used for example to describe the motion of a particle under the influence of gravity, such as a satellite orbiting the Earth or a planet orbiting its sun. Geodesics also appear in nature: for example, when one squirrel chases another up a tree, the path it traces out will have the shape of a helix, which is a geodesic on a cylinder.

Mathematically, we can define a geodesic in terms of the covariant derivative along a curve:

Definition 1.4.1. A curve $\boldsymbol{\gamma}$ is a *geodesic* if its curvature κ is 0 everywhere along the curve.

We now state an important theorem on existence and uniqueness of geodesics:

Theorem 1.4.2. *Let $p \in M, \boldsymbol{v} \in T_pM$. Then there exists a unique maximal geodesic that passes through p with tangent vector $\boldsymbol{v}$.*

This theorem can be proved from Definition 1.4.1; writing the geodesic in terms of its components as $\boldsymbol{\gamma}(u) = (\gamma_1(u), ..., \gamma_n(u))$ the condition $\frac{D}{du}\boldsymbol{\gamma}'(u) = 0$ gives us a system of ODEs for the γ_i' terms with initial data specifying p and $\boldsymbol{v}$. The existence and uniqueness theorem for ODEs then tells us this system can be solved uniquely.

We know that in Euclidean space, the shortest path between two points is given by a straight line. Geodesics have a similar property, which we now examine.

Theorem 1.4.3. *Geodesics are local length minimisers.*

Formally, this theorem says that if a curve $\boldsymbol{\gamma}$ is a geodesic, it is a local minimum of the length functional l: if we perturb $\boldsymbol{\gamma}$ slightly, its length will increase. The word *local* is key here: a curve between two points being a geodesic is not enough to guarantee it is the shortest path between the two points (see Figure 1.1).

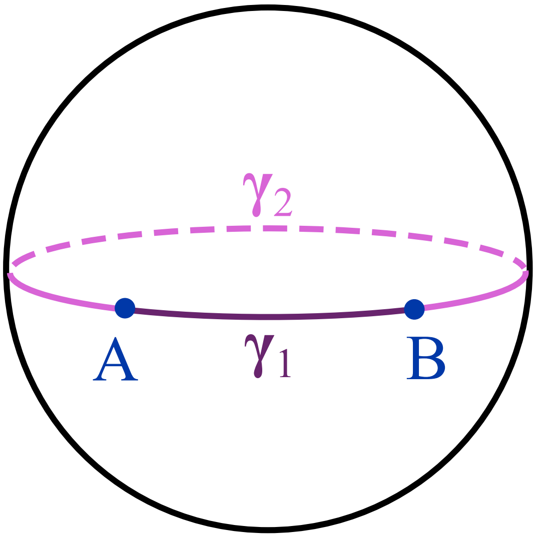

Figure 1.1: Two curves on the round sphere S^2. Both curves are segments of the great circle along the equator, and so are geodesics, and both curves connect points A and B. However, it is clear that $\boldsymbol{\gamma}_1$ is much shorter than $\boldsymbol{\gamma}_2$.

Chapter 2

Curve Shortening Flow in the Plane

In this chapter, after some necessary preliminary definitions, we introduce *curve shortening flow* in the plane, a process which takes a smooth plane curve and modifies it over time by moving each point in the direction of its normal vector at speed proportional to its curvature. This flow has numerous applications, for example in surface evolution [1] and image processing [6]. Particular attention is given to closed embedded curves, for which we state and prove elementary properties of curve shortening flow. We consider the Gage-Hamilton-Grayson theorem, the most significant and powerful result on planar curve shortening flow, and investigate how the shape of curves changes as they flow. We prove that curve shortening flow keeps embedded curves embedded. We end the chapter by looking at the classification of curves which are self-similar under curve shortening flow.

Throughout this chapter we will take M to be the Euclidean plane $\mathbb{R}^2$ with the standard Euclidean metric $g_{\boldsymbol{p}}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \boldsymbol{v}_1 \cdot \boldsymbol{v}_2$ where $\cdot$ is the Euclidean inner product.

2.1 Introduction to Curve Shortening Flow

In this section, we define curve shortening flow, give a simple example of it, and state the Gage-Hamilton-Grayson theorem which concerns the existence of curve shortening flow and the behaviour of closed embedded curves under the flow flow.

For brevity, from this point we will often make reference to functions without explicitly mentioning their arguments; for example, we may write κ instead of $\kappa(u, t)$ when discussing the curvature over a family of curves. It should be clear from context and previous definitions what the missing arguments are.

Definition 2.1.1. Consider the curve $\boldsymbol{\gamma}_0(u)\colon I \to \mathbb{R}^2$, and let $T > 0$. We say that a family of curves $\boldsymbol{\gamma}(u,t)\colon I \times [0,T) \to M$ *evolves under curve shortening flow* if it satisfies the initial value problem

$$\begin{cases} \frac{\partial \boldsymbol{\gamma}}{\partial t} = \kappa \boldsymbol{n}, \\ \boldsymbol{\gamma}(u,0) = \boldsymbol{\gamma}_0(u), \end{cases} \tag{2.1}$$

where $\kappa(u,t)$ is the curvature of $\boldsymbol{\gamma}$ and $\boldsymbol{n}$ is a unit normal vector to $\boldsymbol{\gamma}$, chosen at each point such that the vector field $\boldsymbol{n}$ is continuous along $\boldsymbol{\gamma}$.

Example 2.1.2. Let $\boldsymbol{\gamma}_0$ be the unit circle, centred at the origin. We will derive the formula for its evolution under curve shortening flow. Taking the inward-pointing normal vector and parametrising the circle by

$$\boldsymbol{\gamma}_0(u) = (\cos(u), \sin(u))$$

for u in $[0, 2\pi]$, we can compute the curvature

$$\kappa(u) \equiv 1$$

and the normal vector

$$\boldsymbol{n}(u) = -(\cos(u), \sin(u)) = -\boldsymbol{\gamma}_0(u).$$

By the radial symmetry of the circle, we can see that it will keep its shape under curve shortening flow, so we can write its evolution as

$$\boldsymbol{\gamma}(u,t) = r(t)\,(\cos(u), \sin(u))$$

and we only need to solve for the radius as a function of time $r(t)$. Simple computations show that $\boldsymbol{\gamma}(u,t)$ has curvature

$$\kappa(u,t) = \frac{1}{r(t)}$$

and normal vector

$$\boldsymbol{n}(u,t) = \frac{-1}{r(t)}(\cos(u), \sin(u)) = -\boldsymbol{\gamma}_0(u).$$

Applying (2.1), we get an ODE for the radius of the form

$$\begin{cases} r'(t) = -\frac{1}{r(t)}, \\ r(0) = 1. \end{cases}$$

This can be solved to give

$$r(t) = \sqrt{1-2t},$$

and so the unit circle evolves by curve shortening flow according to

$$\boldsymbol{\gamma}(u,t) = \sqrt{1-2t}\,(\cos(u), \sin(u)).$$

From this we can see that the flow exists up to $t = \frac{1}{2}$, at which point the circle has shrunk to a point and so cannot flow any more. This is illustrated in Figure 2.1.

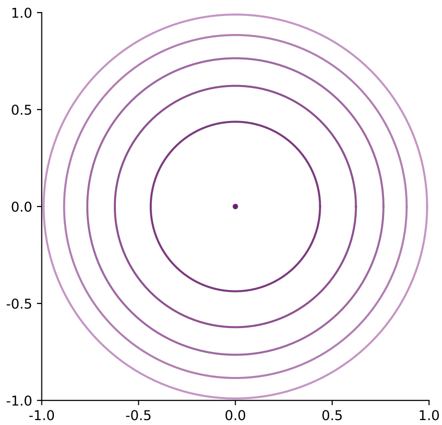

Figure 2.1: The unit circle evolving under curve shortening flow. Smaller circles correspond to later times, with the dot at the origin representing the point the circle shrinks to as $t \to \frac{1}{2}$. The time difference between adjacent circles is kept constant; we can see from the formula for the radius that the area decreases at a constant rate, so the flow appears to go slowly at first then quickly near the end. In fact we will see later in this chapter that this is true for all curves and not just the circle.

We now state a powerful and important theorem, known as the Gage-Hamilton-Grayson theorem, which concerns the existence and behaviour of curve shortening flow on closed embedded curves.

Theorem 2.1.3 (Gage-Hamilton-Grayson Theorem)**.** *Let the curve* $\boldsymbol{\gamma}_0 \colon S^1 \to \mathbb{R}^2$ *be embedded. Then* (2.1) *has a solution up to some maximal finite time* T*. Moreover, the curve* $\boldsymbol{\gamma}_t$ *is smooth for all* t *in* $[0,T)$*, and it converges to a round point as* $t \to T$*.*

Gage and Hamilton proved Theorem 2.1.3 for convex curves in 1986 [7]. In 1987, Grayson extended it to non-convex curves by showing that curve shortening flow will cause any smooth embedded closed curve to become convex in finite time [8]. Later in this chapter we will explicitly calculate the maximal time T that curves can flow for.

2.2 Elementary Properties of Curve Shortening Flow

In this section, we will state and prove some elementary properties of curve shortening flow. These properties can be derived from the defining equation (2.1) of curve shortening flow in a relatively straightforward manner. We will only be looking at closed embedded curves in this section and so, unless stated otherwise, when we talk about a curve we will assume that it is of this form.

Remark 2.2.1. Taking s to parametrise $\boldsymbol{\gamma}$ by arclength, the first part of (2.1) is equivalent to the parabolic PDE

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} = \frac{\partial^2 \boldsymbol{\gamma}}{\partial s}$$

by the definition of curvature. However, since s itself depends on u and t, this is not as simple as it looks and is in fact a nonlinear PDE.

We quickly state and prove some lemmas that will immediately come in handy. These lemmas come from some papers by Gage [9, 10] and one by Gage and Hamilton [7], but all are fully stated and proven in [7] and we prove them in the same way. Throughout the rest of this section we will take u to parametrise $\boldsymbol{\gamma}(\cdot, t) := \boldsymbol{\gamma}_t$ such that u ranges over the interval $[0, 2\pi]$; this will make it easier to proceed with the proofs.

Lemma 2.2.2. *Let s parametrise $\boldsymbol{\gamma}$ by arclength. Then as operators we have*

$$\frac{\partial}{\partial s} = \frac{1}{\left\|\frac{\partial \boldsymbol{\gamma}}{\partial u}\right\|}\frac{\partial}{\partial u}. \tag{2.2}$$

Proof. By definition of arclength, we have:

$$\begin{aligned} \frac{\partial s}{\partial u} &= \frac{\partial}{\partial u}\left(\int_0^u \left\|\frac{\partial \boldsymbol{\gamma}}{\partial \hat{u}}\right\| \mathrm{d}\hat{u}\right) \\ &= \left\|\frac{\partial \boldsymbol{\gamma}}{\partial u}\right\| \qquad \text{by the fundamental theorem of calculus.} \end{aligned}$$

Now,

$$\begin{aligned}\frac{\partial}{\partial s} &= \frac{\partial u}{\partial s}\frac{\partial}{\partial u} \\ &= \left(\frac{\partial s}{\partial u}\right)^{-1}\frac{\partial}{\partial u} \\ &= \frac{1}{\left|\left|\frac{\partial \boldsymbol{\gamma}}{\partial u}\right|\right|}\frac{\partial}{\partial u},\end{aligned}$$

as required. □

For simplicity's sake, we will write $v = \left|\left|\frac{\partial \boldsymbol{\gamma}}{\partial u}\right|\right|$ as in equation (2.2).

In the following two lemmas we assume $\boldsymbol{\gamma}(u,t)$ to evolve under curve shortening flow.

Lemma 2.2.3. *If $\boldsymbol{\gamma}(u,t)$ evolves under curve shortening flow, then*

$$\frac{\partial v}{\partial t} = -\kappa^2 v. \tag{2.3}$$

Proof. In this proof we make use of the well-known Serret-Frenet equations:

$$\begin{cases}\frac{\partial \boldsymbol{t}}{\partial u} = v\kappa\boldsymbol{n} \\ \frac{\partial \boldsymbol{n}}{\partial u} = -v\kappa\boldsymbol{t},\end{cases} \tag{2.4}$$

where $\boldsymbol{t}$ is the unit tangent vector to $\boldsymbol{\gamma}$ (not to be confused with the non-bold t which represents our time variable).

First, note that our choice of u means that u and t are independent co-ordinates, so as operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute.

