PH30101 General Relativity

Prof Tim Birks

"General relativity without tensors"

General relativity (GR) states that spacetime is deformed by gravitating masses. Freely-moving objects follow straight lines (or their nearest equivalents) in this curved spacetime, even in the presence of gravity. GR is our current theory of gravity and, alongside the "standard model" of quantum/particle physics, forms our best account yet of how the Universe fundamentally works.

A complete treatment of GR relies on tensor analysis, a chunk of advanced mathematics we'd need to spend a lot of time learning before even starting the actual physics. But, knowing the curvature of spacetime, we can deduce the motion of particles and light *without knowing about tensors*. (It's still mathematical, but it's maths you already know.) In this unit we will use this approach to examine what curved spacetime means, compare the predictions of GR and Newtonian gravity, and explore the properties of the quintessential GR object: the black hole.

Spacetime curvature is described by *metrics*, which we won't be able to derive without tensors. We'll just take them as given. But, when you learned quantum mechanics, did it bother you (did you even notice) that you never saw where Schrödinger's equation came from?

Revision: You will need material from previous units, including:

special relativity (SR): PH10103 for Physics students, PH20076 for Maths and Physics students, or PH20106/PH20114 for students who transferred to Physics from other courses.

Newtonian mechanics and gravity: including gravitational potential, angular momentum, orbits, planetary motion, impact parameter

calculus: curve sketching, differentiation and integration (including line and multiple integrals), polar plots, ordinary differential equations (separable, forced s.h.m.) and *especially* coordinate systems like spherical polars. But, no vector analysis or complex numbers.

geometry: basic stuff (triangles, circles, parallel lines), curves (ellipses, hyperbolae) and spheres (surface area, latitude and longitude, great circles).

thermal physics and quantum mechanics: entropy, microstates, black-body radiation, the uncertainty principle(s).

I don't schedule office hours, but I'm usually happy to be interrupted. ***To contact me*** outside timetabled contact time, use email (t.a.birks@bath.ac.uk) or visit my office (8W 4.13) if the door is open. ***It's OK*** for students to get in touch and ask me questions!

The Moodle page for the unit contains, amongst other things:

Organisational information (when different lectures and problems classes will be held, etc). I will assume you have read this and will keep checking for changes from time to time, especially if you miss any announcements.

The problem set as a pdf file.

Model answers to the problem set. These will become available after the corresponding problems classes. The idea is to attempt the problems *before* being given the answers ...

A sheet of useful equations, most of which <u>you do not need to memorise</u>.

These notes as a pdf file. On each page of the notes, a header summarises what's on it and indicates the expected (no promises) lecture number when it's covered.

<u>I will not be handing out paper copies of the notes, problem set, model answers or anything else!</u>

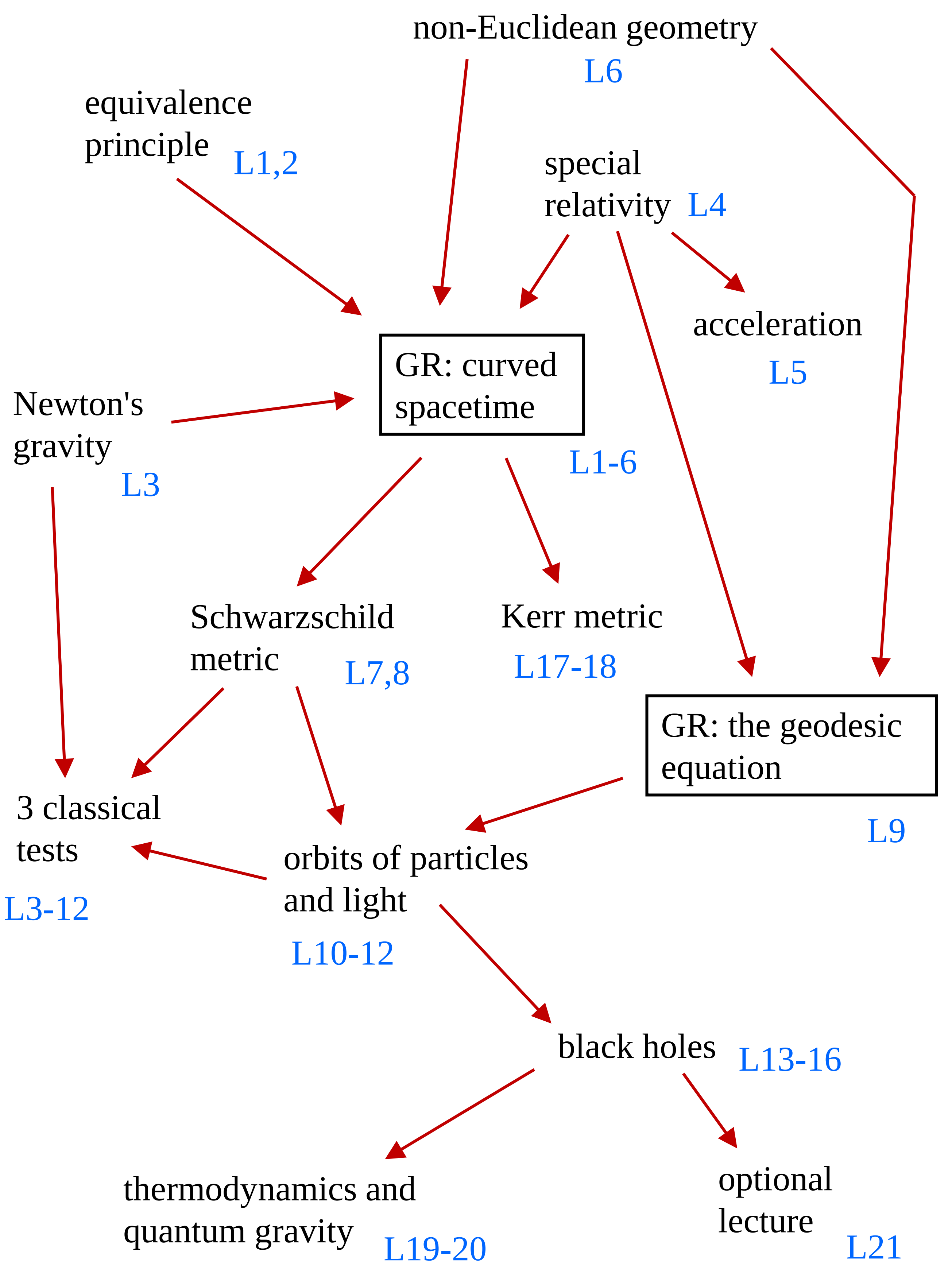
If you decide to print the notes I recommend printing them double-sided, black-and-white, and 2 pages on each side*. In January 2024 this cost £1.90 using University printers. The notes are designed to be comfortable to read at this size.

It cost 22p to print the problem set in the same way.

** If you're using Adobe Reader and University printers: select printer UPS1 or UPS2*
select "Print in grayscale (black and white)"
click "Multiple" and select 2 Pages per sheet, "Print on both sides of paper" and "Flip on short edge".

Remember to deselect these options next time you print something!

Structure of the unit



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1. Introduction: deforming time and space

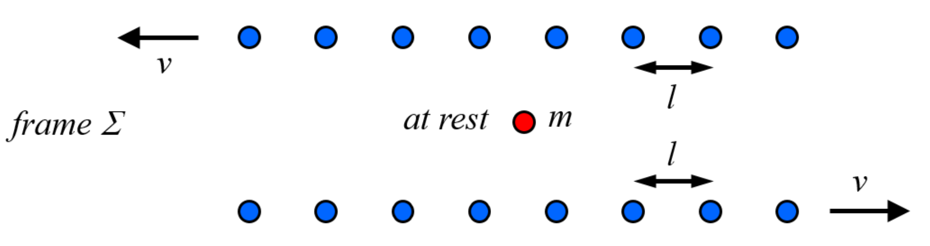
We needed a new theory of gravity!

General Relativity (GR) is Einstein's theory of gravity. So what's wrong with Newton's?

$$F = G\frac{Mm}{r^2}$$

F is the force now, when M and m are r apart. But special relativity (SR) says no influence can travel faster than light, so gravitational influences too must take time to get from M to m. Indeed, r is the difference between the positions of M and m measured simultaneously, but SR says that simultaneity depends on your frame of reference.

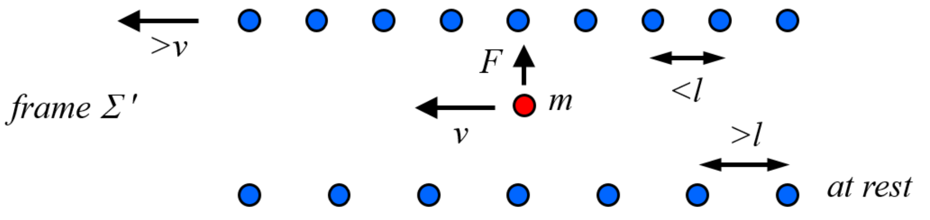
If that criticism is too vague and picky for you, try this thought experiment. In frame of reference Σ, test mass m lies initially at rest half-way between two identical streams of masses, spaced by l, moving at the same speed v in opposite directions:



The net gravitational force on m is zero, by rotation symmetry: the attraction of the upper stream is balanced by the attraction of the lower stream.

$$\Rightarrow \quad m \text{ does not accelerate}$$

Then look at things in frame Σ' moving at speed v to the right relative to Σ. Now the lower stream is at rest, m moves to the left at speed v, and the upper stream moves to the left at a speed greater than v.



According to SR, moving objects undergo length contraction. The upper stream moves faster in Σ' than in Σ and experiences more length contraction, so its spacing is $<l$. The lower stream moves slower in Σ' than in Σ and experiences less length contraction, so its spacing is $>l$.

There's therefore more mass (per unit length) above m than below, and a net upward gravitational force F on m.

$$\Rightarrow \quad m \text{ accelerates upwards}$$

These two viewpoints contradict one another: m can't both move upwards (and eventually collide with the upper stream) and not do so. This exemplifies the fact that:

Newtonian gravity is not consistent with SR.

In our thought experiment, F is balanced by a new *repulsive* force between co-moving masses called "gravito-magnetism", cf the well-known velocity-dependent relativistic force between electric charges. But rather than patching up Newton's theory in this way, Einstein preferred to start from scratch with:

The principle of equivalence

Einstein's thinking on gravity was based on a familiar result from Newton's theory - the acceleration g of test mass m due to mass M is independent on m:

$$\cancel{m}g = G\frac{M\cancel{m}}{r^2} \qquad [ma = F]$$

The fact that all free-falling masses accelerate equally was well known before Newton (Galileo etc) and has been experimentally verified to within one part in 10^{12}. Yet in Newton's theory it is an astonishing coincidence, because the m's on both sides of the above equation represent logically-distinct concepts:

inertial mass m (resistance to acceleration)	gravitational mass m (source of gravitational force)

Why are they identical?