We differentiate v^2:

$$\begin{aligned}\frac{\partial}{\partial t}\left(v^2\right) &= \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{\gamma}}{\partial u}\cdot\frac{\partial \boldsymbol{\gamma}}{\partial u}\right) \\ &= 2\frac{\partial \boldsymbol{\gamma}}{\partial u}\cdot\frac{\partial^2 \boldsymbol{\gamma}}{\partial t\partial u} \quad \text{(since the inner product is symmetric)} \\ &= 2\frac{\partial \boldsymbol{\gamma}}{\partial u}\cdot\frac{\partial^2 \boldsymbol{\gamma}}{\partial u\partial t} \quad \text{(since the derivatives commute)} \\ &= 2v\boldsymbol{t}\cdot\frac{\partial}{\partial u}(\kappa\boldsymbol{n}) \quad \text{(by equation (2.1))} \\ &= 2v\boldsymbol{t}\cdot\left(\frac{\partial \kappa}{\partial u}\boldsymbol{n} - v\kappa^2\boldsymbol{t}\right) \quad \text{(by the product rule and equations (2.4))} \\ &= -2v^2\kappa^2 \quad \text{(since the normal is orthogonal to the tangent).}\end{aligned}$$

It remains only to note that

$$\frac{\partial}{\partial t}\left(v^2\right) = 2v\frac{\partial v}{\partial t},$$

and by cancelling terms we are done. □

Lemma 2.2.4. *As operators, we have that*

$$\frac{\partial^2}{\partial t \partial s} = \frac{\partial^2}{\partial s \partial t} + \kappa^2 \frac{\partial}{\partial s}. \tag{2.5}$$

Proof. The proof follows from some elementary computations:

$$\begin{aligned}
\frac{\partial^2}{\partial t \partial s} &= \frac{\partial}{\partial t}\frac{\partial}{\partial s} \qquad \text{(by definition)} \\
&= \frac{\partial}{\partial t}\frac{1}{v}\frac{\partial}{\partial u} \qquad \text{(by Lemma 2.2.2)} \\
&= \kappa^2 \frac{1}{v}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial t}\frac{\partial}{\partial u} \qquad \text{(by Lemma 2.2.3)} \\
&= \kappa^2 \frac{1}{v}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}\frac{\partial}{\partial t} \qquad \text{(since } u \text{ and } t \text{ are independent)} \\
&= \kappa^2 \frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t} \qquad \text{(from applying Lemma 2.2.2 in the other direction)} \\
&= \frac{\partial^2}{\partial s \partial t} + \kappa^2 \frac{\partial}{\partial s}.
\end{aligned}$$

□

The reason that $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ do not commute is because, as we mentioned in Remark 2.2.1, s and t are not independent co-ordinates.

Lemma 2.2.5. *If $\boldsymbol{\gamma}(u,t)$ evolves under curve shortening flow, then*

$$\frac{\partial \boldsymbol{t}}{\partial t} = \frac{\partial \kappa}{\partial s}\boldsymbol{n}, \text{ and} \tag{2.6}$$

$$\frac{\partial \boldsymbol{n}}{\partial t} = -\frac{\partial \kappa}{\partial s}\boldsymbol{t}. \tag{2.7}$$

Proof. To prove (2.6), we apply the definition of the unit tangent $\boldsymbol{t}$:

$$\begin{aligned}\frac{\partial \boldsymbol{t}}{\partial t} &= \frac{\partial^2 \gamma}{\partial t \partial s} \\ &= \frac{\partial^2 \gamma}{\partial s \partial t} + \kappa^2 \frac{\partial \gamma}{\partial s} \quad \text{(by Lemma 2.2.4)} \\ &= \frac{\partial}{\partial s}(\kappa \boldsymbol{n}) + \kappa^2 \boldsymbol{t} \quad \text{(by (2.1) and definition of the tangent)} \\ &= \frac{\partial \kappa}{\partial s}\boldsymbol{n} \quad \text{(using (2.4) and the product rule).}\end{aligned}$$

To prove (2.7), we take the inner product of the tangent with the normal:

$$\boldsymbol{t} \cdot \boldsymbol{n} = 0.$$

We then differentiate both sides to get

$$\begin{aligned}0 &= \frac{\partial \boldsymbol{t}}{\partial t} \cdot \boldsymbol{n} + \boldsymbol{t} \cdot \frac{\partial \boldsymbol{n}}{\partial t} \\ &= \frac{\partial \kappa}{\partial t} + \boldsymbol{t} \cdot \frac{\partial \boldsymbol{n}}{\partial t} \quad \text{by (2.6).}\end{aligned}$$

Since $\boldsymbol{t}$ and $\boldsymbol{n}$ are orthogonal and so span $\mathbb{R}^2$, this implies that

$$\frac{\partial \boldsymbol{n}}{\partial t} = -\frac{\partial \kappa}{\partial s}\boldsymbol{t} + a\boldsymbol{n},$$

where a is some function of s and t. But since $\boldsymbol{n}$ has unit length we get by differentiating $\boldsymbol{n} \cdot \boldsymbol{n}$ that

$$\boldsymbol{n} \cdot \frac{\partial \boldsymbol{n}}{\partial t} = 0,$$

and so a must be zero. This completes the proof. □

These lemmas allow us to calculate some important properties of the curve shortening flow: namely, how the curvature, length and enclosed area change over time.

Lemma 2.2.6. *The curvature changes with time following the equation*

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3. \tag{2.8}$$

Proof. Let $\boldsymbol{n}$ be the unit normal vector given by an anticlockwise rotation of the unit tangent vector by an angle of $\frac{\pi}{2}$, and take κ to be the curvature induced by this choice of normal vector.

Define $\theta(s,t)$ to be the angle between the unit tangent vector $\boldsymbol{\gamma}'(s,t)$ and the x-axis. Then in co-ordinates,

$$\boldsymbol{t} = (\cos(\theta), \sin(\theta)).$$

Differentiating, we find that

$$\begin{aligned}\frac{\partial \boldsymbol{t}}{\partial t} &= \frac{\partial \theta}{\partial t}(-\sin(\theta), \cos(\theta)) \\ &= \frac{\partial \theta}{\partial t}\boldsymbol{n}. \end{aligned} \tag{2.9}$$

On the other hand, by (2.6) we have that

$$\frac{\partial \boldsymbol{t}}{\partial t} = \frac{\partial \kappa}{\partial s}\boldsymbol{n}. \tag{2.10}$$

Comparing (2.9) and (2.10), we get that

$$\frac{\partial \theta}{\partial t} = \frac{\partial \kappa}{\partial s}. \tag{2.11}$$

Similarly, we have that

$$\frac{\partial \boldsymbol{t}}{\partial s} = \frac{\partial \theta}{\partial s}\boldsymbol{n}. \tag{2.12}$$

By the definition of curvature,

$$\frac{\partial \boldsymbol{t}}{\partial s} = \kappa \boldsymbol{n}. \tag{2.13}$$

Comparing (2.12) and (2.13) gives us the relation

$$\frac{\partial \theta}{\partial s} = \kappa. \tag{2.14}$$

Thus,

$$\begin{aligned}\frac{\partial \kappa}{\partial t} &= \frac{\partial^2 \theta}{\partial t \partial s} \\ &= \frac{\partial^2 \theta}{\partial s \partial t} + \kappa^2 \frac{\partial \theta}{\partial s} \qquad \text{(by Lemma 2.2.4)} \\ &= \frac{\partial}{\partial s}\frac{\partial \theta}{\partial t} + \kappa^2 \frac{\partial \theta}{\partial s} \\ &= \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3 \qquad \text{(by (2.11) and (2.14))}.\end{aligned}$$

If we instead take $\boldsymbol{n}$ to be the normal given by a clockwise rotation of the tangent, going through the same calculations gives us the same result. □

Lemma 2.2.7. *Let $\boldsymbol{\gamma}$ be a closed embedded curve which evolves by curve shortening flow. Let $L(t)$ be the length of $\boldsymbol{\gamma}_t$ and $A(t)$ the area it encloses. Then:*

(i) $\frac{\mathrm{d}L}{\mathrm{d}t} = -\int_{\gamma_t} \kappa^2 \, \mathrm{d}s$.

(ii) $\frac{\mathrm{d}A}{\mathrm{d}t} = -2\pi$.

Proof. To prove (i), we simply use the definition of length:

$$\begin{aligned}
\frac{\partial L}{\partial t} &= \frac{\partial}{\partial t}\left(\int_0^{2\pi} v \, \mathrm{d}u\right) \\
&= \int_0^{2\pi} \frac{\partial v}{\partial t} \, \mathrm{d}u \quad \text{(from taking the derivative under the integral)} \\
&= -\int_0^{2\pi} \kappa^2 v \, \mathrm{d}u \quad \text{(by equation (2.3))} \\
&= -\int_{\gamma_t} \kappa^2 \, \mathrm{d}s \quad \text{(by change of variables).}
\end{aligned}$$

To prove (ii), we start by writing $\boldsymbol{\gamma}$ in co-ordinates as $\boldsymbol{\gamma}(u,t) = (\gamma_1(u,t), \gamma_2(u,t))$ and choose $\boldsymbol{n}$ to be the normal vector pointing inwards. We then use Green's theorem to get an expression for $A(t)$:

$$\begin{aligned}
A(t) &= \frac{1}{2}\int_0^{2\pi} \left(\gamma_1 \frac{\partial \gamma_2}{\partial u} - \gamma_2 \frac{\partial \gamma_1}{\partial u}\right) \mathrm{d}u \\
&= -\frac{1}{2}\int_0^{2\pi} v\boldsymbol{\gamma} \cdot \boldsymbol{n} \, \mathrm{d}u
\end{aligned}$$

We now differentiate both sides by t, bring the derivative under the integral sign and use the product rule to obtain

$$\begin{aligned}
\frac{\partial A}{\partial t} &= -\frac{1}{2}\int_0^{2\pi} \frac{\partial \boldsymbol{\gamma}}{\partial t} \cdot v\boldsymbol{n} + \boldsymbol{\gamma} \cdot \frac{\partial v}{\partial t}\boldsymbol{n} + \boldsymbol{\gamma} \cdot v\frac{\partial \boldsymbol{n}}{\partial t} \, \mathrm{d}u. \\
&= -\frac{1}{2}\int_0^{2\pi} \kappa v - \boldsymbol{\gamma} \cdot \kappa^2 v\boldsymbol{n} - \boldsymbol{\gamma} \cdot \frac{\partial \kappa}{\partial u}\boldsymbol{t} \, \mathrm{d}u \quad \text{(by (2.1), (2.3), and (2.7))} \\
&= -\frac{1}{2}\int_0^{2\pi} \kappa v - \boldsymbol{\gamma} \cdot \kappa^2 v\boldsymbol{n} + \kappa v + \boldsymbol{\gamma} \cdot v\kappa^2\boldsymbol{n} \, \mathrm{d}u \quad \text{(using integration by parts)} \\
&= -\int_0^{2\pi} \kappa v \, \mathrm{d}u \\
&= -\int_{\gamma_t} \kappa \, \mathrm{d}s.
\end{aligned}$$

The result follows from the fact that, with our choice of $\boldsymbol{n}$, the total curvature of a closed embedded curve is equal to 2π. $\square$

Remark 2.2.8. Lemma 2.2.7 together with Theorem 2.1.3 let us explicitly evaluate the maximal time T up to which the curve shortening flow can exist in terms of the area A_0 enclosed by the initial curve $\boldsymbol{\gamma}_0$. As the area enclosed by a curve is necessarily non-negative, the flow cannot exist beyond $T = \frac{A_0}{2\pi}$. We also know the flow exists until the curve shrinks to a point, at which point it must have zero area. Thus the flow cannot stop before $T = \frac{A_0}{2\pi}$. Combining these two results shows that the maximal T is indeed $\frac{A_0}{2\pi}$.

Remark 2.2.9. If we consider the curves $\boldsymbol{\gamma}_t$ as *unparametrised curves*– that is, we care only about their images in $\mathbb{R}^2$ and ignore the dependence on u– then we do not need to consider how the curve 'spreads out along itself' as it flows, since this does not change its image. That is, the component of $\frac{\partial \boldsymbol{\gamma}}{\partial t}$ in the direction of the tangent vector $\boldsymbol{t}$ does not affect the shape of the curve under curve shortening flow, but only its parametrisation, which we can ignore as we are looking at unparametrised curves. Since the normal and tangent vectors together form a basis of $\mathbb{R}^2$, for the purposes of curve shortening flow we only need to consider the flow in the direction normal to the curve. Thus, a family of curves satisfying the equation

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} \cdot \boldsymbol{n} = \kappa \boldsymbol{n} \cdot \boldsymbol{n} = \kappa, \tag{2.15}$$

where the second equality is immediately true since the unit normal vector has unit length, can be said to evolve under curve shortening flow.

2.3 Closed Embedded Plane Curves Shrink to Round Points

In this section we will again be considering closed embedded curves. We discuss the limiting shape of such curves as they shrink to points under the curve shortening flow. We make the notion of "convergence to round points" completely rigorous and discuss some of its implications.

In Theorem 2.1.3, we saw that curve shortening flow shrinks closed embedded curves to round points. Formally, this means that as it shrinks to a point, the curve becomes circular, in the sense that as $t \to T$ the following occur:

- The ratio of the inscribed radius of $\boldsymbol{\gamma}_t$ to its circumscribed radius approaches 1.
- The ratio of the maximum curvature of $\boldsymbol{\gamma}_t$ to its minimum curvature approaches 1.

- All space derivatives of κ converge uniformly to 0.

Here, the first bullet point, proved in [10], can be considered as "C^0" convergence to a circle. The second and third, proved in [7], can be considered as "C^2" and "C^∞" convergence respectively. As with Theorem 2.1.3, these were originally proved for convex curves only; the extension to all plane curves came when Grayson showed that a non-convex curve will flow to become convex in finite time [8].

The fact that convex curves become round is perhaps not so surprising: consider for example an ellipse with high eccentricity, given by the equation

$$(\varepsilon x)^2 + \left(\frac{y}{\varepsilon}\right)^2 = 1$$

for small $\varepsilon > 0$ (see Figure 2.2). Although the ellipse starts off very far from round, it is clear that its curvature at the left and right tips (i.e. the points at the major axis of the ellipse) is proportionally far greater than the curvature at the top and bottom tips (the points at the minor axis of the ellipse). In fact, parametrising the ellipse by

$$\left(\frac{1}{\varepsilon}\cos(t), \varepsilon\sin(t)\right)$$

for t in $[0, 2\pi]$, we can explicitly calculate that $\kappa = \varepsilon^3$ at the left and right tips of the ellipse and $\kappa = \varepsilon^{-3}$ at the top and bottom tips.

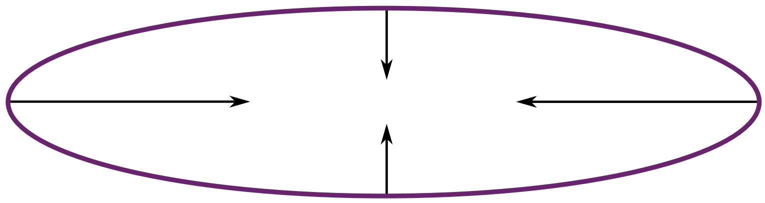

Figure 2.2: An ellipse with $\varepsilon = \frac{1}{2}$ drawn to scale.

In Figure 2.2, the arrows show the direction of the curvature vectors $\kappa \boldsymbol{n}$ at the tips of the ellipse, but these are not accurately scaled; if they had been, the vectors at the left and right tips would be 64 times larger than those at the top and bottom tips! This illustrates how a convex curve will become round, as the points further away from the centre are pushed inwards very rapidly.

However, it is not obvious that non-convex curves should also become round as they flow. This is illustrated well by a spiral, such as the one shown in Figure 2.3, which must 'unwind' before it collapses. In fact, by giving our spiral very thin arms and lots of turns, we can make an arbitrarily long and windy spiral which encloses an arbitrarily small area; then by Lemma 2.2.7, the time it takes to unwind can also be made arbitrarily small!

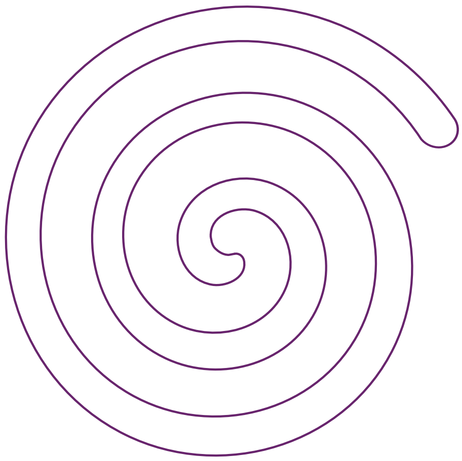

Figure 2.3: A spiral. It might seem surprising that the curve shortening flow can unravel this curve before it collapses to a point.

2.4 Curve Shortening Flow Keeps Embedded Curves Embedded

In this section, we follow section 3.2 in [7] in proving the following theorem:

Theorem 2.4.1. *Let* $\boldsymbol{\gamma}\colon S^1 \times [0, T')$ *be a family of closed curves satisfying equation* (2.1)*. If the initial curve* $\boldsymbol{\gamma}_0$ *is embedded and if there exists* $c \in \mathbb{R}$ *such that* $\kappa(u,t) \leq c$ *for all* $(u,t) \in S^1 \times [0, T')$*, then* $\boldsymbol{\gamma}_t$ *is an embedded curve for each* $t \in [0, T')$.

Note that the T' term in Theorem 2.4.1 is not the same as the T term in Theorem 2.1.3: when a curve shrinks to a point, its curvature approaches ∞ and so it is not uniformly bounded.

Intuitively, Theorem 2.4.1 makes sense: if an embedded curve was to develop a self-intersection under the curve shortening flow, then at the first time t_0 when intersection occurs, it must occur tangentially, or else there would be an earlier time when there was an intersection. Assuming the two intersecting parts of the curve have different curvatures, the motion of the curve shortening flow would separate the points (see Figure 2.4). But this implies that slightly before t_0, the curves must have had an intersection, which is a contradiction. Gage [11] argues this point in slightly less detail.

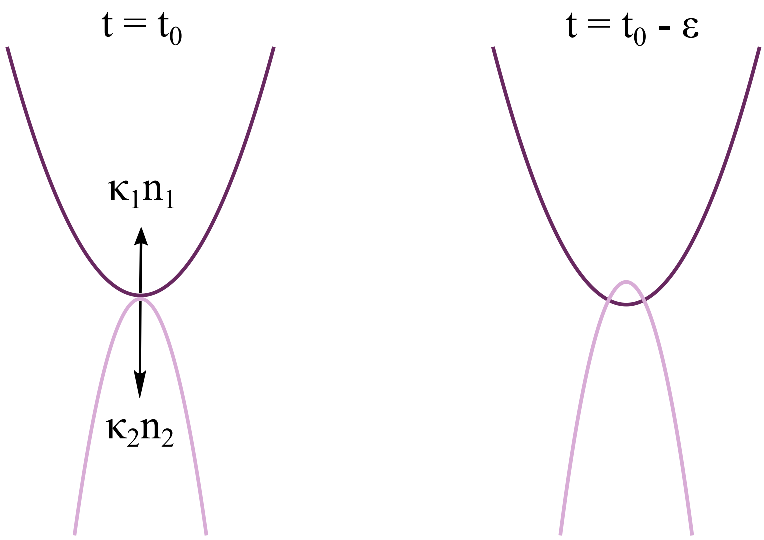

Figure 2.4: Two parts of a curve at the 'first' instant of touching t_0, and what they would have to look like slightly before. This gives a very rough idea of why we expect Theorem 2.4.1 to be true.

One of the main ingredients in the proof of Theorem 2.4.1 will be the function $f\colon S^1 \times S^1 \times [0, T') \to \mathbb{R}$ defined by

$$f(u_1, u_2, t) = |\boldsymbol{\gamma}(u_1, t) - \boldsymbol{\gamma}(u_2, t)|^2.$$

We will show that $f = 0$ only when $u_1 = u_2$; thus, for all t in $[0, T')$ we have that $\boldsymbol{\gamma}_t$ has no self-intersections and so indeed it is an embedded curve.

Before we begin proving Theorem 2.4.1, we will need two lemmas and a corollary. The first lemma concerns the behaviour of f, and the second uses curvature in a clever way to allow us to compare the distances of two line segments. The corollary uses the second lemma to give an upper bound on f in terms of a lower bound of the curvature of $\boldsymbol{\gamma}$.

Lemma 2.4.2. *The function f satisfies the heat equation*

$$\frac{\partial f}{\partial t} - \Delta f = -4, \tag{2.16}$$

where

$$\Delta f = \frac{\partial^2 f}{\partial s_1^2} + \frac{\partial^2 f}{\partial s_2^2},$$

and s_1 and s_2 are arclength parameters.

Proof. By definition,

$$f(s_1, s_2, t) = (\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \cdot (\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)).$$

Taking the t-derivative, we find that

$$\begin{aligned} \frac{\partial f}{\partial t} &= 2(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \cdot \frac{\partial}{\partial t}(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \\ &= 2(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \cdot (\kappa \boldsymbol{n}(s_1, t) - \kappa \boldsymbol{n}(s_2, t)) \quad \text{by equation (2.1)}, \end{aligned}$$

where we have slightly abused notation in favour of brevity by writing $\kappa \boldsymbol{n}(s, t)$ to mean $\kappa(s, t)\boldsymbol{n}(s, t)$.
Taking s_1-derivatives, we find that

$$\begin{aligned} \frac{\partial f}{\partial s_1} &= 2(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \cdot \frac{\partial}{\partial s_1}(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \\ &= 2(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \cdot \boldsymbol{t}(s_1, t), \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial^2 f}{\partial s_1^2} &= 2\left(\boldsymbol{t}(s_1, t) \cdot \boldsymbol{t}(s_1, t) + (\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)) \cdot \kappa \boldsymbol{n}(s_1, t)\right) \\ &= 2 + 2\left(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)\right) \cdot \kappa \boldsymbol{n}(s_1, t), \end{aligned}$$

where we have used equation (2.1) again.

Exploiting the antisymmetry of f immediately gives us

$$\frac{\partial^2 f}{\partial s_2^2} = 2 - 2\left(\boldsymbol{\gamma}(s_1, t) - \boldsymbol{\gamma}(s_2, t)\right) \cdot \kappa \boldsymbol{n}(s_2, t),$$

and a direct comparison of terms gives us the result. □

Since f satisfies a heat equation, we can apply the maximum principle to it. This is the main component of the proof of Theorem 2.4.1.

The next lemma, illustrated in Figure 2.5, is attributed to A. Schur and E. Schmidt [12].

Lemma 2.4.3. *Let γ_1 be a smooth curve of length L from A_1 to B_1 such that γ_1 together with the chord connecting A_1 and B_1 is a convex curve. Let γ_2 be a smooth curve of length L from A_2 to B_2. Assume that the absolute value of the curvature at each point of γ_1 is greater than or equal to the absolute value of the curvature at each corresponding point of γ_2: that is, for all s in $[0, L]$ we have that $|\kappa_1(s)| \geq |\kappa_2(s)|$. Then $d(A_1, B_1) \leq d(A_2, B_2)$.*

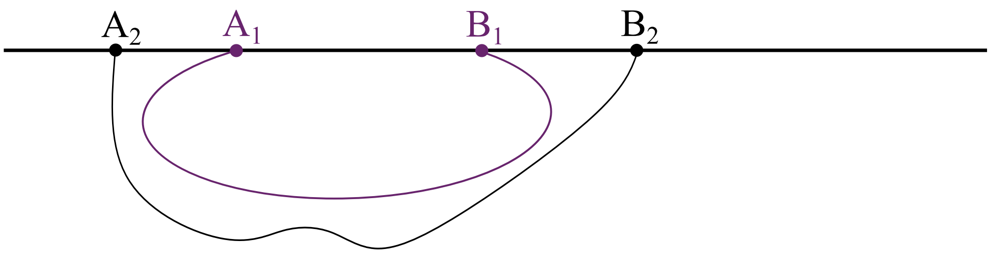

Figure 2.5: An illustration of Lemma 2.4.3; see Figure 1 in [7].

Proof. Parametrise both curves by the arclength parameter s. Without loss of generality, let γ_1 be oriented anticlockwise; if instead it is oriented clockwise, the proof is the same up to a change of sign in the curvature of γ_1. Take the unit normal vector to be given by an anticlockwise rotation of the unit tangent by $\frac{\pi}{2}$. With this choice of normal vector, since γ_1 is convex and oriented anticlockwise its curvature is non-negative.
Arrange the curves so A_1B_1 and A_2B_2 lie on the x-axis (see Figure 2.6). Define $\theta_i(s)$, $i \in \{1, 2\}$, to be the angle formed by the unit tangent vector to γ_i, so writing the tangents as column vectors in co-ordinates we have

$$\gamma_i'(s) = \begin{pmatrix} \cos(\theta_i(s)) \\ \sin(\theta_i(s)) \end{pmatrix}$$

Note that by definition of curvature,

$$\frac{d\theta_i}{ds} = \kappa_i,$$

and so by assumption we have an inequality of derivatives:

$$\frac{d\theta_1}{ds} \geq \left| \frac{d\theta_2}{ds} \right|.$$

Also, we are taking γ_1 to be oriented anticlockwise, so $\theta_1(0) < 0$. This together with our assumption that κ_1 is non-negative lets us conclude by monotonicity

that there is a unique minimal s_0 such that $\theta_1(s_0) = 0$, where θ_1 is negative for $s < s_0$ and non-negative for $s \geq s_0$.
If $s > s_0$, integrating from s_0 to s gives us

$$\begin{aligned}\int_{s_0}^{s} \frac{d\theta_1}{ds}\,\mathrm{d}s &\geq \int_{s_0}^{s} \left|\frac{d\theta_2}{ds}\right| \mathrm{d}s \\ &\geq \left|\int_{s_0}^{s} \frac{d\theta_2}{ds}\,\mathrm{d}s\right|,\end{aligned}$$

and so by evaluating these integrals with the fundamental theorem of calculus we obtain the inequality

$$\theta_1(s) \geq \left|\theta_2(s) - \theta_2(s_0)\right|.$$

On the other hand, if $s < s_0$ then performing the same integral yields

$$-\theta_1(s) \geq \left|\theta_2(s) - \theta_2(s_0)\right|.$$

Putting these two results together, we conclude that

$$|\theta_1(s)| \geq \left|\theta_2(s) - \theta_2(s_0)\right|.$$

Now, since $\boldsymbol{\gamma}_1$ is convex, for $s \in [0, L]$ we have that $|\theta_1(s)| \leq \pi$. In this range of values, cos is a decreasing function, so

$$\begin{aligned}d(A_1, B_1) = \int_0^L \cos\left(|\theta_1(s)|\right)\,\mathrm{d}s &\leq \int_0^L \cos\left(|\theta_2(s) - \theta_2(s_0)|\right)\,\mathrm{d}s \\ &\leq \int_0^L \cos\left(\theta_2(s) - \theta_2(s_0)\right)\,\mathrm{d}s \\ &\leq d(A_2, B_2),\end{aligned}$$

where the last inequality holds since the right-hand integral is the expression for the length of the projection of A_2B_2 onto the tangent line to $\boldsymbol{\gamma}_2(s_0)$.

□

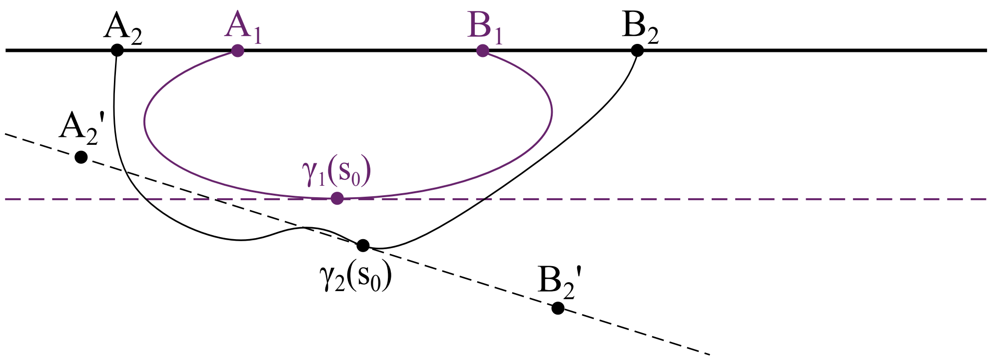

Figure 2.6: Figure 2.5 labelled as in the proof of Lemma 2.4.3; see Figure 1 in [7].