Actually there are other forces that accelerate independently of mass. For example, "g-forces" that push you backwards in an accelerating rollercoaster, centrifugal forces that pull outward on a curved path and Coriolis forces that spin weather systems. What all these forces have in common is that they don't exist... They are *pseudo-forces* that appear only in accelerating (ie, non-inertial) frames of reference. The acceleration that all masses seem to have in common is merely the acceleration of the frame itself.

Einstein's big idea was that the gross "9.8 ms^{-2}" effect of gravity is also a pseudo-force. It appears only because, standing on the ground, we're really using an upwardly-accelerating frame of reference. He realised this by imagining that a man falling from a building feels no gravity as he falls, which Einstein called the "happiest thought" of his life.

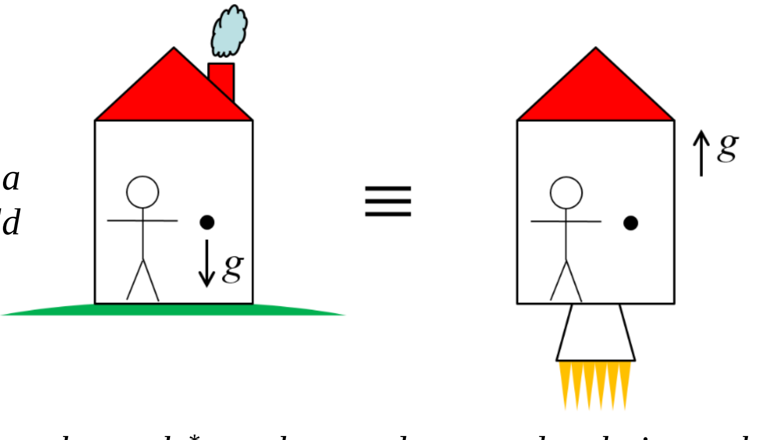
$$\boxed{\text{Free-falling frames} \equiv \text{inertial frames}}$$

It follows that being at rest (in the conventional sense) in a gravitational field is the same as being accelerated relative to a freely-falling (and hence inertial) frame. Einstein generalised this mechanical result to all physical phenomena, leading to

> The Principle of Equivalence: no local experiment can distinguish between gravity and accelerated motion.

According to the principle, experiments at rest on the ground yield the same results as they would in a rocket accelerating smoothly at 9.8 ms^{-2} far from gravitating masses. Without external interactions (for example, no looking out of windows), you can't tell the difference.

at rest in a gravitational field

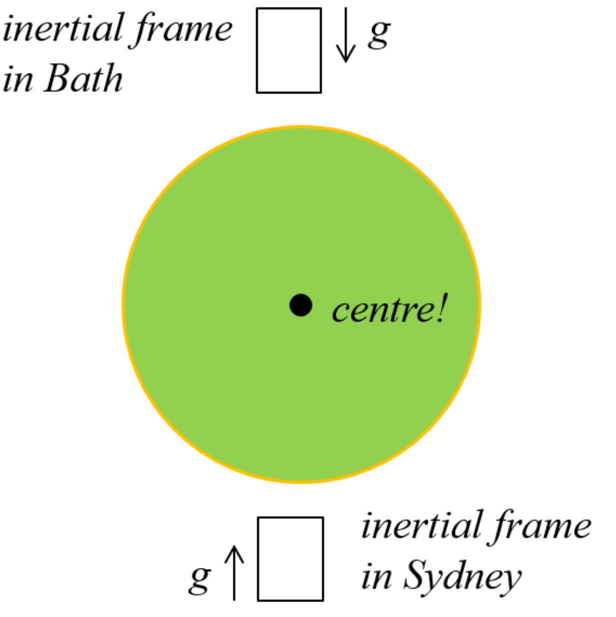


in an accelerating rocket, no gravity

in both cases, the apple accelerates downwards relative to the observer*

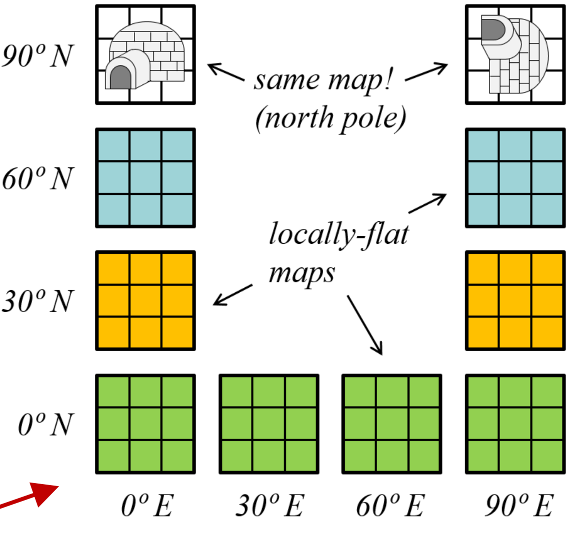
* *We're studying gravity - tradition demands it's an apple.*

The word "local" is important. Unlike uniform acceleration, gravity has a centre. Free-falling frames have vector accelerations that vary from place to place, in magnitude (g is slightly bigger at your feet than your head) and direction (the vertical in Bath is ~1½° away from the vertical in London). A big-enough experiment can use this spatial variation to distinguish gravity and accelerated motion, and we can't find a common frame of reference that eliminates gravity everywhere:



There are no universal inertial frames, only local ones.

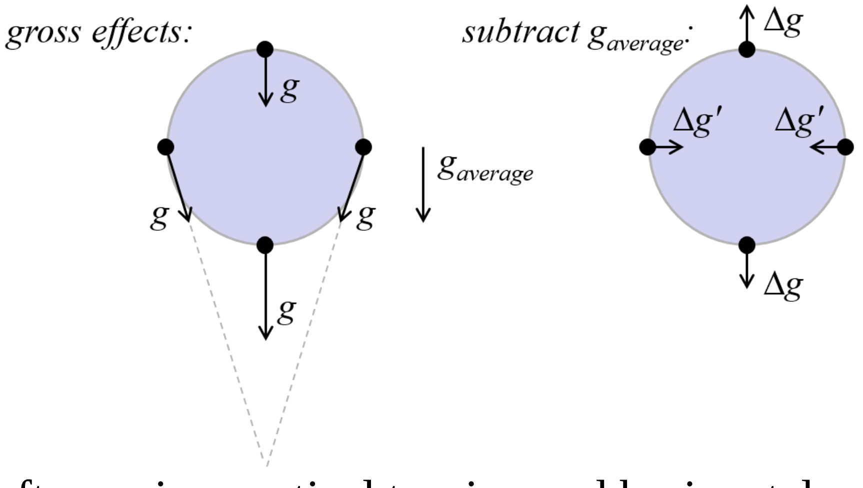
Stitching together neighbouring local inertial frames across an extended region turns the flat *Minkowski spacetime* of SR into the *curved spacetime* of GR. (Like the way stitching together lots of flat city-scale maps produces a spherical surface on a continental scale.)



Cut out the U-shape of stitched-together small-scale maps and lay the two maps for the north pole over each other (with the right orientation). The paper strip forms part of a globe, even though each individual map is approximately flat.

The essence of gravity in GR - what can't be eliminated by moving to a new frame of reference - is the spatial variation left over when the gross effect of gravity is subtracted by moving to a freely-falling local inertial frame.

Imagine a free-falling sphere of loose gravel, ignoring air resistance etc. The bottom of the sphere has bigger g than the top, and at the sides the directions of g converge slightly to point to the centre of the gravitating mass*. Shifting to the (inertial) frame of the centre of the sphere means subtracting $g_{average}$:



What's left over is a vertical tension and horizontal compression, tending to deform the sphere into an ellipsoid. If the gravel was water, with a rocky ball inside rotating once per day, the whole lot in free fall towards the Moon, you might recognise these left-over forces as the *tides*.

Tidal forces are the essence of gravity.

They encapsulate the spatial variations discussed on the previous page. In our unit we won't study tidal forces much, but this concept is central to the tensor formulation of GR as a whole.

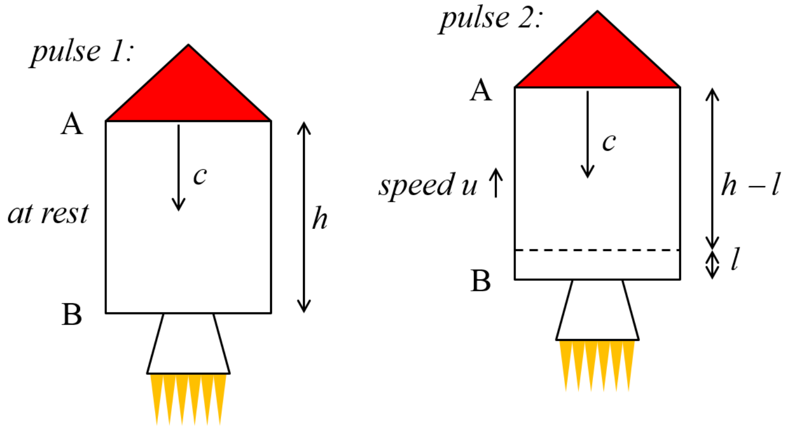
* *The gravel doesn't have to actually hit the gravitating mass. Free fall is just motion without forces other than gravity, and can be upwards or sideways (like an orbit) as well as the classic vertical drop.*

Consequences of the principle of equivalence

The principle of equivalence quickly leads to two surprising consequences about space and time in the presence of gravity.

- Consequence #1: gravitational time dilation

Rocket R undergoes constant acceleration g far from gravitating masses. Time t is measured by inertial observer O. Pulses of light leave the ceiling A every period Δt_A. What is the period Δt_B of the pulses reaching the floor B, if h is the height of the room?



Let pulse 1 leave A when R is instantaneously at rest relative to O. It reaches B after time* $t_1 = h/c$, the time of flight for light to travel distance h.