Corollary 2.4.4. *Suppose $\boldsymbol{\gamma}$ has uniformly bounded curvature: that is, $|\kappa(u,t)| \leq c$ for $c \in \mathbb{R}$. Define the function $l(u_1, u_2, t)$ to be the distance along the curve $\boldsymbol{\gamma}_t$ between points u_1 and u_2. Then the following equation holds:*

$$f(u_1, u_2, t) \geq \left(\frac{2}{c}\sin\left(\frac{c}{2}l(u_1, u_2, t)\right)\right)^2. \tag{2.17}$$

Proof. Apply Lemma 2.4.3, setting $\boldsymbol{\gamma}_1$ to be the arc of length $l(u_1, u_2, t)$ on the circle of radius $\frac{1}{c}$ and $\boldsymbol{\gamma}_2$ to be the segment of $\boldsymbol{\gamma}$ with endpoints $\boldsymbol{\gamma}(u_1)$ and $\boldsymbol{\gamma}(u_2)$. The expression for $d(A_1, B_1)$ follows immediately from the length of a chord of a circle, and is less than or equal to $d(A_2, B_2)$ which is by definition equal to $f(u_1, u_2, t)$. □

This corollary means that, when $l(u_1, u_2, t) < \frac{2\pi}{c}$, then $f(u_1, u_2, t) = 0$ if and only if $u_1 = u_2$. Geometrically, this can be interpreted as saying that $\boldsymbol{\gamma}$ has no self-intersections from local 'kinks', where the tighter our bound on the curvature, the shorter the range in which the word "locally" applies.

With this knowledge, we are well-equipped to prove Theorem 2.4.1. As with the other proofs in this section we follow the methods of Gage [7], but in particular our proof is much more explicit explaining why each step is logically valid, and contains some original justifications.

Proof of Theorem 2.4.1. First, split the domain $S^1 \times S^1 \times [0, T')$ of f into two parts:

$$E = \{(u_1, u_2, t) \,|\, l(u_1, u_2, t) < \frac{\pi}{c}\},$$

$$D = \left(S^1 \times S^1 \times [0, T')\right) \backslash E.$$

By Corollary 2.4.4, a necessary condition for $f(u_1, u_2, t) = 0$ is that $l(u_1, u_2, t)$ is an integer multiple of $\frac{2\pi}{c}$. On the set E, this is only possible when $l = 0$, i.e.

$u_1 = u_2$.
Now, the boundary of D is given by $D_1 \cup D_2$, where

$$D_1 = \{(u_1, u_2, t) \,|\, l(u_1, u_2, t) = \frac{\pi}{c}\},$$

$$D_2 = \{(u_1, u_2, 0) \,|\, l(u_1, u_2, t) \geq \frac{\pi}{c}\}.$$

On D_1, Corollary 2.4.4 implies that $f \geq \frac{4}{c^2}$. On D_2, since the initial curve is embedded then the minimum of f must be positive.
Take m to be the smaller of these two minima. Let $\varepsilon > 0$ be arbitrary, and introduce the function

$$g(u_1, u_2, t) = f(u_1, u_2, t) + \varepsilon t.$$

It immediately follows from Lemma 2.4.2 that g satisfies the heat equation

$$\frac{\partial g}{\partial t} - \Delta g = -4 + \varepsilon, \tag{2.18}$$

where as in Lemma 2.4.2 we take Δg with reference to the arclength parameter s. Note also that the s-derivatives of g are the same as the s-derivatives of f.
Take $\delta \in (0, m)$ and assume for a contradiction that g takes the value $m - \delta$ somewhere in D. Since $m - \delta < m$, this cannot happen on the boundary of D and so must happen somewhere in the interior. Continuity of g and compactness of D together imply that this happens for the first time at some point $(\hat{u}_1, \hat{u}_2, \hat{t})$. At this point, we must have $\frac{\partial g}{\partial t} \leq 0$. Indeed, assume for a contradiction this is not true, so there is some time $t_0 < \hat{t}$ with $g(\hat{u}_1, \hat{u}_2, t_0) < m - \delta$. Fixing $\hat{u}_1$ and $\hat{u}_2$, we can see that $g(\hat{u}_1, \hat{u}_2, t)$ is a continuous function of t which is at least m at $t = 0$, and thus by the intermediate value theorem there exists $t_1 \in (0, t_0)$ such that $g(\hat{u}_1, \hat{u}_2, t_1) = m - \delta$. This implies $\hat{t}$ is not the first time that g attains the value $m - \delta$, a contradiction.
A similar argument implies that $(\hat{u}_1, \hat{u}_2, \hat{t})$ is the point at which the function $g(u_1, u_2, \hat{t})$ of the two space variables u_1 and u_2 attains its minimum: if not, we could find some $(v_1, v_2, \hat{t})$ with $g(v_1, v_2, \hat{t}) < m - \delta$ and derive a contradiction with the same approach as used above. Therefore, at $(\hat{u}_1, \hat{u}_2, \hat{t}) = (\hat{s}_1, \hat{s}_2, \hat{t})$,

$$\frac{\partial^2 g}{\partial s_1^2}, \frac{\partial^2 g}{\partial s_2^2} \geq 0,$$

and by the second partial derivative test the inequality

$$\frac{\partial^2 g}{\partial s_1^2}\frac{\partial^2 g}{\partial s_2^2} - \left(\frac{\partial^2 g}{\partial s_1 \partial s_2}\right)^2 \geq 0 \tag{2.19}$$

holds, where we have changed variables to the arclength parameter s so we can easily work with explicit formulas for the partial derivatives.

At a minimum point of g we must have that the tangents $\boldsymbol{t}(s_1,t)$ and $\boldsymbol{t}(s_2,t)$ are parallel; this is clear from the geometric interpretation of f. Taking derivatives in the same way we did in Lemma 2.4.2, we calculate that

$$\begin{aligned}\frac{\partial^2 g}{\partial s_1 \partial s_2} &= -2\boldsymbol{t}(s_1,t)\cdot\boldsymbol{t}(s_2,t)\\ &= \pm 2. \end{aligned} \tag{2.20}$$

Now,

$$\begin{aligned}\Delta g &= \frac{\partial^2 g}{\partial s_1^2} + \frac{\partial^2 g}{\partial s_2^2} \quad \text{(by definition)}\\ &\geq 2\sqrt{\frac{\partial^2 g}{\partial s_1^2}\frac{\partial^2 g}{\partial s_2^2}} \quad \text{(by the AM-GM inequality)}\\ &\geq 2\left|\frac{\partial^2 g}{\partial s_1 \partial s_2}\right| \quad \text{(by (2.19))}\\ &\geq 4 \quad \text{(by (2.20))},\end{aligned}$$

and so equation (2.18) implies that $\frac{\partial g}{\partial t} > 0$, which contradicts the fact $\frac{\partial g}{\partial t} \leq 0$ we derived before.

Thus, $g \geq m$ on D, and letting $\varepsilon \to 0$ we conclude that $f \geq m > 0$ on D. This means that $f = 0$ on $S^1 \times S^1 \times [0, T')$ only where $u_1 = u_2$, which finishes our proof. □

Remark 2.4.5. A similar argument can be used to show "avoidance principles": if one smooth embedded curve is contained within another, then as they evolve under curve shortening flow they will never touch. Also, if an initial curve $\boldsymbol{\gamma}_0$ is contained in a convex set, then as it evolves under curve shortening flow it cannot leave the convex set.

2.5 Self-Similar Curves in Planar Curve Shortening Flow

In this section we look at self-similar solutions to curve shortening flow in $\mathbb{R}^2$. Such curves, which are not necessarily closed or embedded, do not change under curve shortening flow up to an affine transformation. Clearly, the straight line (which has zero curvature and so is unchanged) and the circle (which shrinks to a point while maintaining its shape as seen in Example 2.1.2) are examples, but we will see that there are in fact other more exotic families of curves which exhibit this behaviour.

Suppose we have a smooth family of curves $\boldsymbol{\gamma}(u,t)$ satisfying the first part of equation (2.1). Suppose also that $\boldsymbol{\gamma}$ moves under curve shortening flow by vertical translation at a constant unit rate. A curve moving in this way is an example of a *soliton*, a type of wave studied in mathematical physics. Assume we can parametrise $\boldsymbol{\gamma}$ as the graph of a function:

$$\boldsymbol{\gamma}(u,t) = (u, f(u) + t)\,.$$

This implies that

$$\begin{aligned}\frac{\partial \gamma}{\partial t} &= (0,1)\,,\\ \kappa(u,t) &= \frac{f''(u)}{\left(1+(f'(u))^2\right)^{\frac{3}{2}}},\\ \boldsymbol{n}(u,t) &= \frac{1}{\left(1+(f'(u))^2\right)^{\frac{1}{2}}}\left(-f'(u)\,,1\right).\end{aligned}$$

Then using (2.15) we have a differential equation for f of the form

$$\frac{f''(u)}{\left(1+(f'(u))^2\right)} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\arctan\left(f'(u)\right)\right) = 1.$$

The above differential equation can be solved to give us

$$f(u) = c_1 - \log\left(\cos\left(u + c_2\right)\right),$$

with c_1 and c_2 real constants of integration that define the vertical and horizontal positioning of $\boldsymbol{\gamma}$ respectively. This curve is called the *Grim Reaper* (see Figure 2.7), discovered by Calabi [8, 13]; if any closed curve is inside the Grim Reaper, then under curve shortening flow it will shrink to a point and 'die' before the Grim Reaper sweeps past it (like a scythe)! This can be shown using the avoidance principles mentioned in Remark 2.4.5.

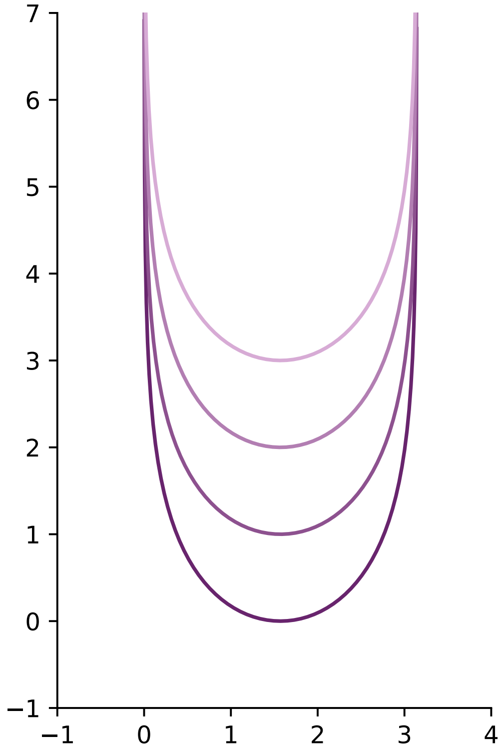

Figure 2.7: The Grim Reaper curve evolving under curve shortening flow. Shown here are the curves at times $t \in \{0, 1, 2, 3\}$ where the colour of the curve gets lighter as time goes on. We can clearly see the curve flows at constant speed.

Other curves exist which under curve shortening flow do not change their shape but will scale, rotate, or scale and rotate: a complete classification of all such curves is given in [13]. The method used is the same as we present below, with the minor technicality that the curves are considered as belonging to $\mathbb{C}$ rather than $\mathbb{R}^2$ and so the machinery of complex numbers is used instead of matrices; the difference is purely notational though and does not change any of the final results.

Suppose $\boldsymbol{\gamma}(u, t)$ evolves under curve-shortening flow in a self-similar manner, so at each $t \in \mathbb{R}^{\geq 0}$ for which it is defined the curve $\boldsymbol{\gamma}_t(u)$ is the image of $\boldsymbol{\gamma}_0(u)$ under some transformation. Here, "a self-similar manner" means that the transformation is affine: either a translation, a rotation about the origin, a dilation centred at the origin, or some combination thereof. We can explicitly write

$$\boldsymbol{\gamma}(u, t) = g(t) \begin{pmatrix} \cos(f(t)) & -\sin(f(t)) \\ \sin(f(t)) & \cos(f(t)) \end{pmatrix} \boldsymbol{\gamma}_0(u) + \boldsymbol{h}(t), \tag{2.21}$$

where g and f are scalar-valued functions representing the dilation and rotation terms respectively, $\boldsymbol{h}$ is a vector-valued function representing the translation term, and these functions satisfy $f(0) = 0$, $g(0) = 1$, and $\boldsymbol{h}(0) = \mathbf{0}$ so that the equation holds at $t = 0$.

We now consider the translation term $\boldsymbol{h}$. If our transformation is purely a translation, i.e. $g(t) = 1$ and $f(t) = 0$, then we have two cases to consider. Firstly, in the trivial case $\boldsymbol{h}(t) = \mathbf{0}$, then $\boldsymbol{\gamma}$ is unchanged under curve shortening flow and must be a geodesic i.e. a straight line. Secondly, if $\boldsymbol{h}(t) \neq \mathbf{0}$, [14] shows that it is the Grim Reaper.
If, on the other hand, our transformation is a combination of a translation and a rotation and/or dilation, [13] shows we can remove the translation term by shifting the plane and taking a new choice of origin. So we can simplify matters by taking $\boldsymbol{h}(t) \equiv \mathbf{0}$.

Without loss of generality, take $\boldsymbol{n}$ to be the normal vector given by an anticlockwise rotation by $\frac{\pi}{2}$ of the unit tangent $\boldsymbol{t}$.