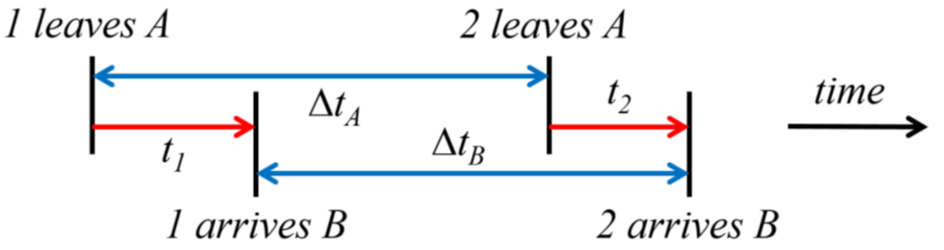
Pulse 2 is emitted time Δt_A later. By then, R has accelerated to speed* $u = g\,\Delta t_A$. As the pulse travels downwards, the floor travels upwards through distance* $l \approx u\,t_1 = g\,\Delta t_A \times h/c$ (using t_1 as our first approximation for the time of flight of pulse 2).

Pulse 2 therefore travels a shorter distance $h - l$ to B, with a shorter time of flight (our second approximation) of*

$$t_2 = \frac{(h-l)}{c} = t_1 - \frac{gh}{c^2}\Delta t_A$$

** From the definitions of velocity and acceleration - or "suvat" if you prefer.*

Here's a time-line for these various events:

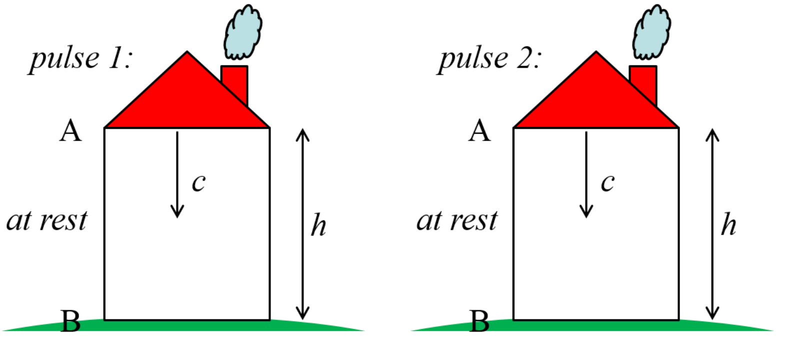


The time Δt_B between the two pulses reaching B is therefore

$$\Delta t_B = \Delta t_A + t_2 - t_1 = \Delta t_A + t_1 - \frac{gh}{c^2}\Delta t_A - t_1 = \left(1 - \frac{gh}{c^2}\right)\Delta t_A$$

So $\Delta t_B < \Delta t_A$: the pulses arrive at B more often than they leave A. Nothing paradoxical so far: the times of flight of the pulses are clearly different, just giving a fancy kind of Doppler shift.

<u>But</u> the equivalence principle says we get the same result in a room R' at rest in a gravitational field g.



Now the room doesn't move. The times of flight are now the same. But the equivalence principle demands $\Delta t_B < \Delta t_A$ still. An inhabitant of the room, measuring these times, concludes:

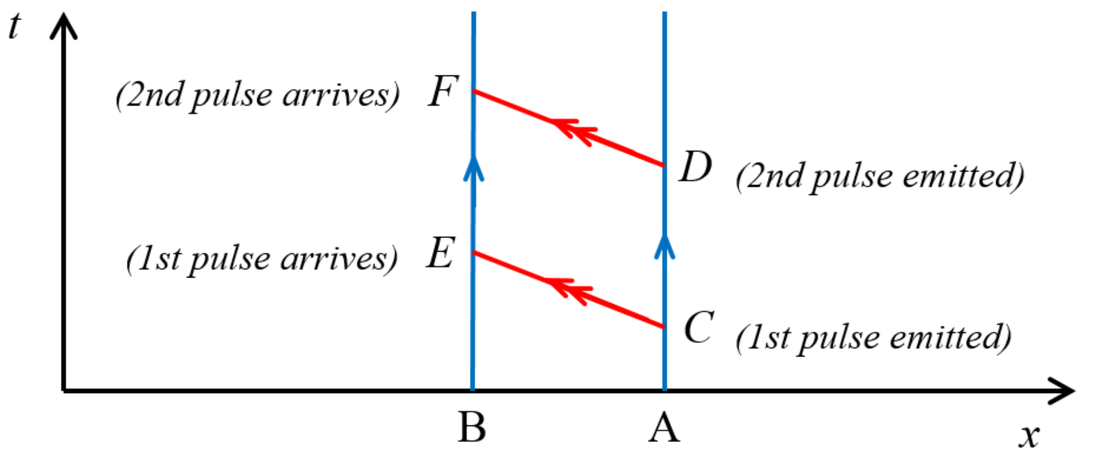
Gravitational time dilation: time passes more slowly lower down in a gravitational field*.

** The difference is tiny on Earth: ~1 μs per century between a typical ceiling and floor.*

- Consequence #2: curved spacetime

In a (t versus x) spacetime diagram* of the events in R', A and B are at rest so their worldlines are vertical, with constant values of x separated by h. This means lines CD and EF are parallel.

The two pulses of light travel at the same speed c, so their worldlines make the same angle to the axes. This means lines CE and DF are also parallel. By definition, the quadrilateral CDFE is therefore a parallelogram.



But CD = Δt_A and EF = Δt_B, so the time dilation result means

$$CD \neq EF$$

CDFE is a parallelogram with a pair of unequal opposite sides! Obviously this contradicts a basic theorem of plane geometry as developed by the ancient Greeks, such as Euclid. In fact such a shape cannot be accurately drawn on a flat sheet like the above diagram. The sheet would need to be *curved*.

Gravity causes spacetime to be curved.

The geometry of spacetime is "non-Euclidean".

* *x is the height coord: it's a shame spacetime diagrams always put space horizontally!*

General Relativity

J Wheeler summarised GR in two parts like this:

> " Spacetime tells matter how to move, ②
>
> matter tells spacetime how to curve. " ①

Part ① (it logically comes first...) means that sources of gravity cause the geometry of spacetime to depart from the flatness of the Minkowski spacetime of SR. This is described by the fundamental equation of GR, the *Einstein field equation*:

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$\mu, \nu = 0, 1, 2, 3$
4-D components, usually 0 is time

Einstein tensor describes the curvature of spacetime (in a differentiated form)

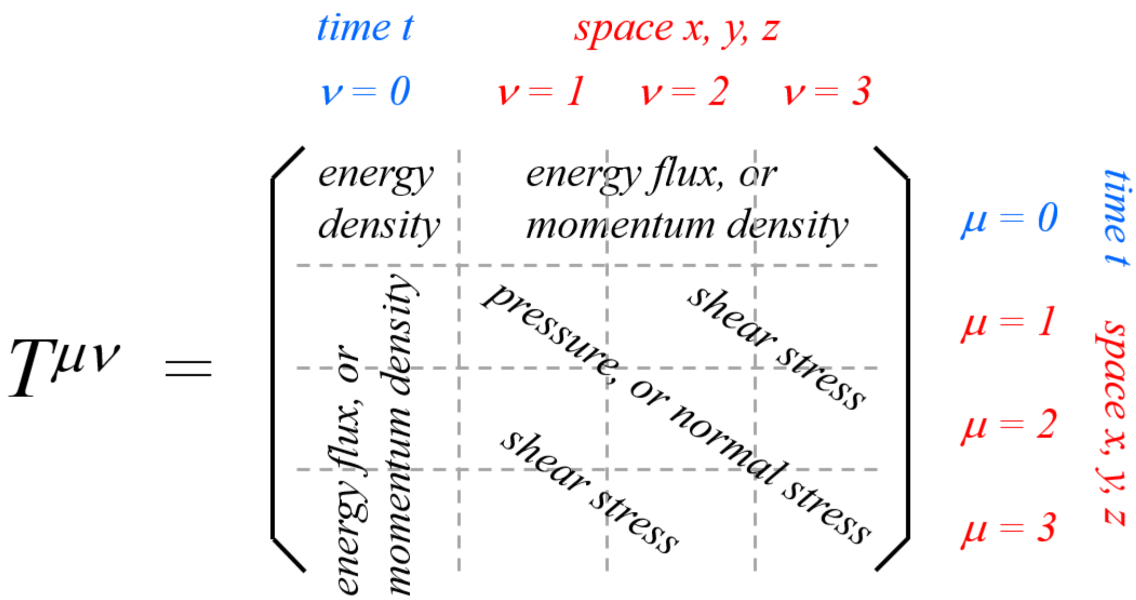
gravitational constant

stress-energy tensor represents sources of gravity

This simple-looking equation plays the role of a force law, like $F = GMm/r^2$, telling us how a gravitational field (represented by the curvature of spacetime) is produced by sources of gravity.

However, it is not as simple as it looks! It's written in the language of tensor analysis. The two tensors $G^{\mu\nu}$ and $T^{\mu\nu}$ can each be thought of as a 4×4 symmetric matrix, with 10 independent components*. Just for the sake of curiosity - you are <u>not</u> expected to remember this, or even the Einstein field equation itself, for the exam - take a look at $T^{\mu\nu}$ on the RHS:

* *μ and ν act as matrix row and column indices - not exponents / powers.*



Ordinary mass (represented by energy density) is merely T^{00}, the time-time component of $T^{\mu\nu}$. Other sources of gravity in GR are momentum density (space-time components) and pressure and stress (space-space components). Meanwhile, on the LHS, $G^{\mu\nu}$ is a set of complicated derivatives of something called the *metric*, which describes the spacetime curvature geometrically.

The use of tensors is an elegant way to express the *principle of general covariance*, which states that the laws of physics should be valid in all frames of reference, not just inertial ones. But written out in full, the Einstein equation becomes 10 coupled nonlinear partial differential equations in non-cartesian coordinates. Actually solving these equations, to get the metric, is hideously complicated. Indeed, it has only ever been done exactly in a handful of very simple cases. We'd need to spend most of the semester learning tensor analysis before even beginning any physics. Therefore, I regret to inform you that

In this unit*, we will not learn about how "matter tells spacetime how to curve".

** To study GR with tensors, and learn about matter telling spacetime how to curve, take PH40112 Relativistic Cosmology.*

Part ② of Wheeler's comment means that the motion of particles (and light) is determined by the metric found in part ①. The rule is that SR remains valid locally, and that free-falling particles follow *geodesic* world-lines. (In a curved space or spacetime, a geodesic is the nearest thing to a straight line.) This plays the role of an equation of motion, like $F = ma$, telling us how the particle moves in a given spacetime.

Although the metric is really a tensor, it is actually possible to write it as a so-called "line element" without knowing anything about tensors. Here's an important example, the Schwarzschild metric for a spherically-symmetric mass M:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

It relates a small spacetime interval ds^2 (as in SR) to small changes in four coordinates. The coordinates *resemble* spherical polars (r, θ, ϕ) with time t tacked on. The metric generalises the Pythagoras theorem, and is a geometrical description of how the mass warps spacetime in its vicinity.

Note there are no tensors to be seen, just ordinary calculus. You don't need tensors to perform calculations with it either. Tensors are elegant but for part ② they are optional - and we will opt out*! I am therefore pleased to inform you that

In this unit, we will take spacetime metrics derived elsewhere to learn about how "spacetime tells matter how to move".

** Other omitted GR topics: for cosmology, gravitational waves and other astrophysics, see PH40112 Relativistic Cosmology and PH40113 High Energy Astrophysics; for exotic matter and warp drives - get back to me when they've become science.*

In the next few lectures, we'll do some background and revision work using Newtonian gravity, special relativity and the geometry of curved spaces and spacetimes. Then we'll be ready for some actual GR.

2. Newtonian gravity

We will look at Newtonian gravity, partly to introduce some useful tools and partly to see where it differs from GR. Einstein originally proposed 3 observations, which were realistic to make using the technology of his day, where GR's predictions differ from the Newtonian ones. They are called the *three classical tests of GR*. To contrast the two theories, we need to know what the Newtonian predictions are.

Time

The prediction for one test can be written down right away:

Classical test #1, gravitational time dilation

Newton*: There is no time dilation - time is absolute.

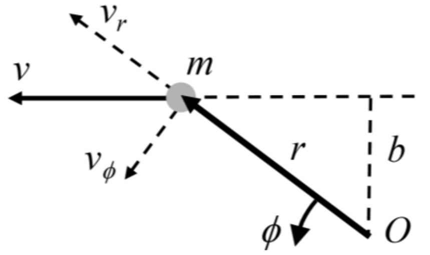
Newton wrote, “Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external.” In contrast, we have already seen how the equivalence principle predicts an inequable flow of time between the ceiling and the floor.

For the other two tests, we need to study the "planetary" motion of particles around a gravitating mass - ie, *orbits*. Most of this should be revision for you!

** ie this is what Newton's theory says; Sir Isaac knew nothing of time dilation ...*

The effective potential

A test particle m moves near a gravitating point-mass M at O.



- The angular momentum of m, defined by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, has a magnitude L that can be thought of in two ways [revision!]:

$$L = mr^2 \frac{d\phi}{dt} = mvb$$

$I\omega$

1. angular version of $p = mv$:

2. moment of momentum: $p \times \perp$ distance

For central forces like gravity, angular momentum is conserved. The size of m is rarely important, so we'll work with angular momentum per unit mass or *specific angular momentum* l:

$$l = \frac{L}{m} = r^2 \frac{d\phi}{dt} = vb \qquad \text{(i)}$$

- The energy of m = KE + PE $= \frac{1}{2}mv^2 - \frac{GMm}{r}$

Likewise, energy per unit mass or *specific energy* E_N. If we write speed v in its radial and azimuthal* components v_r and v_ϕ:

$$E_N = \frac{1}{2}\underbrace{\left(\frac{dr}{dt}\right)^2}_{v_r = dr/dt} + \frac{1}{2}r^2\underbrace{\left(\frac{d\phi}{dt}\right)^2}_{v_\phi = r\omega} - \frac{GM}{r} \qquad [v^2 = (v_r)^2 + (v_\phi)^2]$$

(N for "Newtonian")

= radial KE/m + azimuthal KE/m + grav. potential

* *"radial" = the r direction = in or out from the origin = vertical*
"azimuthal" = the ϕ direction = perpendicular to the radial direction = horizontal

This is the energy equation for a particle undergoing *2-D orbital* motion. But, if we substitute $d\phi/dt = l/r^2$ using the angular momentum formula (i) we can pretend that the particle is undergoing *1-D radial* motion, by combining the azimuthal KE with the true potential to make an *effective potential* $V_N(r)$. The key feature is that the form of V_N depends only on position r not velocity, which is what we expect from a potential function:

$$E_N = K_N + V_N$$

$$= \frac{1}{2}\left(\frac{dr}{dt}\right)^2 \underbrace{- \frac{GM}{r} + \frac{l^2}{2r^2}}_{} \qquad \text{(ii)}$$

K_N = radial KE/m

$$\text{effective potential} \quad V_N(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} \qquad \text{(iii)}$$

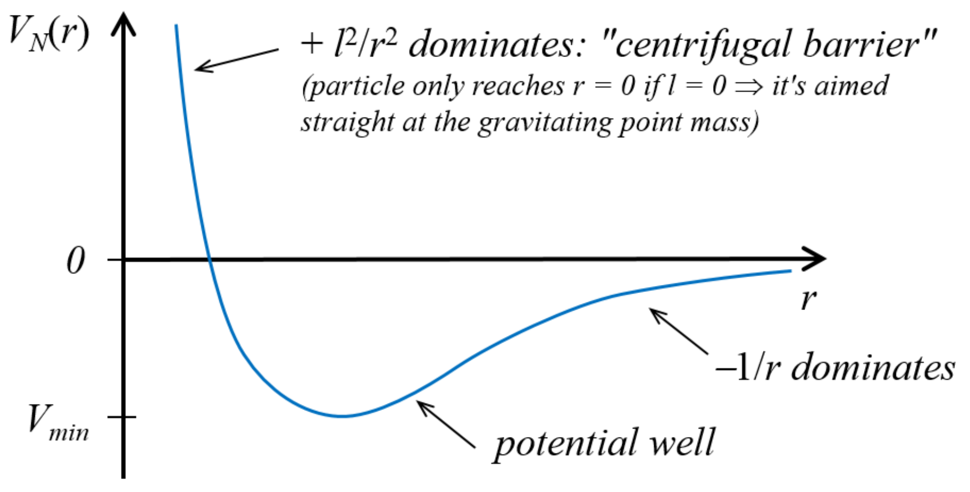
true gravitational potential

"centrifugal term"
(from azimuthal KE)

Now we can analyse the radial part of the particle's motion as if it was just moving along r subject to the effective potential V_N. [When we need the azimuthal part of the motion, we can solve the angular momentum formula (i).]

Note that K_N can never be negative - it's something squared. So the particle can only be where $E_N \geq V_N(r)$, and the gap between E_N and V_N relates to the particle's radial (r direction) speed there. A plot of $V_N(r)$ therefore tells us a lot about the possible orbits.

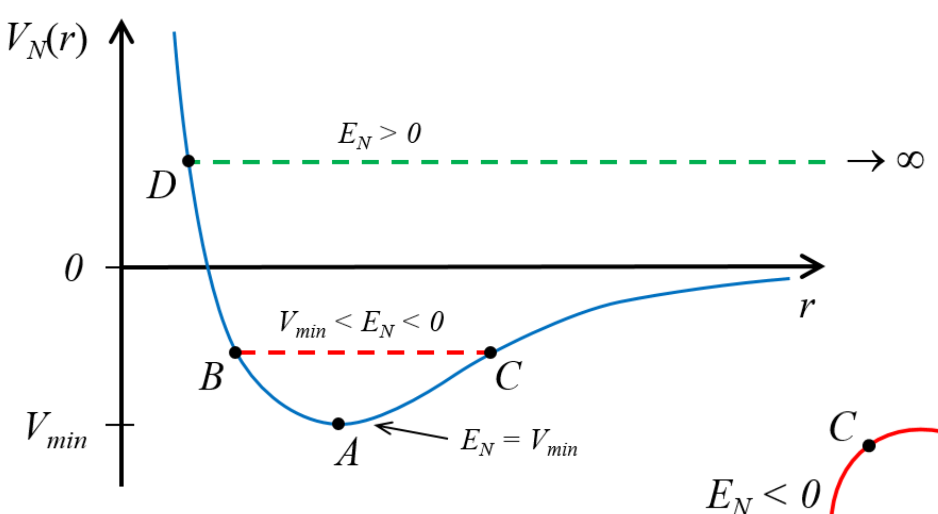
Plot $V_N(r)$ for a fixed l, and possible orbits for different E_N:



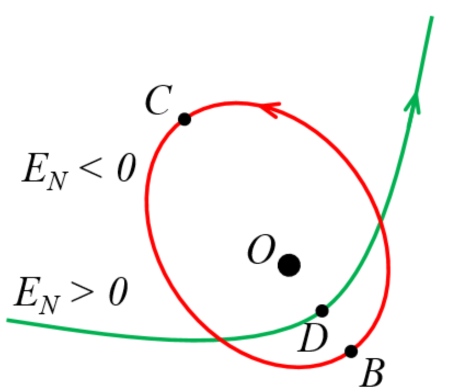
$E_N = V_{min} \Rightarrow$ stable circular bound orbit at A. This is the only value of r allowed for this energy.

$V_{min} < E_N < 0 \Rightarrow$ elliptical bound orbit between *perihelion* (closest point) B and *aphelion* (furthest point) C.

$E_N \geq 0 \Rightarrow$ hyperbolic or parabolic escape orbit with perihelion D and no aphelion.



what the orbits might look like in space:



Only if $l = 0$ can we get plunge orbits that reach $r = 0$.

<u>Shapes of orbits</u> (the *Kepler problem*)

Differential equations (i) and (ii) can in principle be solved to find $r(t)$ and $\phi(t)$. But if we're only interested in the shape of the orbit not the timing, then finding the "polar plot" $r(\phi)$ instead is enough. (If you think the following mathematical trickery is not obvious - you're right!)