Now, we can exploit the self-similarity of the flow to get simplified equations for the tangent, normal and curvature in terms of their values at $t = 0$:

$$\boldsymbol{t}(u,t) = \begin{pmatrix} \cos(f(t)) & -\sin(f(t)) \\ \sin(f(t)) & \cos(f(t)) \end{pmatrix} \boldsymbol{t}_0(u), \tag{2.22}$$

$$\boldsymbol{n}(u,t) = \begin{pmatrix} \cos(f(t)) & -\sin(f(t)) \\ \sin(f(t)) & \cos(f(t)) \end{pmatrix} \boldsymbol{n}_0(u), \tag{2.23}$$

$$\kappa(u,t) = \frac{\kappa_0(u)}{g(t)}. \tag{2.24}$$

We will now use these equations to derive the conditions under which $\boldsymbol{\gamma}$ evolves by curve shortening flow. By Remark 2.2.9, a sufficient condition is that $\frac{d\boldsymbol{\gamma}}{dt} \cdot \boldsymbol{n} = \kappa$. First, we take the derivative of the rotation matrix:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \cos(f(t)) & -\sin(f(t)) \\ \sin(f(t)) & \cos(f(t)) \end{pmatrix} &= -f'(t) \begin{pmatrix} \sin(f(t)) & \cos(f(t)) \\ -\cos(f(t)) & \sin(f(t)) \end{pmatrix} \\ &= -f'(t) \begin{pmatrix} \cos(f(t) - \frac{\pi}{2}) & -\sin(f(t) - \frac{\pi}{2}) \\ \sin(f(t) - \frac{\pi}{2}) & \cos(f(t) - \frac{\pi}{2}) \end{pmatrix}. \end{aligned}$$

Writing these as operators,

$$\frac{\mathrm{d}}{\mathrm{dt}} R_{f(t)} = -f'(t) R_{f(t)-\frac{\pi}{2}},$$

where R_θ denotes rotation anticlockwise by θ.

Now, taking the partial derivative with respect to t of equation (2.21) with $\boldsymbol{h}$ set to 0, we compute

$$\frac{\partial \gamma}{\partial t}(u,t) = \left(g'(t)R_{f(t)} - f'(t)g(t)R_{\frac{\pi}{2}-f(t)}\right)\gamma_0(u).$$

Taking the dot product with the unit normal immediately gives

$$\kappa(u,t) = \left(g'(t)R_{f(t)} - f'(t)g(t)R_{f(t)-\frac{\pi}{2}}\right)\gamma_0(u) \cdot \boldsymbol{n}(u,t).$$

Using equations (2.22)–(2.24), this simplifies to

$$\begin{aligned}
\frac{\kappa_0(u)}{g(t)} &= \left(g'(t)R_{f(t)} - f'(t)g(t)R_{f(t)-\frac{\pi}{2}}\right)\gamma_0(u) \cdot R_{f(t)}\boldsymbol{n}_0(u) \\
&= g'(t)\boldsymbol{\gamma}_0(u) \cdot \boldsymbol{n}_0(u) - f'(t)g(t)R_{-\frac{\pi}{2}}\boldsymbol{\gamma}_0(u) \cdot \boldsymbol{n}_0(u) \\
&= g'(t)\boldsymbol{\gamma}_0(u) \cdot \boldsymbol{n}_0(u) - f'(t)g(t)\boldsymbol{\gamma}_0(u) \cdot R_{\frac{\pi}{2}}\boldsymbol{n}_0(u) \\
&= g'(t)\boldsymbol{\gamma}_0(u) \cdot \boldsymbol{n}_0(u) + f'(t)g(t)\boldsymbol{\gamma}_0(u) \cdot \boldsymbol{t}_0(u),
\end{aligned}$$

where in the last two lines we have used the fact that the dot product $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$ is invariant when both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are rotated by the same amount, and that $\boldsymbol{n} = R_{\frac{\pi}{2}}\boldsymbol{t}$.

These equations must hold for all time if our curve is to be self-similar under curve shortening flow, and so in particular they must hold at $t = 0$. Substituting this into the equation above, and applying our initial conditions on f and g, we can see the initial curve $\boldsymbol{\gamma}$ must satisfy

$$f'(0)\boldsymbol{\gamma}_0(u) \cdot \boldsymbol{t}_0(u) + g'(0)\boldsymbol{\gamma}_0(u) \cdot \boldsymbol{n}_0(u) = \kappa_0(u). \tag{2.25}$$

We will now show that for any pair $(A, B) := (f'(0), g'(0))$, there exists a curve $\boldsymbol{\gamma}_0$ satisfying equation (2.25); we do so by explicitly constructing the curve. Since we are no longer working with a time parameter, we will drop the subscripts from our notation to make things easier to read. Thus, equation (2.25) becomes:

$$A\boldsymbol{\gamma}(u) \cdot \boldsymbol{t}(u) + B\boldsymbol{\gamma}(u) \cdot \boldsymbol{n}(u) = \kappa(u). \tag{2.26}$$

Parametrising by arclength, we define two functions x and y to be the unique solutions of the ODE

$$\begin{cases} x' = xy + A, \\ y' = -x^2 - B, \end{cases}$$

for any initial values $x(0)$, $y(0)$.
In addition, define

$$\theta(s) = \int_0^s x(\tilde{s})\,\mathrm{d}\tilde{s} + \theta_0,$$

for any θ_0.
Note that the definition of θ implies by the fundamental theorem of calculus that $\theta'(s) = x(s)$. It will turn out that, up to equivalence by translation or rotation (or both), the three values $x(0)$, $y(0)$ and θ_0 define the same curve and so we really can pick any values for them.
Now, writing points in $\mathbb{R}^2$ as column vectors rather than row vectors to save space, define our curve by

$$\boldsymbol{\gamma}(s) = \frac{1}{A^2+B^2}\begin{pmatrix} \cos(\theta(s)) & -\sin(\theta(s)) \\ \sin(\theta(s)) & \cos(\theta(s)) \end{pmatrix}\begin{pmatrix} Ax(s) - By(s) \\ Bx(s) + Ay(s) \end{pmatrix}.$$

Differentiating this using the product rule and substituting in our relations for x', y', and θ' in terms of x and y gives us the expression

$$\boldsymbol{\gamma}'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix},$$

and we compute that $\kappa(s) = \theta'(s) = x(s)$ using the Serret-Frenet equations.
On the other hand, after some routine computations we obtain that

$$A\boldsymbol{\gamma}(s)\cdot\boldsymbol{t}(s) + B\boldsymbol{\gamma}(s)\cdot\boldsymbol{n}(s) = x(s) = \kappa(s),$$

i.e. equation (2.26) is satisfied.

Thus, we have shown that for every pair (A, B) there exists a curve $\boldsymbol{\gamma}$ satisfying (2.26). In fact, Halldorsson [13] goes even further by characterising how the curves behave under curve shortening flow depending on (A, B). The following result is obtained:

- If $A = B = 0$, the curve is a straight line and is invariant under curve shortening flow.
- If $A = 0$, $B > 0$, the curve expands.
- If $A = 0$, $B < 0$, the curve shrinks.
- If $A \neq 0$, $B > 0$, the curve rotates and expands.
- If $A \neq 0$, $B = 0$, the curve rotates.
- If $A \neq 0$, $B < 0$, the curve rotates and shrinks.

In the last three cases, the rotation is clockwise if A is negative and anticlockwise if A is positive.

In Figures 2.8 and 2.9, two curves which rotate in the anticlockwise direction under the flow are shown. Such curves form a one-dimensional family parametrised by their distance d to the origin; when this distance becomes 0, we get Altschuler's yin-yang curve [15].

These cases combined with the Grim Reaper give us every possible curve which is invariant under the curve shortening flow in $\mathbb{R}^2$. Note that not all such curves are embedded, and not all curves exist forever under the flow.

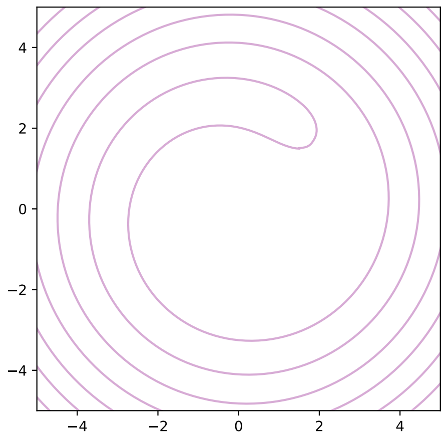

Figure 2.8: A curve which rotates under the curve shortening flow with distance from the origin $d = \frac{3}{2}\sqrt{2}$. The tip of this curve looks extremely similar to the Grim Reaper.

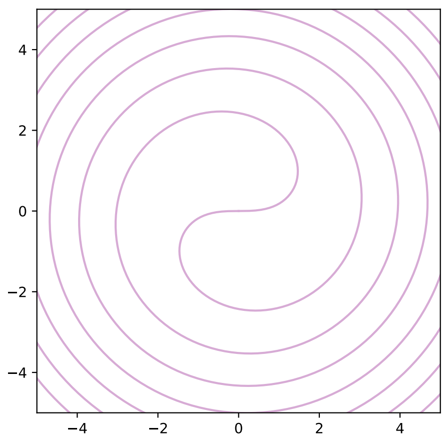

Figure 2.9: Altschuler's yin-yang curve, which rotates under the curve shortening flow with distance from the origin $d = 0$.

Chapter 3

Curve Shortening Flow on Surfaces

In this chapter, we generalise the notion of curve shortening flow from $\mathbb{R}^2$ to a general two-dimensional complete orientable Riemannian manifold, which we call a *surface*. We look at some properties of the curve shortening flow, both extending our existing results from Chapter 2 and including some new results which are not relevant in the plane. We state an important result, conjectured by Gage & Hamilton [7] then proved by Grayson [16], that under suitable conditions a closed embedded curve either evolves forever under curve shortening flow and approaches a closed embedded geodesic or shrinks to a point in finite time. Finally, we discuss what is currently known about curves which are self-similar under curve shortening flow in manifolds other than $\mathbb{R}^2$.

3.1 Introduction to Curve Shortening Flow on a Surface

In this section, we define curve shortening flow on a surface, then look at the existence and long-term behaviour of the flow.

Definition 3.1.1. For a two-dimensional complete orientable manifold M, let the curve $\boldsymbol{\gamma}_0(u) : I \to M$ be smooth, and let $T > 0$. We say that a family of curves $\boldsymbol{\gamma}(u,t) : I \times [0,T) \to M$ *evolves under curve shortening flow* if it satisfies the initial value problem

$$\begin{cases} \frac{\partial \boldsymbol{\gamma}}{\partial t} = \kappa \boldsymbol{n}, \\ \boldsymbol{\gamma}(u,0) = \boldsymbol{\gamma}_0(u), \end{cases} \tag{3.1}$$

where $\kappa(u,t)$ is the curvature of $\boldsymbol{\gamma}$ and $\boldsymbol{n}$ is a consistent unit normal vector to $\boldsymbol{\gamma}$.

We restrict ourselves to orientable manifolds since equation (3.1) makes use of a *consistent* normal vector; if our manifold was not orientable, then by definition there would be no such consistent vector and the equation would not even be well-defined. We restrict ourselves to complete manifolds so the curve shortening flow does not run into any issues from 'hitting the boundary' of the manifold.

There are as always two choices for which normal vector field we take. However, as discussed in Chapter 1, the quantity $\kappa \boldsymbol{n}$ does not depend which choice we take, and so the curve shortening flow is indeed well-defined.

The main result we discuss for curve shortening flow on a surface concerns the long-term existence and behaviour of the curve shortening flow on closed embedded curves, providing an analogue to the Gage-Hamilton-Grayson theorem for the plane. It was conjectured by Gage and Hamilton in [7] and proved by Grayson [16]:

Theorem 3.1.2. *Let (M, g) be a smooth orientable two-dimensional Riemannian manifold which is convex at infinity. Let $\boldsymbol{\gamma}_0 \colon S^1 \to M$ be a smooth closed embedded curve. Then there exists a maximal $T \in \mathbb{R}^+ \cup \{\infty\}$ such that (3.1) has a solution for $t \in [0, T)$, and the following hold:*

- *If T is finite, γ shrinks to a point.*
- *If T is infinite, the curvature of γ and all its t-derivatives converge to 0 and γ approaches a geodesic.*

Recall here from Definition 1.2.8 that the condition of *convexity at infinity* means that every compact subset of M has a convex hull that is itself compact. This condition is used to 'trap' the flowing curves in a compact set: the initial curve $\boldsymbol{\gamma}_0$ is itself compact, and thus by convexity at infinity its convex hull V, convex by definition, is compact also. We will see later by extending Remark 2.4.5 to manifolds that if the initial curve $\boldsymbol{\gamma}_0$ is contained in a convex set, then so too is $\boldsymbol{\gamma}_t$ for all t in $(0, T)$. Thus, each $\boldsymbol{\gamma}_t$ is contained in the compact set V. This stops the curves from "sliding off to infinity" as they evolve.

If M is not convex at infinity, then the evolution of $\boldsymbol{\gamma}_0$ under curve shortening flow will not necessarily be contained in a compact set, which gives rise to counterexamples to Theorem 3.1.2. For example, [16] gives the example of a punctured torus with a complete finite-volume hyperbolic structure: this looks like a torus with a horn attached which gets thinner and thinner as it moves out to infinity, where the puncture is located (see Figure 3.1). In this case, horocycles around the cusp of the horn will move out along the horn forever. These curves are not contractible and so do not shrink to a point, but their curvature $\kappa \equiv 1$ is kept constant at 1 and so they do not approach a geodesic either.

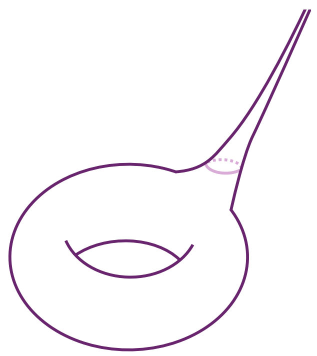

Figure 3.1: An illustration of part of a punctured torus with puncture "at infinity".

3.2 Properties of Curve Shortening Flow on Surfaces

In this section, we extend our results from Chapter 2 to apply to surfaces. We see that some properties carry over simply, applying exactly as they did in the plane, but some are more complicated and require modification to work on surfaces.

Remark 3.2.1. The following properties of curve shortening flow hold, with proofs identical (or nearly identical) to those from Chapter 2:

$$\text{(i)} \quad \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = \kappa^2 \frac{\partial}{\partial s}, \tag{3.2}$$

$$\text{(ii)} \ \nabla_{\frac{\partial}{\partial t}} \boldsymbol{t} = \frac{\partial \kappa}{\partial s}\boldsymbol{n}\,,\ \nabla_{\frac{\partial}{\partial t}} \boldsymbol{n} = -\frac{\partial \kappa}{\partial s}\boldsymbol{t}, \tag{3.3}$$

$$\text{(iii)} \ \frac{\mathrm{d}L}{\mathrm{d}t} = -\int_{\gamma_t} \kappa^2 \,\mathrm{d}s, \tag{3.4}$$

where $[\cdot,\cdot]$ is the commutator and ∇ is the covariant derivative.

These results still hold because their proofs did not use any properties specific to $\mathbb{R}^2$ which do not hold in a general manifold. However, not all of the results from Chapter 2 carry on in the same way. Specifically, the formulae for the evolution of curvature and enclosed area must be updated to account for the manifold we are in.

Lemma 3.2.2. *The curvature changes with time following the equation*

$$\nabla_{\frac{\partial}{\partial t}} \boldsymbol{t} = \frac{\partial \kappa}{\partial s}\boldsymbol{n}. \tag{3.5}$$

This lemma has its origins in Section 1 of [11], whose proof we follow.

Proof. Using the properties of the covariant derivative, we first calculate

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}}\kappa\boldsymbol{n} &= \frac{\partial\kappa}{\partial t}\boldsymbol{n} + \kappa\nabla_{\frac{\partial}{\partial t}}\boldsymbol{n} \\ &= \frac{\partial\kappa}{\partial t}\boldsymbol{n} - \kappa\frac{\partial\kappa}{\partial s}\boldsymbol{t} \text{ by (3.3).}\end{aligned}$$

We can also compute the same quantity in a different way: denoting the Riemann curvature tensor by R, we get that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}}\kappa\boldsymbol{n} &= \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\boldsymbol{t} \quad \text{(by definition of curvature)} \\ &= R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)\boldsymbol{t} + \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\boldsymbol{t} + \nabla_{\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]}\boldsymbol{t} \quad \text{(by definition of } R\text{)} \\ &= R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)\boldsymbol{t} + \nabla_{\frac{\partial}{\partial s}}\frac{\partial\kappa}{\partial s}\boldsymbol{n} + \nabla_{\kappa^2\frac{\partial}{\partial s}}\boldsymbol{t} \quad \text{(by (3.2) and (3.3))} \\ &= R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)\boldsymbol{t} + \frac{\partial^2\kappa}{\partial s^2}\boldsymbol{n} - \kappa\frac{\partial\kappa}{\partial s}\boldsymbol{t} + \kappa^2\nabla_{\frac{\partial}{\partial s}}\boldsymbol{t} \quad \text{(by (3.3))} \\ &= R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)\boldsymbol{t} + \frac{\partial^2\kappa}{\partial s^2}\boldsymbol{n} - \kappa\frac{\partial\kappa}{\partial s}\boldsymbol{t} + \kappa^3\boldsymbol{n} \quad \text{(by the definition of curvature.)}\end{aligned}$$

We now compare the components in these two expressions, resulting in the equation

$$\frac{\partial\kappa}{\partial t}\boldsymbol{n} = R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)\boldsymbol{t} + \frac{\partial^2\kappa}{\partial s^2}\boldsymbol{n} + \kappa^3\boldsymbol{n}.$$

Taking the inner product of both sides with $\boldsymbol{n}$ and applying the definition of sectional curvature gives us the result. □

Lemma 3.2.3. *Let $\boldsymbol{\gamma}_0\colon S^1 \to M$ be a closed embedded curve, and consider its evolution under curve shortening flow. Define Ω_t to be the region enclosed by $\boldsymbol{\gamma}_t$, where the normal vector to $\boldsymbol{\gamma}_t$ points inside of Ω_t, and let $A(t)$ denote the area of Ω_t. Suppose that Ω_t is diffeomorphic to a disc. Then the derivative of A is given by*

$$\frac{\mathrm{d}A}{\mathrm{dt}} = \int_{\Omega_t} K\,\mathrm{d}A - 2\pi, \tag{3.6}$$

where K represents the Gaussian curvature of M.

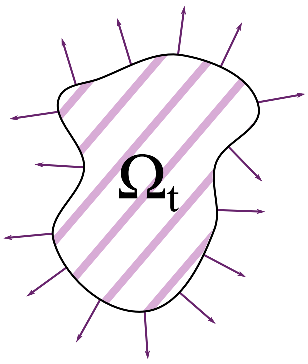

Figure 3.2: An example of a region Ω_t as in Lemma 3.2.3.

Proof. With notation as above,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\int_{\gamma_t} v \, \mathrm{d}s,$$

where v is the velocity in the normal direction. By definition, we have $v = \kappa$, so

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\int_{\gamma_t} \kappa \, \mathrm{d}s.$$

Applying the Gauss-Bonnet theorem, and noting that as Ω_t is diffeomorphic to a disc it has Euler characteristic equal to 1, we get that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \int_{\Omega_t} K \, \mathrm{d}A - 2\pi$$

as required. □

Remark 3.2.4. Lemma 3.2.3 implies that if M is a manifold which has non-positive Gaussian curvature everywhere, then every closed embedded curve bounding a finite disc-like region Ω will shrink to a point in finite time and thus will not approach a geodesic. This is easy to see: the rate of change of area enclosed by the curve will be bounded above by a negative number less than or equal to 2π, and so the area must eventually reach 0. Even on manifolds which admit positive Gaussian curvature, it is sometimes possible to use a similar trick to show that a curve must shrink to a point. On the other hand, in some circumstances Lemma 3.2.3 lets us show that certain types of curves will exist forever under curve shortening flow and therefore necessarily approach a geodesic.

Example 3.2.5. The unit sphere S^2 serves as a good example to illustrate the latter parts of Remark 3.2.4. It is well-known that the unit sphere has constant

Gaussian curvature $K \equiv 1$, and so equation (3.6) becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t} = A(t) - 2\pi,$$

a first order linear ODE which can easily be solved to give

$$A(t) = ce^t + 2\pi.$$

Our initial conditions force c to equal $A(0) - 2\pi$, so this equation becomes

$$A(t) = (A(0) - 2\pi)e^t + 2\pi.$$

If $A(0) < 2\pi$, then $A(t)$ is a strictly decreasing function and we can work out that when $t = \log(2\pi) - \log(2\pi - A(0))$, $A(t) = 0$ and so our curve must shrink to a point. If $A(0) > 2\pi$, we can switch our choice of normal vector so $S^2 \backslash \Omega_t$ becomes the 'new' Ω_t); then the area of $S^2 \backslash \Omega_0$) is equal to $4\pi - A(0) < 4\pi - 2\pi = 2\pi$ and so the first situation applies again. Finally, the initial condition $A(0) = 2\pi$ means that $c = 0$ and so $A(t)$ is constant: in particular, our initial curve can never shrink to a point! Note that 2π is exactly half the area of the sphere: this example tells us that an embedded curve on the sphere exists forever under curve shortening flow if and only if it splits the sphere into two pieces of equal area.

It is also possible to extend the result of Theorem 2.4.1 to surfaces, which is done in the following theorem:

Theorem 3.2.6. *Let $\boldsymbol{\gamma}\colon S^1 \times [0, T') \to M$ be a family of closed curves satisfying equation (3.1). If the initial curve $\boldsymbol{\gamma}_0$ is embedded and there exists $c \in \mathbb{R}$ such that $\kappa(u,t) \leq c$ for all $(u,t) \in S^1 \times [0,T')$, then $\boldsymbol{\gamma}_t$ is an embedded curve for each $t \in [0, T')$.*

This theorem, proved by Gage in Section 3 of [11], extends Theorem 2.4.1 from the plane to a general 2-dimensional orientable manifold. The overall idea behind the proof of Theorem 2.4.1 is the same as that in Theorem 2.4.1– use of the maximum principle to show that the "first" time when the curve intersects itself was actualy not the first time at all, and so it cannot have happened– but working in a general two-dimensional manifold instead of $\mathbb{R}^2$ means that the details are much more complicated. This results in a technical and lengthy proof (5 pages long in [11], which goes into far less detail than we would need to), so we will not show it here.

Remark 3.2.7. The avoidance principles of Remark 2.4.5 also extend to manifolds.

3.3 Convergence to Geodesics

In this section we discuss the relation between curve shortening flow and geodesics, and see why a curve that can flow forever under curve shortening flow will approach a geodesic. Note that in [16], convergence to a *unique* geodesic was not shown; it was only shown that the curvature of the curve converges to 0 in the C^∞ norm, in the sense that as $t \to \infty$ the curvature and all its time-derivatives uniformly approach 0. This is not as trivial as it may initially seem: given only this knowledge it is possible, for example, that a curve converges to an infinite set of closed geodesics and "vibrates" between them. However, it has been shown that in this case the limiting geodesics all have the same length (see Lemma 3.3.1 below) and that any two of them share at least one intersection point, with the number of intersection points not depending on the geodesics chosen [16, Section 7]. Grayson [16] conjectured that an analytic Riemannian metric is a sufficient condition for a unique limit. In [11] it was shown that on the unit two-sphere, the limiting geodesic is unique. Section 5 of [17] contains an example on $\mathbb{R}^2 \times S^1$ where the curve shortening flow has a non-unique limit.

Lemma 3.3.1. *If γ_0 is a closed embedded curve which evolves under curve shortening flow, with $T = \infty$ as in Theorem 3.1.2, then the length of the curve $L(\gamma_t)$ converges to some $L_\infty > 0$ as $t \to \infty$.*

Proof. First, we note that $L(t)$ is a decreasing function which is bounded below by 0, so it must converge to some value $L_\infty \geq 0$.
Assume for a contradiction that $L_\infty = 0$.
Parametrising γ_0 with space parameter $u \in [0, 2\pi]$ as we did in Chapter 2, the image γ_0 is the image of a compact set $[0, 2\pi]$ under a continuous function. Thus $\gamma_0 \subset M$ is compact as a subset of M.
Since we are assuming M to be convex at infinity, the convex hull V of γ_0 is compact. Also, V is convex by definition, and so by Remark 3.2.7 γ_0 cannot leave V as it evolves under curve shortening flow. In particular, $\gamma_t \subset V$ for all $t \geq 0$ such that γ_t is defined.
By compactness, the injectivity radius is bounded away from 0 by some constant inj_V on V. Pick t_0 sufficiently large such that $L(t) \leq \mathrm{inj}_V$ for $t \geq t_0$, so the exponential map is a diffeomorphism on the set of tangent vectors with length at most $L(t_0)$. Taking $p = p(t) = \gamma(u, t)$ for some fixed u, it is clear that $\gamma_t \subset \hat{B}_{L(t)}(p) := \exp(B_{L(t)}(p))$.
Now, $\hat{B}_{l(t)}(p)$ is the diffeomorphic image of an open ball in $T_pM \cong \mathbb{R}^2$, so its boundary is diffeomorphic to the circle S^1. In particular, the boundary is compact. Let W be the convex hull of the boundary; again, as M is convex at infinity then W is compact. By compactness, the sectional (Gaussian) curvature K is bounded on W, and so is bounded on $\hat{B}_{L(t)}(p)$, which is a subset of W by the definition of convex hull.

Define Ω_t to be the region enclosed by γ_t. It immediately follows that as sets, $\Omega_t \subset \hat{B}_{L(t)}(p)$.
Let $\tilde{M}$ be the manifold of constant Gaussian curvature $K_{\min}$. Then we have that

$$0 \leq \text{vol } \Omega_t \leq \text{vol } \hat{B}_{L(t)}(p) \leq \text{vol } \tilde{B}_{L(t)},$$

where $\tilde{B}_{L(t)}$ is the (geodesic) ball of radius $l(t)$ around any point in $\tilde{M}$; the last inequality follows from the Bishop-Gromov inequality.
We know that vol $\tilde{B}_{L(t)} \to 0$ as $t \to \infty$, and so vol $\Omega_t \to 0$ tends to 0 as well.
Furthermore, Ω_t is the interior of $\boldsymbol{\gamma}_t$, an embedding which lies inside the diffeomorphic image of an open ball $\hat{B}_{L(t)}(p)$. Thus, it is itself diffeomorphic to an open ball and so has an Euler characteristic of 1, which means that by the Gauss-Bonnet theorem we obtain the relation

$$\int_0^{L(t)} \kappa \, \mathrm{d}s = 2\pi - \int_{\Omega_t} K \, \mathrm{d}A. \tag{3.7}$$

As K is bounded on W, we have that

$$K_{\min} \int_{\Omega_t} \mathrm{d}A \leq \int_{\Omega_t} K \, \mathrm{d}A \leq K_{\max} \int_{\Omega_t} \mathrm{d}A,$$

and by taking the limit as $t \to \infty$ and squeezing we have that the middle term in (3.7) decreases to 0. This implies that for arbitrary $\varepsilon > 0$, there is a time $t_1 \geq t_0$ such that for all $t \geq t_1$,

$$\int_0^{L(t)} \kappa \, \mathrm{d}s \geq 2\pi - \varepsilon.$$

We now get a chain of inequalities:

$$\begin{aligned}
2\pi - \varepsilon &\leq \int_0^{L(t)} \kappa \, \mathrm{d}s \\
&\leq \left(\int_0^{L(t)} \kappa^2 \, \mathrm{d}s \int_0^{L(t)} 1 \, \mathrm{d}s \right)^{\frac{1}{2}} && \text{by the Hölder inequality} \\
&\leq \int_0^{L(t)} \kappa^2 \, \mathrm{d}s \int_0^{L(t)} 1 \, \mathrm{d}s && \text{as this quantity is greater than 1} \\
&= -L(t)L'(t) \text{ by (3.4)} \\
&\leq -L(0)L'(t) && \text{as } L(t) \text{ is decreasing.}
\end{aligned}$$

This means that $L'(t)$ is bounded away from 0, and so $L(t)$ will become 0 in finite time, i.e. $\boldsymbol{\gamma}_t$ will shrink to a point in finite time. But this contradicts our assumption that $T = \infty$. Therefore if $T = \infty$, the limiting length L_∞ must be positive. □

The above proof is similar to the one found in Section 2.2.2 of [18], but we have made it more general as we do not assume M is compact. The original proof, found in section 7 of [16], is along the same lines but does not go into as much detail as we have done here.