Work with $u = 1/r$ instead of r, and write dr/dt as

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\phi}\frac{d\phi}{dt} \qquad \text{[chain rule, twice]}$$

$$= -l\frac{du}{d\phi} \qquad \text{[subst (i) and } du/dr\text{]}$$

Substitute into (ii):

$$\frac{E_N}{l^2} = \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 - \frac{GM}{l^2}u + \frac{1}{2}u^2$$

Differentiate this w.r.t. ϕ, E_N and l being constants. [NB chain rule when differentiating $(du/d\phi)^2$, then $du/d\phi$ cancels]:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{l^2}$$

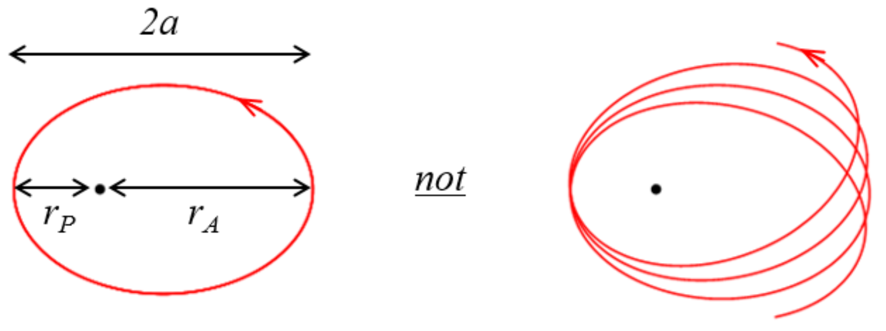
Our trickery has given us something easy: the equation for forced s.h.m.. The general solution follows using YR1 methods:

$$u(\phi) \equiv \frac{1}{r(\phi)} = \frac{GM}{l^2}\left(1 + \varepsilon\sin\phi\right) \qquad \text{(iv)}$$

where ε is one constant of integration, and the other went to define $\phi = \pi/2$ to be the perihelion (the maximum of $u(\phi)$ if we take $\varepsilon \geq 0$). You may recall ε being the eccentricity of the orbit.

- Bound orbits

If $\varepsilon < 1$, u is greater than 0 for all ϕ so $r = 1/u$ is always finite. This corresponds to a bound orbit. The curve turns out to be an ellipse of course (or a circle if $\varepsilon = 0$), but the important thing for us is that $u(\phi)$ in (iv) has a period of exactly 2π. This means that the particle returns to where it started in r after each complete revolution in ϕ, so a bound orbit is a *closed curve*. If nothing else disturbs the orbit, each perihelion is at the same place in space.



And this is the second classical test:

Classical test #2, perihelion shift
Newton: In the absence of other influences, a bound orbit is closed and its perihelion does not shift.

Incidentally, while we're here, let's capture some results from (iv) for future reference:

perihelion distance, $1/u$ at $\phi = \pi/2$, is $\quad r_P = \dfrac{l^2}{GM(1+\varepsilon)}$

aphelion distance, $1/u$ at $\phi = -\pi/2$, is $\quad r_A = \dfrac{l^2}{GM(1-\varepsilon)}$

$\Rightarrow$ semi-major axis is $\quad a \equiv \dfrac{r_P + r_A}{2} = \dfrac{l^2}{GM(1-\varepsilon^2)} \qquad$ (v)

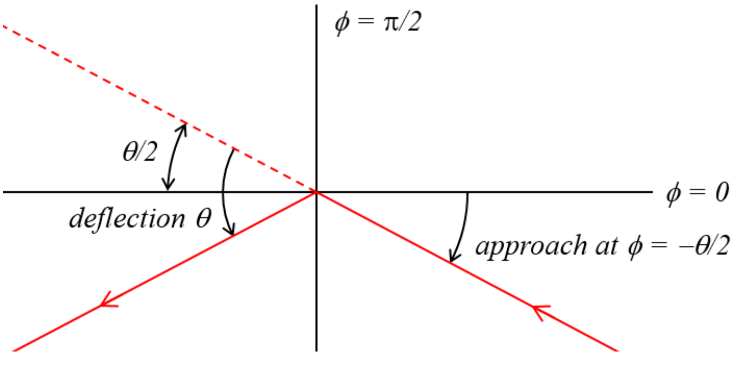
- Escape orbits

If $\varepsilon > 1$, r in (iv) becomes infinite ($u = 0$) for some ϕ, corresponding to an escape orbit with a hyperbolic shape. Let's look at the extreme case where the particle approaches at high speed v with a large impact parameter b (see p. 19), so that it will be deflected only slightly by gravity and its speed is roughly constant. Our aim is to find the small deflection angle θ.

Big scale view:

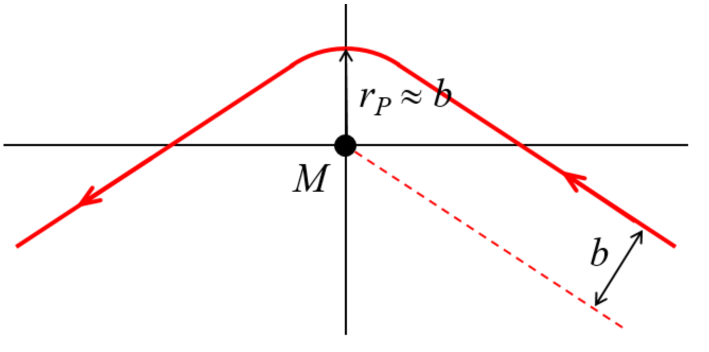
perihelion (and line of symmetry) at $\phi = \pi/2$

$r \to \infty$ when $\phi = -\theta/2$



Small scale view:

$r \approx b$ when $\phi = \pi/2$
(a very good approx for small θ)



At perihelion ($\phi = \pi/2$), substitute $r = b$ and $l = vb$ (from (i) on p. 19) into (iv) to get ε:

$$\frac{1}{b} = \frac{GM}{v^2 b^2}(1+\varepsilon)$$

$$\Rightarrow \varepsilon = \frac{v^2 b}{GM} - 1 \approx \frac{v^2 b}{GM} \qquad \text{[large } v \text{ and } b\text{]}$$

As the particle approaches from a long way away ($\phi = -\theta/2$), substitute $r \to \infty$ into (iv) to relate θ to ε:

$$0 = 1 + \varepsilon \sin\phi$$

$$= 1 - \varepsilon \sin(\theta/2) \approx 1 - \frac{\varepsilon\theta}{2} \qquad \text{[small } \theta\text{]}$$

Then use our value for ε to get θ:

$$\theta = \frac{2}{\varepsilon} = \frac{2GM}{v^2 b}$$

The final classical test is about the deflection of light by gravity. In Newtonian gravity we'll treat light as a particle with a speed of $v = c$:

Classical test #3, deflection of light

Newton: Light approaching mass M with (large) impact parameter b is deflected by an angle of

$$\theta = \frac{2GM}{c^2 b}$$

We now have the Newtonian results for all three of Einstein's classical tests, ready to compare with the predictions of GR.

3. Special relativity

My first lecture on SR will revise Dr Sloan's teaching, and the second lecture will be new material on accelerated motion.

I will adopt two conventions that differ from Dr Sloan's. Here's the first one:

Relativistic units

Relativity is about spacetime, a unified 4-D continuum, so we really should measure all four dimensions using the same units. We therefore adopt SI but with *time measured in metres*.

1 metre of time is the time it takes light to travel one metre:

$$1 \text{ metre} = \frac{1}{2.998\times 10^{8}}\ s \approx 3\ ns \text{ (ie, very small)}$$

$$\Rightarrow \text{speed of light} \quad c = \frac{\text{distance}}{\text{time}} = \frac{1 \text{ metre}}{1 \text{ metre}} \equiv 1 \text{ metre per metre}$$

These units are therefore sometimes called "c = 1 units". In fact all velocities v are expressed as unit-less fractions of c.

To convert a physical quantity with ordinary SI units into c = 1 units, multiply or divide by whatever power of (conventional) c eliminates seconds from the units. For example, G:

$$G = \frac{6.672\times 10^{-11}\ m^{3}s^{-2}kg^{-1}}{(2.998\times 10^{8}\ ms^{-1})^{2}}$$
$$= 7.423\times 10^{-28}\ mkg^{-1}$$

no "s"

NB sometimes the seconds are hidden inside derived SI units, in which case use the dimensions of the unit to find them.

eg [joules] = $\mathrm{M\,L^2\,T^{-2}} \rightarrow kg\ m^2\ s^{-2}$

so the unit of energy becomes kg - does that surprise you?

To convert the other way, from a value in $c = 1$ units into ordinary units, do the opposite: multiply or divide the value by whatever power of (conventional) c restores the right SI unit.

These units simplify relativity (eg, no more "*ct*" on spacetime diagrams) but conversion is undeniably awkward. When evaluating formulae that use $c = 1$, any values you plug in must have the right units. You need to get used to this, and may be tested on it in the exam. (However, it is not my intention to set exam questions to deliberately catch you out.)

Some people go one step further and adopt the metre as the unit of mass as well. For them, 1 metre of mass is the mass that has a Schwarzschild radius of 2 metres, or $G = c = 1$. The motivations for $G = 1$ are much less compelling than for $c = 1$, however, since mass is not another dimension of spacetime. I will keep kilograms and $G \neq 1$, but be aware that various combinations of these conventions are in use "out there" - be careful when pulling equations from books, papers and online sources. (Don't get me started on how these people butcher electromagnetic units!)

From this point on, unless otherwise noted, all our derivations, formulae, problems and exam questions will be in $c = 1$ units.

Spacetime

- Event: a particular place at a particular time, specified by a set of four spacetime coordinates like t, x, y and z.

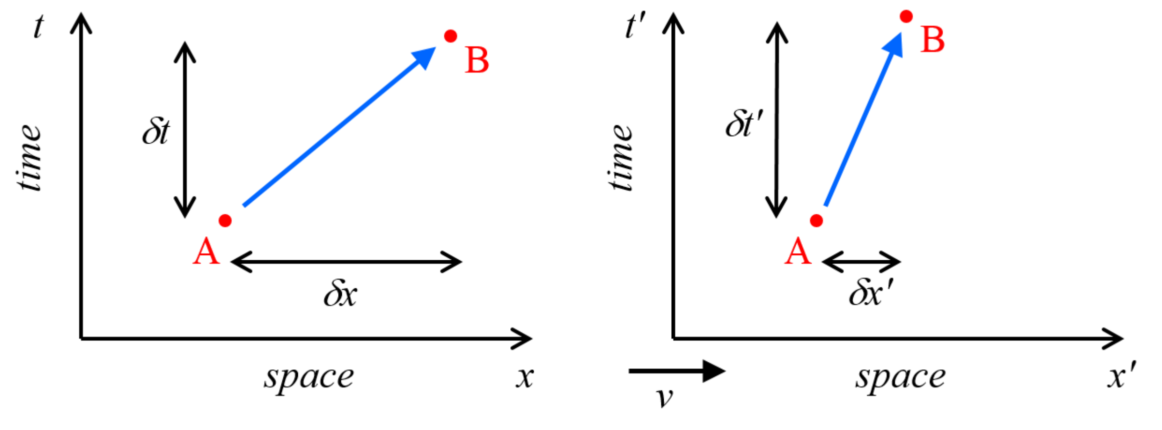
- Proper distance σ: that which is measured by rulers. The ruler measures the distance between two events *at the same time* (simultaneous in the ruler's frame of reference).