To show that the curvature approaches 0 in the C^∞ norm, the following theorem must be proved:

Theorem 3.3.2. *If the closed embedded curve γ_0 evolves under curve shortening flow forever, then for all $n \in \mathbb{N} \cup \{0\}$ the following limit holds:*

$$\lim_{t\to\infty} \max(\kappa^{(n)}) = 0,$$

where we use the convention that $\kappa^{(n)}$ denotes the n^{th} time derivative of κ, i.e.

$$\kappa^{(n)} = \frac{\partial^n}{\partial t^n}\kappa,$$

and we are considering it as a function of u for fixed t.

Originally, this theorem was stated with sup instead of max. However, we know that κ must be bounded on each γ_t, as the curve shortening flow exists forever by assumption and so equation (3.1) must be well-defined. Moreover, each γ_t is compact, and so κ attains its maximum on it. Thus we can exchange sup for max with no issues.

We will not prove this theorem in full but instead outline how it can be proven. For a detailed proof the reader can consult Section 7 of [16] or Section 2.2.2 of [18].

The proof of Theorem 3.3.2 proceeds by considering the L^2 norms of the curvature and its time-derivatives, where we parametrise the family of curves by arclength. The first step is to prove a lemma concerning the long-term behaviour of the curvature and its first derivative:

Lemma 3.3.3. *The following limits hold:*

$$\lim_{t\to\infty} \int_0^{L(t)} \kappa^2 \,\mathrm{d}s = 0,$$

$$\lim_{t\to\infty} \int_0^{L(t)} \left(\kappa^{(1)}\right)^2 \mathrm{d}s = 0.$$

This lemma is proved by bounding the time derivatives of each integral in terms of the integral itself. An elementary ODE result then tells us that the integrals converge to 0; specifically, they converge at a rate which is at least exponential.

We can then apply this lemma to prove a strictly stronger lemma:

Lemma 3.3.4. *For all $n \in \mathbb{N}$, the following limit holds:*

$$\lim_{t\to\infty} \int_0^{L(t)} \left(\kappa^{(n)}\right)^2 \,\mathrm{d}s = 0.$$

The proof of this lemma is by induction; Lemma 3.3.3 provides the 'base case'. The same approach as before is used to show convergence, and again we have that it happens at a rate no slower than exponential.

With this lemma established, to show that the maxima converge to 0 we can bound them from above in terms of the above integrals, which immediately proves the result.

3.4 Self-Similar Curves in Curve Shortening Flow

It is worth briefly mentioning some papers concerning the analogue of the results of Section 2.5 in other manifolds. In [19], examples of curves which are self-similar under curve shortening flow– also known as *soliton solutions of curve shortening flow*– are found on the unit sphere S^2, one-sheet hyperboloid, and the helicoid. In [20], curves in $\mathbb{R}^n$ which are self-similar under curve shortening flow are classified (in this report we have focused on two-dimensional curve shortening flow but the extension to curve shortening flow in higher dimensions comes naturally). Most recently, all curves on the unit sphere S^2 which are self-similar under curve shortening flow have been classified in a paper from 2019 [21]. Currently, to the best of our knowledge not much is known about the soliton solutions on other manifolds, even on well-understood surfaces such as the hyperbolic surface $\mathbb{H}^2$ of constant negative Gaussian curvature.

Chapter 4

Applications of Curve Shortening Flow

In this chapter, we look at some applications of curve shortening flow in both pure and applied mathematics as well as in other fields. Specifically, we discuss how curve shortening flow can be used to prove two theorems: the *theorem of the three geodesics* and the *tennis ball theorem*. We then discuss the connections between curve shortening flow and isoperimetric inequalities, including a proof of the planar isoperimetric inequality, before examining some uses of curve shortening flow with applications in computer science and chemistry. We conclude with a brief overview of *mean curvature flow*, a generalisation of curve shortening flow which can be applied to manifolds of dimension greater than one.

4.1 The Theorem of the Three Geodesics

This section is concerned with the below theorem:

Theorem 4.1.1 (Theorem of the Three Geodesics [22])**.** *Every smooth Riemannian metric g on the 2-sphere S^2 admits at least 3 closed embedded geodesics.*

This theorem, first conjectured by Poincaré in 1905 [23], had a proof outlined in 1929 by Soviet mathematicians Lazar Lyusternik and Lev Schnirelmann [22]. However, this proof was found to be slightly flawed and was not repaired until 1978 when Ballmann published a complete proof [24]. The main idea behind the proof, the details of which rely on algebraic topology to a level beyond what is suitable for this report, comes from looking at the homology of a space of closed embedded curves on the manifold (S^2, g). Three homology classes of this space can be found, each of which can be represented by a shortening cycle which leads us to the required geodesics [18].

In fact, though the theorem is most commonly stated in the form of Theorem 4.1.1 above, the original theorem is more detailed and states that one of the following cases occurs [25]:

- There are exactly three closed embedded geodesics, each with different length.
- There is a one-parameter family of closed embedded geodesics which cover (S^2, g), all of the same length, and one embedded closed geodesic of a different length.
- Every geodesic is closed and embedded, and they all have the same length.

In 1989, Grayson used curve shortening flow to prove the theorem of the three geodesics, after Uhlenbeck suggested the approach [16]. This approach was similar to Ballmann's, which does involve the deformation of curves to make them shorter but does not use curve shortening flow. In the same 1989 article, Grayson argues that such methods are more dangerous than those using curve shortening flow, which is in some sense a "natural" way to deform a curve and so does not run into the same problems that other methods do. For example, we have already seen in Theorem 3.2.6 that curve shortening flow keeps embedded curves embedded, which is essential in the proof of the theorem; other methods of deforming curves have to be very carefully treated in order to exhibit this behaviour.

4.2 The Tennis Ball Theorem

This section is concerned with the below theorem:

Theorem 4.2.1 (Tennis Ball Theorem)**.** *Any smooth embedded curve on the round sphere S^2 which divides the sphere into two parts of equal area must have at least four inflection points.*

The theorem gets its name from the white curve on the surface of a tennis ball: this curve– when modelled as a one-dimensional curve of zero thickness rather than the two-dimensional strip it is in reality– is exactly such a curve.

We will not give a full proof of Theorem 4.2.1 here, but instead direct the reader to [26], where it is proven in full. The same article also states without proof analogous theorems for curves in $\mathbb{R}^2$ and on the real projective plane $\mathbb{RP}^2$. Proofs of all three theorems can be found in [27]; all of them rely on a theorem of Sturm [26, p. 2; 28, p. 4] relating to parabolic PDEs.

The proof begins by noting, as we did in example 3.2.5, that under curve shortening flow such a curve must approach a geodesic; moreover, as mentioned in

Section 3.3, an argument in [11] shows that this geodesic is unique. Denote the limiting geodesic by γ_∞.
Taking this geodesic to be the equator – which we may do because of the rotational symmetry of the sphere – and removing both poles, we can project the sphere onto the cylinder C given implicitly by $x^2 + y^2 = 1$, which we parametrise by cylindrical co-ordinates (φ, z).
For large enough t we can write the projection of γ_t onto C as a graph given by $z = u(\varphi, t)$. With this equation, it is possible to show that γ_t has at least 4 inflection points by applying the definition of curve shortening flow from equation (3.1) and linearising it about γ_∞. The aforementioned theorem of Sturm implies that the number of inflection points of γ_t is a decreasing function of time, which in turn implies that the initial curve γ_0 must have had at least 4 inflection points as well, finishing the proof.

4.3 Isoperimetric Inequalities

In this section we discuss the famous planar isoperimetric inequality and a generalisation of it to certain manifolds. A lot of theory on curve shortening makes use of the inequality; for example, Gage uses it in [9] and [10] to show that the curve shortening flow makes convex plane curves more circular, as we discussed in Chapter 2. However, it is also possible to "go the other way around" and use elementary properties of curve shortening flow to derive some isoperimetric inequalities.

Theorem 4.3.1 (Planar Isoperimetric Inequality)**.** *Let $\gamma_0 \colon S^1 \to \mathbb{R}^2$ be a closed embedded curve. Let A denote the area it encloses and L denote its length. Then the following inequality holds:*

$$4\pi A \leq L^2. \tag{4.1}$$

Proof. Consider the evolution of γ_0 under curve shortening flow. By Lemma 2.2.7,

$$\frac{dA}{dt} = -2\pi$$

and

$$\frac{dL}{dt} = -\int_{\gamma_t} \kappa^2 \, \mathrm{d}s.$$

Thus,

$$
\begin{aligned}
-4\pi \frac{dA}{dt} &= 2(2\pi)^2 \\
&= 2 \left(\int_{\gamma_t} \kappa \, \mathrm{d}s \right)^2 \\
&\leq 2 \int_{\gamma_t} 1 \, \mathrm{d}s \int_{\gamma_t} \kappa^2 \, \mathrm{d}s \qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= -2L(t) \frac{\mathrm{d}L}{\mathrm{d}t} \\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \left(L(t)^2 \right).
\end{aligned}
$$

Integrating between $t = 0$ and $t = T$, the time at which $\boldsymbol{\gamma}$ shrinks to a point and so has zero area and length, gives us the desired result. □

Remark 4.3.2. In fact, Theorem 4.3.1 holds for all closed curves; we do not require them to be simple or smooth. However, to prove the theorem using curve shortening flow we make use of the Gage-Hamilton-Grayson theorem when we use the fact that the curves shrink to a point, and so we must assume the conditions of this theorem hold.

The above proof comes from a paper by Topping [29], in which a more general inequality is proven for manifolds of the form (D, g), where D is the closed two-dimensional disc and g is any smooth metric. This naturally extends to an inequality for curves on manifolds, provided the area they enclose acts as such a D. Writing $A = A(D)$ for the area of the disc and $L = L(\partial D)$ for the length of the boundary, we introduce the function $K^*\colon (0, A) \to \mathbb{R}$ as a rearrangement of the sectional curvature K, defined as the unique decreasing function such that

$$
A(\{\boldsymbol{x} \in D \mid K(\boldsymbol{x}) \geq s\}) = |\{y \in (0, A) \mid K^*(y) \geq s\}| .
$$

Then the isoperimetric inequality

$$
4\pi A \leq L^2 + 2 \int_0^A (A - x) K^*(x) \, \mathrm{d}x
$$

holds.

To prove the inequality using curve shortening flow, two more assumptions are required: namely, that the disc D satisfies

$$
\int_D K_+ \, \mathrm{d}A < 2\pi,
$$

where $K_+ = \max\{K, 0\}$, and that the boundary of the disc ∂D is a convex curve. However, the inequality remains true without these assumptions; a complete proof of this is given in [30].

4.4 Practical Applications of Curve Shortening Flow

In this section we look at some of the ways in which curve shortening flow can be applied to the real world. Interestingly, the study of curve shortening flow first arose in materials science, starting in the 1920s with the general expression first appearing in 1957 in [1], where Mullins used it to describe how grooves develop on the surfaces of hot polycrystals.

One use of curve shortening flow is in image processing. Specifically, it can be used to smooth images, by removing unnecessary noise without compromising the amount of information conveyed, and to enhance images, by emphasising particular parts of them. In [6] a variant of curve shortening flow is developed for this use, where the flow is applied to curves given by the level sets of $I(x, y)$, the intensity of a black and white image. Based on local properties of the image, the κ term in equation (2.1) is either retained, replaced by $\max(\kappa, 0)$, or replaced by $\min(\kappa, 0)$. This "level set curvature flow", implemented using results from [31] and [32], was particularly useful due to reasons set out in [6]. One reason is that is based on only one parameter of the curve (its curvature κ), making it computationally quicker. Another is that the flow will naturally 'stop' after a certain number of steps, so it is not necessary to continue to apply the flow for an excessive amount of time. Due to the aforementioned max-min variation, it does not necessarily stop when the curves have shrunk to a point and so this is of practical use.

Many figures are contained in [6] which illustrate the power of this flow: Figure 2 compares it to a Gaussian smoothing, a more traditional approach at the time, and shows that it is effective at smoothing an initial image while maintaining a high level of detail. Along with Figures 3 and 4, it also illustrates the use of the flow in restoring images that have been corrupted by noise, even when the level of corruption is quite high. Finally, Figures 5 and 6 illustrate the flow applied to medical imaging.

This application of curve shortening flow is particularly interesting due to the relatively short time gap between the major development of the mathematical theory of curve shortening flow (the significant results from Gage, Hamilton, Grayson and others came about in the mid-1980's and late 1980's) and its use in image processing (developed in the late 1980's).

Another application of curve shortening flow can be found in [33] where it is applied in the context of chemical reactions modelled by a specific form of the reaction-diffusion equation. The equation is constructed to model systems which have a very high reaction rate (proportional to ε^{-1} for small $\varepsilon > 0$) and a very slow diffusion rate (proportional to ε). Such a system naturally leads to the formation of fronts– effectively curves in the two-dimensional case– along which the bulk of the reaction takes place. These fronts change over time as the reaction goes on; Sections 3 and 4 of [33] explain that this change over time is well modelled by curve shortening flow. Section 4 goes further in showing, for manifolds with boundary $\Omega \subset \mathbb{R}^2$ taken with g as the induced Euclidean metric, certain conditions under which geodesics are stable or unstable equilibria of the curve shortening flow.

4.5 Mean Curvature Flow

In Chapter 3, we generalised curve shortening flow from $\mathbb{R}^2$ to two-dimensional orientable manifolds. In this section, we will give an overview of *mean curvature flow*, another generalisation of the curve shortening flow where instead of studying one-dimensional submanifolds (curves) in $\mathbb{R}^2$, we study n-dimensional submanifolds in $\mathbb{R}^{n+1}$. For the reader looking for a more mathematically detailed account of mean curvature flow, there are many introductions such as [17], [34], [35], and [36].

We first define a (Euclidean) *hypersurface*, a specific type of manifold on which it makes sense to discuss mean curvature flow. The following definition is taken from [3]:

Definition 4.5.1. Let M be a smooth manifold of dimension n, and let $f\colon M \to \mathbb{R}^{n+1}$ be an embedding. The manifold $f(M) \subset \mathbb{R}^{n+1}$ together with its Riemannian metric induced from the Euclidean metric of $\mathbb{R}^{n+1}$ is called a *hypersurface*.

Examples of hypersurfaces include the n-sphere $S^n \subset \mathbb{R}^n$, which can be defined by the equation $x_1^2 + \ldots + x_n^2 = 1$, and the $(n-1)$-dimensional Euclidean space $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, which can be defined by the equation $x_n = 0$. Another example is the n-torus $T^n := S^1 \times S^1 \times \ldots \times S^1$ (n times), which can be embedded into $\mathbb{R}^{n+1}$ [37].

Remark 4.5.2. All C^∞ smooth hypersurfaces without boundary are orientable.

A short, self-contained proof of this remark can be found in [38]. The main consequence of the remark lies in the fact that it excludes non-orientable manifolds without boundary, such as the Klein bottle, from consideration. Thus we can give a consistent choice of normal vector to a hypersurface without boundary, in the same way as we did to orientable two-dimensional manifolds in Chapter 3.

As we did in the case of normal vectors on curves, when picking a consistent normal vector field to an orientable hypersurface we can choose between two vectors, both of which lie in a one-dimensional vector space (i.e. we have a choice up to a $\pm$ sign). This well-known result, outlined in equation (2) of [39], is obtained via the implicit function theorem; we define the normal to be the (normalised) gradient vector of some function f which locally defines M as a level set.
Analogously to how the quantity $\kappa \boldsymbol{n}$ in Chapters 2 & 3 did not depend on the choice of $\boldsymbol{n}$, in this case the quantity $H\boldsymbol{n}$ does not depend on the choice of $\boldsymbol{n}$, where H is the mean curvature. This is because H itself depends on $\boldsymbol{n}$, so if we change our choice of normal $\boldsymbol{n} \mapsto -\boldsymbol{n}$, we also get a change $H \mapsto -H$ and the two negative signs cancel each other out. For the rest of this chapter, we will take $\boldsymbol{n}$ to be the "inward-pointing" normal vector, with the convention that the mean curvature of a convex hypersurface is positive. We note however that this choice is purely aesthetic, as we could have just as easily taken the "outward-pointing" normal vector instead.

With this machinery all in place, we are able to define mean curvature flow:

Definition 4.5.3. Let M be an orientable n-dimensional manifold embedded in $\mathbb{R}^{n+1}$ under a smooth embedding F_0. We say that the family of smooth embeddings

$$F\colon M \times [0,T) \to \mathbb{R}^{n+1}$$

evolves by mean curvature flow if it satisfies the initial value problem

$$\begin{cases} \frac{\partial F}{\partial t} = H\boldsymbol{n}, \\ F(\boldsymbol{p}, 0) = F_0(\boldsymbol{p}), \end{cases} \tag{4.2}$$

for all $\boldsymbol{p}$ in M and t in $[0,T)$, where H is the mean curvature of $M_t := F(M,t)$ and $\boldsymbol{n}$ is the inward-pointing normal vector to M_t.

It should be noted that across the literature there are different conventions for which normal to use and how to represent the mean curvature: for example, sometimes the mean curvature differs by a factor of -1, so equation (4.2) is read with a term of $-H\boldsymbol{n}$ in instead of the $H\boldsymbol{n}$ we use. However, the difference is purely superficial and the flow is exactly the same in each case.

Again, we will slightly abuse notation by referring to the embedding $M_0 = F_0(M)$ as a hypersurface, and say that "M_0 evolves under mean curvature flow". This is technically not quite accurate, as M_0 is the embedding of the hypersurface M rather than a hypersurface itself, but it is clear what is meant. Because embeddings preserve topological properties, we may also refer to topological properties of M_0 when we are actually talking about the topology of the underlying manifold

M.

There are immediate similarities between equation (4.2) and equation (2.1). This is not a coincidence: indeed, if we take M to be some curve embedded in the Euclidean plane $\mathbb{R}^2$, then the two equations are exactly the same. We were able to generalise equation (2.1) to equation (3.1) by considering curves embedded in a two-dimensional non-Euclidean manifold as well; in the same way, mean curvature flow can be generalised by considering hypersurfaces immersed in non-Euclidean manifolds. Some articles which cover this are [40], [41], and [42]. For the purposes of this section, however, we will restrict our attention to the 'Euclidean' mean curvature flow as defined in definition 4.5.3.

We have the following result for mean curvature flow, analogous to the rate of change of length formula displayed in lemma 2.2.7 and remark 3.2.1:

Lemma 4.5.4. *Let M evolve by mean curvature flow, and $N_t \subseteq M_t$ be relatively compact (i.e. the closure of N_t is compact in M_t). Then the area of N_t evolves by the following formula:*

$$\frac{\mathrm{d}}{\mathrm{d}t}A(N_t) = -\int_{N_t} H^2 \, \mathrm{d}A. \tag{4.3}$$

This immediately implies that if a hypersurface has 0 mean curvature everywhere, it is invariant under mean curvature flow. In $\mathbb{R}^3$, such a two-dimensional hypersurface is called a *minimal surface*, because it is a local minimiser of surface area. Minimal surfaces were first discovered in 1776 by French mathematician Jean Baptiste Meusnier [43]. Note the similarity to the one-dimensional geodesics, which have zero curvature everywhere and are local length minimisers.
A trivial example of a minimal surface is the plane, but more exciting examples are the helicoid, shown in Figure 4.1, and catenoid. Minimal surfaces can be 'constructed' in a real-world sense by making a thin wire frame in the shape of the boundary of the minimal surface; dipping the frame in soapy water will cause a soap film to form on the frame exactly in the shape of the minimal surface!

Figure 4.1: A segment of a helicoid, a two-dimensional surface in $\mathbb{R}^3$ which has 0 mean curvature everywhere and thus is unchanged by mean curvature flow. Any small deformation of the surface will increase its area.

Another theorem, proved by Huisken [44], generalises Theorem 2.1.3 to the context of mean curvature flow:

Theorem 4.5.5. *Let the hypersurface M_0 be convex and compact. Then under mean curvature flow, M_0 shrinks to a point in finite time, becoming round as it does so.*

Formally, by "becoming round" we mean that if the flow is rescaled to keep the area of M_t constant then it converges smoothly to a round n-sphere.

Both conditions on M_0 in this theorem are necessary: for example, the cylinder $S^1 \times \mathbb{R} \subset \mathbb{R}^3$ is compact but not convex, and under mean curve shortening flow it retains its shape as it shrinks to a *line*. A less exciting example is the plane $\mathbb{R}^2 \subset \mathbb{R}^3$, which is convex but not compact, and as discussed before is not affected by mean curvature flow. That compactness is necessary is not surprising: intuitively, it makes little sense that an 'infinite' hypersurface such as a cylinder could be compressed into a finite point.
More interesting counterexamples can be found in the case of hypersurfaces which are compact but not convex. In the case of curve shortening flow, we know from Theorem 2.1.3 that non-convex curves will become convex in finite time, and so will also become round as they shrink to a point. This, however, does not extend to mean curvature flow: a counterexample is given by *Grayson's dumbbell* [45], a two-dimensional surface which looks (very roughly) like a dumbbell with a long, thin bar (see Figure 4.2). In more mathematical terms, it looks like two spheres

connected by a cylinder. The bar will shrink and develop a singularity before the two 'weights' on either end have contracted much, and at the time of the singularity developing the surface is neither round nor has it contracted to a single point! This example illustrates the point that although mean curvature flow is the natural extension of curve shortening flow to higher dimensions, it is nowhere near as simple and not all the results carry over.

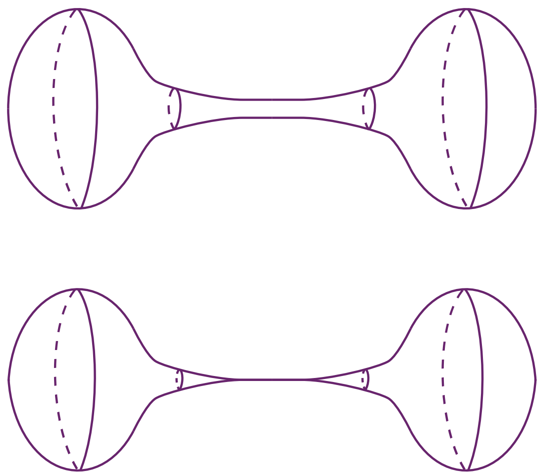

Figure 4.2: An illustration of a dumbbell evolving under mean curvature flow. In the second image we see it develops a singularity under mean curvature flow before it can shrink to a point; the thin handle has quickly been squashed. It is possible to define 'weak' solutions to the mean curvature flow which extend past singularities; see for example [46] or [47]. In this case, extending the flow past the singularity will split the dumbbell into two components, both of which will shrink to points.

Another result which generalises from curve shortening flow is contained in the following theorem:

Theorem 4.5.6. *Let the hypersurface M_0 evolve under mean curvature flow such that no singularities develop in the time interval $[0, T)$. Then for each $t \in [0, T)$, M_t is embedded.*

This is the extension of Theorem 3.2.6, and it tells us that a hypersurface will not develop any self-intersections under the mean curvature flow so long as it is 'well-behaved'. Here, though, it is much easier for bad behaviour to occur, for example as in Figure 4.2 when the dumbbell develops a singularity and ceases to be embedded.

Conclusion

In this report, we have looked at the history, motivations, results and uses of curve shortening flow. Starting with the restricted case of curve shortening flow in the plane, we have seen explicitly how the elementary properties of its behaviour were developed and proved, and we have investigated the main results of the more advanced theory, notably in Section 2.4 where we proved Theorem 2.4.1 in full detail. The main result here was the surprising Theorem 2.1.3, known as the Gage-Hamilton-Grayson Theorem, which states that curve shortening flow collapses closed embedded plane curves to round points in finite time. We have also looked at the theory of plane curves which are self-similar under curve shortening flow and seen some visually appealing plots of such curves.

Continuing on from this, we have generalised curve shortening flow to work on any two-dimensional orientable manifold, and have extended our results from the plane to apply in this context as well. Some of these results have carried over straightforwardly, while some have become significantly more exotic: the main item of interest here is Theorem 3.1.2, conjectured by Gage and Hamilton then proved by Grayson, which says that under curve shortening flow a closed embedded curve will either collapse to a point in finite time or exist forever and approach a geodesic. As well as generalising curve shortening flow into other two-dimensional spaces, we have also generalised it to higher dimensions, giving a brief overview of Euclidean mean curvature flow, and again we have discussed how we can carry over our existing results concerning curve shortening flow. Mean curvature flow, however, is much more complicated, and not all the results extend without issue– in particular, it is much more common for singularities to develop before the flow is complete.

After developing the theory behind curve shortening flow, we have seen some of its applications in both pure and applied maths. Although there are certainly applications of curve shortening we have not been able to cover, we have looked at a wide range of its uses in many different contexts. Possibly the most significant among these from a pure mathematical point of view is its application as a tool to prove the *theorem of the three geodesics*. Curve shortening flow is an example of a *geometric flow*; the study of geometric flows is a very active area of mathematics, as they are key components of many well-known and important theorems as well as being extremely interesting in their own right.

Although most of the literature referenced in this report is not extremely recent, with most of the main results on curve shortening flow developed in the 1980s, it is by no means a finished subject and there are many ongoing developments in a wide range of areas. For example, the material we covered on self-similar curves originated in a 2012 paper [13], and a more recent article is a 2019 paper classifying all solutions to curve shortening flow in the plane which are convex and exist backwards in time forever without singularities [48]. This depth of potential study, combined with the elegance of curve shortening flow and its beautiful visual geometric representation, are what make it such a fascinating and exciting topic to the author of this report.

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COVID19 IMPACT SHEET
Project 3/4
Department of Mathematical Sciences

Student Name: Ed Gallagher

Year group (3/4): 4

Project Topic: Curve Shortening Flow (Original Title: Geodesics)

Project supervisor(s): Dr Wilhelm Klingenberg (Michaelmas Term); Dr Fernando Galaz García (Epiphany Term)

Did Covid19 prevent you from completing part of your project report (Yes/No): No

If 'Yes', please indicate what it prevented you from doing (max 100 words): N/A

Please summarise the action taken in response (max 100 words): N/A