- Proper time τ: that which is measured by clocks. The clock measures the time between two events *at the same place* (co-located in the clock's frame of reference).

- Inertial frame of reference: a frame of reference in which Newton's first law of motion holds; an unaccelerated frame.

- Spacetime diagram: a map of t versus x, y and z. (The limitations of 2-D paper usually restrict us to x.)

events A and B, seen in two different frames of reference:



frame Σ, coords t and x

frame Σ', coords t' and x', moving at velocity v relative to Σ

- Special relativity: the laws of physics (including the speed of light) are the same in all inertial frames. This causes the space (δx and $\delta x'$) <u>and</u> time (δt and $\delta t'$) increments between two events to be different in different frames.

- Lorentz transformation: relates coordinate increments in different frames

$$\left.\begin{aligned} \delta x' &= \gamma(\delta x - v\delta t) \qquad & \delta x &= \gamma(\delta x' + v\delta t') \\ \delta t' &= \gamma(\delta t - v\delta x) \qquad & \delta t &= \gamma(\delta t' + v\delta x') \end{aligned}\right\} \quad c = 1, \text{ remember...}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}} \geq 1$$

- Simultaneity: observers in different frames may disagree about which events are simultaneous (or, indeed, co-located).

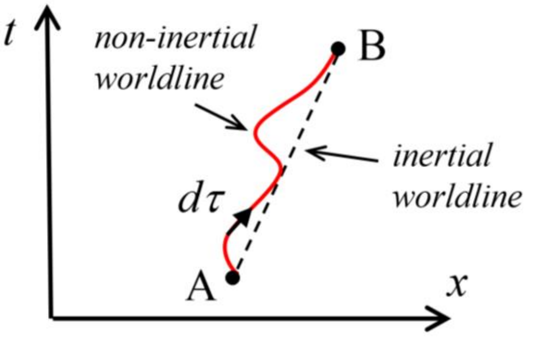
- Length contraction: moving rulers shorten by γ along the direction of motion.

- Time dilation: moving clocks slow by $dt/d\tau = \gamma$.

- Rest frame (of a particle): the inertial frame in which the particle is *instantaneously* at rest, even if it is accelerating*.

- Proper time τ: time measured in the rest frame of the particle, ie the particle's "personal", "experienced" or "wrist-watch" time.

- Worldline: a path in spacetime. Along a non-inertial worldline, τ is the integral of $d\tau$'s of infinitesimal inertial segments.



$$\tau = \int_A^B d\tau$$

- **The principle of maximal proper time: the inertial worldline between two events maximises proper time.**

** When we loosely talk about a particle or observer being "in a frame" or having "their own frame", we mean their rest frame. All observers and particles are in all frames!*

- Relativistic energy: $E = \gamma m$ [$E = \gamma mc^2$ in $c = 1$ units]

$\Rightarrow$ *specific* energy $e \equiv E/m = \gamma = dt/d\tau$ [energy per unit mass]

<u>The interval δs^2 and the Minkowski metric</u>

- A kind of "displacement in spacetime":

space *time*

$$\delta s^2 = -\delta t^2 + \delta x^2 \qquad [1+1\text{ D}]$$

$$\delta s^2 = -\delta t^2 + \delta x^2 + \delta y^2 + \delta z^2 \qquad [3+1\text{ D}]$$

Almost Pythagoras in 4-D, but minus signs distinguish time and space dimensions. Whether the minus signs are attached to the time or the space increments

We'll use this one

$$\delta s^2 = -\delta t^2 + \delta x^2 \quad \textit{versus} \quad \delta s^2 = \delta t^2 - \delta x^2 \qquad [1+1\text{ D}]$$

is a convention: the "metric signature". It's a free choice (both are in use) but, once chosen, we must stick to it. I have chosen the "space-like" or (–, +, +, +) signature in which time has the minus sign, so that concepts of curved space link more directly to curved spacetime. This is the other way (besides units) in which my conventions differ from Dr Sloan's!

- The expression $\delta s^2 = -\delta t^2 + \delta x^2 + \delta y^2 + \delta z^2$ is called the *Minkowski metric* and it describes *Minkowski spacetime*: the flat spacetime of gravity-free SR. Like the more-general metrics we'll meet in GR, it relates a *physical* measurement δs^2 (a proper distance or proper time, as measured by a ruler or clock) to *mathematical* coordinates like δx and δt. The coordinate increments change if you transform to other coordinates (eg a Lorentz transformation, or cartesian-to-polar), but:

- Lorentz transformations $\Rightarrow$ the interval between two nearby events is *absolute* - the same for all frames:

$$\delta s^2 = -\delta t^2 + \delta x^2 + \delta y^2 + \delta z^2 = -\delta t'^2 + \delta x'^2 + \delta y'^2 + \delta z'^2$$

- Recall (p. 28) that proper distance $\delta\sigma$ is the distance between two events in a frame where they are simultaneous. That is, $\delta\sigma$ is δx in a frame where $\delta t = 0$. From $\delta s^2 = -\delta t^2 + \delta x^2$:

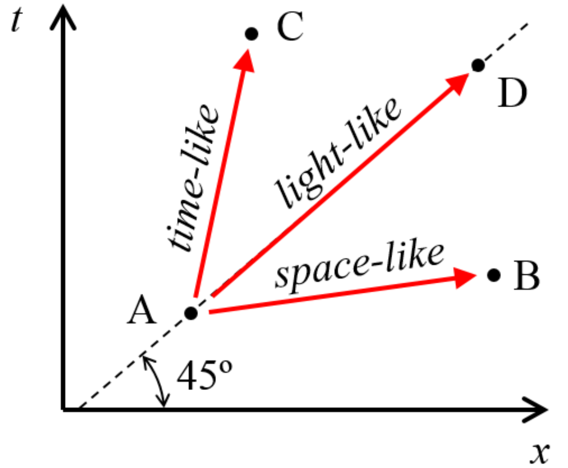
$$\delta\sigma^2 = \delta s^2$$

- Similarly, recall that proper time $\delta\tau$ is the time between two events in a frame where they are co-located. That is, $\delta\tau$ is δt in a frame where $\delta x = 0$. From $\delta s^2 = -\delta t^2 + \delta x^2$:

$$\delta\tau^2 = -\delta s^2$$

- Though written as a square, δs^2 can be positive, negative or zero. This suggests there's a thing called δs that can sometimes be imaginary. Resist that suggestion! There are no complex numbers in GR*. We don't ever square-root δs^2 without first replacing it with whichever of $\delta\sigma^2$ or $-\delta\tau^2$ makes the square root real (see below). Otherwise, just leave it squared and treat it as an object that can have either sign.

What the sign of the interval means:



$$\delta s^2 = -\delta t^2 + \delta x^2$$

* *Unless they're grafted in from other branches of physics, like wave equations.*

- Space-like interval AB: $\delta s^2 > 0$

$-\delta t^2 + \delta x^2 > 0 \Rightarrow$ more space $|\delta x|$ than time $|\delta t|$.

Observers in different frames may disagree on the time order of A and B, $\delta t > 0$ (B after A) vs $\delta t < 0$ (B before A), and there's a frame in which $\delta t = 0$ (A and B are simultaneous).

All observers agree on the space order of A and B, $\delta x > 0$ (B is always to the right of A).

The proper distance $\delta\sigma$ between A and B is $\delta\sigma = \sqrt{\delta s^2}$

The proper time between A and B is not defined, since there is no frame in which they are co-located.

- Time-like interval AC: $\delta s^2 < 0$

$-\delta t^2 + \delta x^2 < 0 \Rightarrow$ more time $|\delta t|$ than space $|\delta x|$.

Observers in different frames may disagree on the space order of A and C, $\delta x > 0$ (C right of A) vs $\delta x < 0$ (C left of A), and there's a frame in which $\delta x = 0$ (A and C are co-located).

All observers agree on the time order of A and C, $\delta t > 0$ (C is always after A).

The proper time $\delta\tau$ between A and C is $\delta\tau = \sqrt{-\delta s^2}$

The proper distance between A and C is not defined, since there is no frame in which they are simultaneous.

- Light-like (or null) interval AD: $\delta s^2 = 0$ even if $A \neq D$

All observers agree on the time and space order of A and D, $\delta t > 0$ (D always after A) and $\delta x > 0$ (D always right of A).

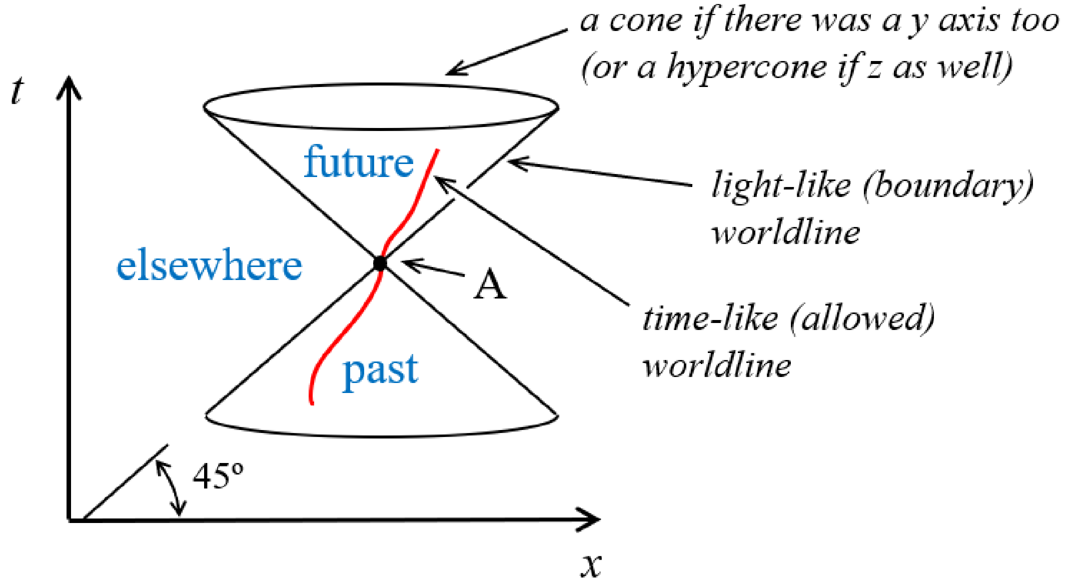
The proper distance $\delta\sigma$ and proper time $\delta\tau$ between A and D are both zero.

- Because δs^2 is invariant, so is its sign $\Rightarrow$ an interval that is time-like (etc) in one frame is time-like (etc) in all frames.

- Matter always travels slower than light, so $\delta x^2 < \delta t^2$, $\delta s^2 < 0$, and its worldline is always time-like.

- *Causality* is the idea that a *cause* must precede its *effect* according to all observers. It is an essential part of the concept of "time". Therefore, no influences can pass between events separated by a space-like interval because different observers will disagree about the time order of the events, and some will say the events are simultaneous. *Causally-connected* events must be separated by a time-like or light-like interval.

- Light cone: the set of all time-like worldlines through a given event A, bounded by the light-like worldlines through A. All events causally connected to A, or reachable from A (if in the future), or from which A can be reached (if in the past), lie within its light cone.



Did you notice this vertical bar on the last 3 pages? It marks the most conceptually-important revision material for this unit, which you must know AND THOROUGHLY UNDERSTAND to do well at GR!!

Acceleration in SR

This is a bit of a side-track from GR. But acceleration mimics the gross effects of gravity, so studying accelerated motion in SR will give us some hints about gravity before starting GR.

- Proper acceleration α: the acceleration of a particle (or rocket) R measured *in its own (instantaneous) rest frame* Σ' :

$$\alpha = \frac{d^2\sigma}{d\tau^2} = \frac{d^2x'}{dt'^2} \qquad \text{while} \qquad \frac{dx'}{dt'} = 0$$

This is the acceleration that R itself feels. If $\alpha = 9.8\ \mathrm{ms}^{-2}$, it will feel like gravity on the surface of the Earth.

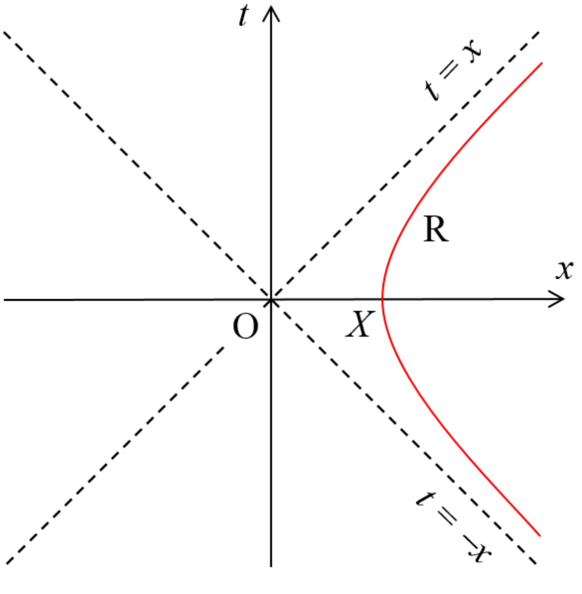
In a different inertial frame (in which R is not at rest), the same expression gives a mere "coordinate acceleration", whose value does not relate simply to what it feels like to R.

- Uniform acceleration

Say R's worldline in inertial frame Σ is the positive-x branch of

$$x^2 - t^2 = X^2$$

where X is a constant. The worldline is a hyperbola with 45° asymptotes through O: $x = X$ at $t = 0$, and $x \to \pm t$ as $t \to \pm\infty$.



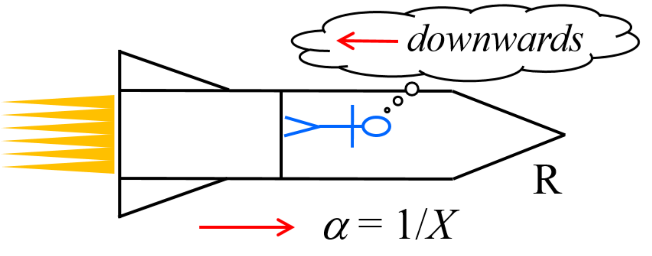
It can be shown (see the problem set, A9) that R has an instantaneous velocity (measured in Σ) of

$$v = \frac{t}{x} \tag{vi}$$

and a constant proper acceleration of

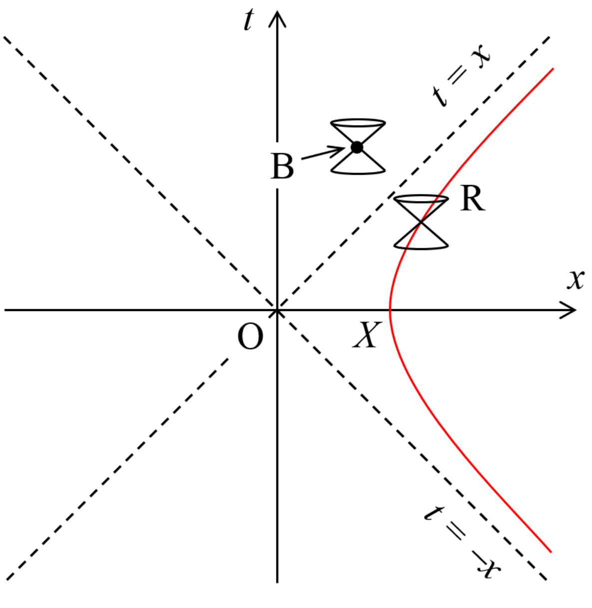
$$\alpha = \frac{1}{X}$$

R is like a rocket that accelerates uniformly for all time



- Causality of eternal acceleration

Consider event B above the line $t = x$. No part of R's worldline is inside the future light cone of B, and B is never inside the past light cone of any event on R. This means that no objects or signals originating at B can ever reach, communicate with or influence R. Objects or signals from R can travel across the line $t = x$, but then can never return to R.

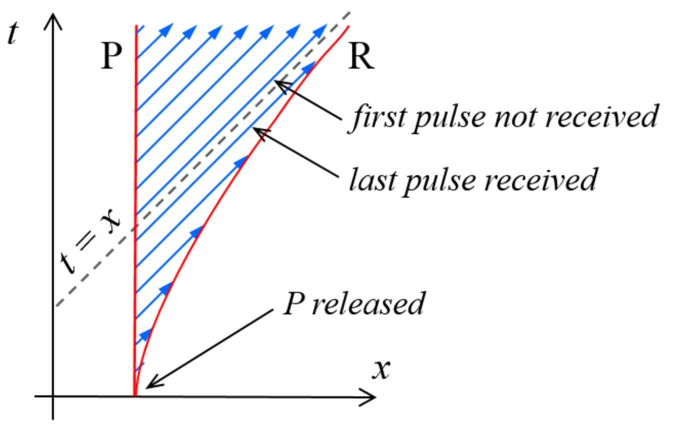


Therefore

$$\boxed{\text{The line } t = x \text{ is an } \textit{event horizon}^{*} \text{ for R.}}$$

A probe P is released from R at $t = 0$, when R is instantaneously at rest. P is unpowered, so remains at rest with a vertical worldline. Meanwhile R continues its acceleration and sees P drop "downwards" in the negative-x direction, as if falling off a cliff on a world with acceleration-due-to-gravity $\alpha = 1/X$.

Like Sputnik, all P does is emit periodic light (or radio) pulses, to be detected by R. Plot the worldlines of R, P and the pulses on a spacetime diagram:



The pulses (blue lines at 45º) are emitted regularly by P, but the time between the arrival of the pulses at R lengthens, and pulses emitted beyond the horizon $t = x$ never arrive at all. R sees P approach, but never reach, the horizon, and interprets this as time running ever more slowly for P.

Yet P just sits at rest in inertial frame Σ. Nothing special or even noticeable happens to P, at the horizon or afterwards.

So, the accelerating rocket experiences a gravity-like field with many of the properties we associate with black holes, yet described entirely by SR.

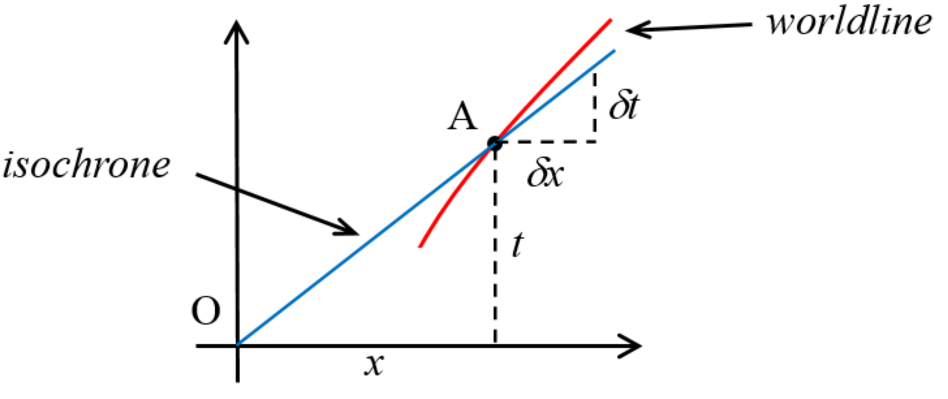
* *Likewise, ponder the causal status of the line $t = -x$ and the events below it for R ...*

- The Rindler frame - an accelerating frame of reference

Say Σ' is the rest frame of R at a particular event A, when its velocity is v relative to Σ. Events simultaneous with A define an *isochrone* - a line of constant time according to R. The (reverse) Lorentz transformations with $\delta t' = 0$ give

$$\delta x = \gamma(\delta x' + v\delta t') = \gamma\delta x'$$
$$\delta t = \gamma(\delta t' + v\delta x') = \gamma v\delta x'$$

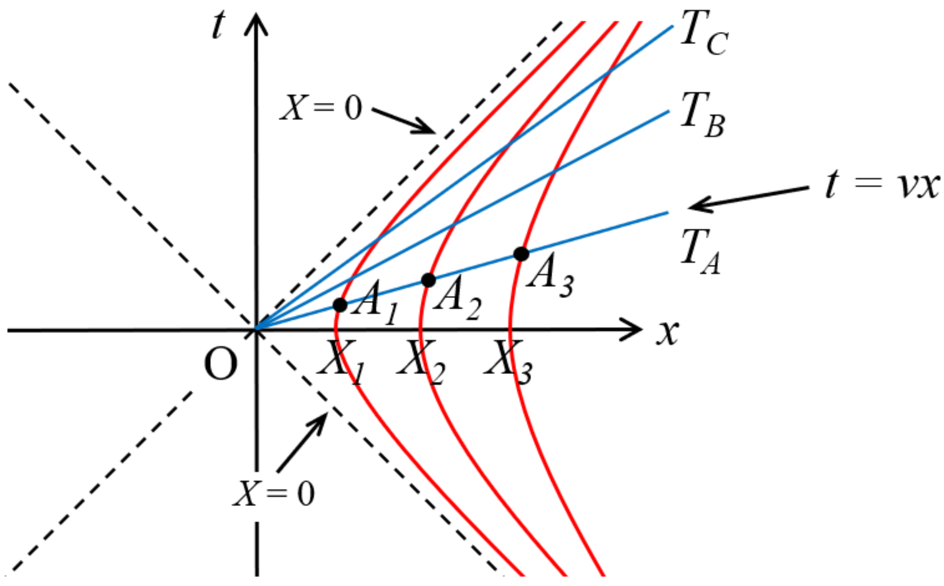
Dividing gives the slope of the isochrone through A: $dt/dx = v$. However, the line joining O to A also has a slope of v because (vi) on p. 35 tells us that $t/x = v$. Therefore straight lines $t = vx$ through the origin are isochrones for R when its velocity is v.



This result does not depend on X, which determines the acceleration of R and the value of x when $t = 0$. A whole *family of accelerating rockets* R_1, R_2, R_3 ... with similar worldlines but different $X = X_1, X_2, X_3$ ... therefore shares the same isochrones, each one through simultaneous events like A_1, A_2, A_3 ... at which the rockets share the same velocity v. Consequently:

1. The R_i are always at rest with each other, since at a given time (ie on an isochrone) they have the same velocity v. Instead of a family of rockets, the R_i could be *parts of one (tall) rocket*.

2. Their relative positions are X_i at $t = 0$, and therefore always.



3. X therefore defines a *position co-ordinate* in an *accelerating frame of reference* (the *Rindler frame*) in which the R_i are at rest. The hyperbolae $x^2 - t^2 = X^2$ are lines* of constant X, and the event horizon $t = x$ is a degenerate hyperbola with $X = 0$.

4. The straight lines $t = vx$ are lines of constant time in this frame, and so conceptually define the frame's *coordinate time* T. (We'd need to do more work to quantify T, eg equating it to the proper time of one of the R_i, but we won't bother.)

5. How can the R_i always be at rest relative to each other while having *different proper accelerations* $\alpha = 1/X$? Between adjacent isochrones (ie for a given increment δT of coordinate time T), all R_i change velocity by the same δv.

But from the definition of acceleration (using proper variables):

$$\delta v = \alpha \ \delta\tau$$

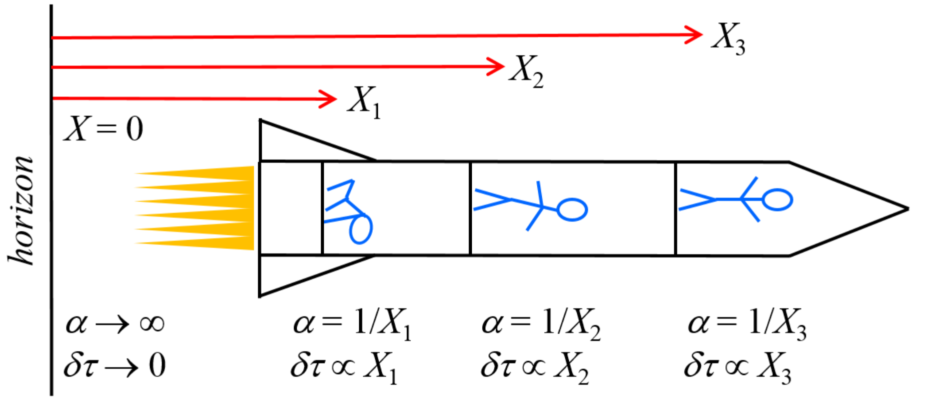
Therefore, in the *coordinate time* increment δT, the *proper time* increments $\delta\tau$ for different X values must differ

$$\delta\tau \propto \frac{1}{\alpha} = X$$

* *Dunno what the fancy word for that is: iso-X'es?*

It's *"gravitational" time dilation* again: time passes more slowly "lower down" at smaller X, but bigger proper acceleration acting for smaller proper time yields the same change of velocity.

6. $X = 0$ means $\delta\tau = 0$: time for the accelerating frame stands still at the event horizon where $v \to 1$, the speed of light. This is a *coordinate singularity*: the Rindler coordinates (T, X) fail at the horizon, but spacetime itself is well-behaved there physically (the probe P doesn't even notice it).



In summary, GR and black holes versus SR and accelerating frames:

effect	GR (gravity)	SR (Rindler)
non-inertial frames	✓	✓
more weight ($m\alpha$) lower down	✓	✓
more time dilation lower down	✓	✓
an event horizon	✓	✓
coordinate singularity at horizon	✓	✓
uneventful fall through horizon	✓	✓
a centre of attraction	✓	✗
curved spacetime	✓	✗
tidal forces	✓	✗
physical singularity at centre	✓	✗
peace, joy and long life beyond the horizon	✗	✓
Hawking radiation	✓	✓*

* *It's called Unruh radiation instead, but it's analogous.*

- Coordinate time and proper time

In relativity we use the *metric* to relate coordinate time to proper time. The Rindler frame illustrates the distinction between them. In everyday life we're used to τ being $T_2 - T_1$ (it's $\tau = 1$ hour between T_1 = 8:00 pm and T_2 = 9:00 pm) but in relativity that ain't necessarily so.

Coordinate time (T in the Rindler case) is time as in "what's the time?" It's a label which, with 3 space coordinates, uniquely specifies an event, a point in spacetime. Like a runner's position in a race, it may put events in the right order (eg 2nd before 3rd) and identify simultaneous events (eg joint 4th) for a given frame, but may not relate simply to the passage of time. Its value may be different in different coordinate systems, may not be in time units, and may not even have physical meaning.

A spatial analogy would be using house numbers on a street to identify the location of a bus stop or a particularly interesting dead squirrel. You expect house numbers to get things in the right order (the squirrel at no. 17 is further along than the bus stop at no. 9) but they don't tell you distances. A coordinate singularity would be like the length of a house tending to zero so that an infinity of house numbers is crammed into zero distance - without doing anything at all to the actual street.

Proper time τ is time as in "how much time?" It's the physical time between events, as measured by a clock on a particular worldline. A different worldline may yield a different τ between the same events, but it's the same for a given worldline in all coordinate systems (because it's given by the interval δs^2, and δs^2 is invariant), and it's always measured in time units (for us, metres).

4. Geometry

Here's another Wheeler quote: “Gravity is geometry.”

We're familiar with flat space; what about curved spacetime? Two routes:

flat space

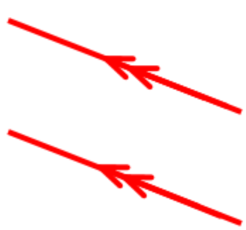
curved space　　　　　　　　flat spacetime

curved spacetime

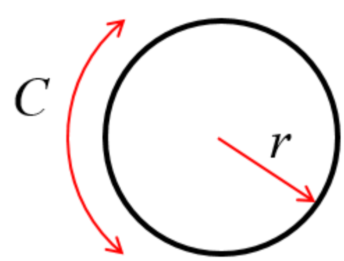
This webpage, by a philosopher of science:
http://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/index.html
contains a very readable introduction to curved spacetime. At least read ch. 24 "General Relativity" just over half-way down. Ch. 18-23, on non-Euclidean geometry, provide a fuller background.

Flat* space

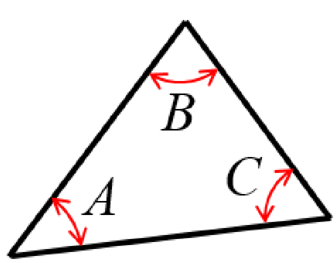
This is the "Euclidean geometry" of the ancient Greeks (like Euclid) that you learned at school. Of course you *know* that the following propositions are true:



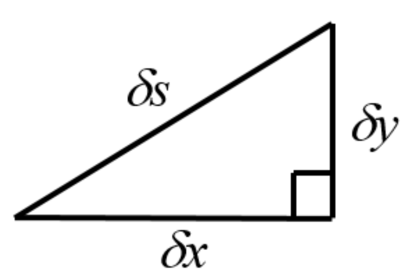
parallel lines never meet



circumference: $C = 2\,\pi\, r$



interior angles: $A + B + C = 180°$

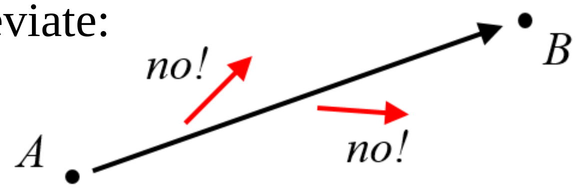


Pythagoras: $\delta s^2 = \delta x^2 + \delta y^2$

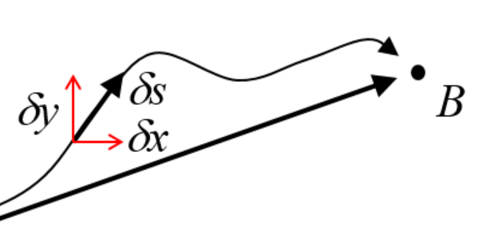
* *NB "flat" here doesn't mean* *two-dimensional*; *it means* *not curved*.

- Straight lines - two equivalent definitions:

1. Keep moving forward, don't deviate:



2. Shortest distance: minimise $\int_A^B ds$



- Pythagoras in 3-D / "solid" geometry

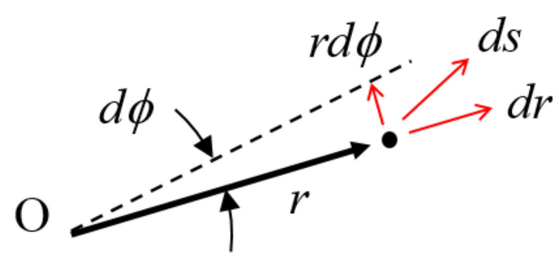
$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2$$

the *metric of flat 3-D space* in cartesian coords.

- The metric of flat space in other coordinate systems*

2-D plane polars

$$ds^2 = dr^2 + r^2 d\phi^2$$



3-D cylindrical polars

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

[plane polars with *dz*]

3-D spherical polars [problem set, B3]

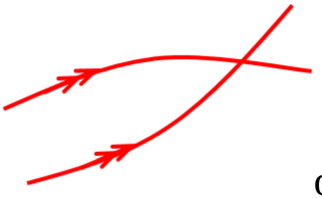
$$ds^2 = \underbrace{\;\;}\, dr^2 + \underbrace{r^2}\, d\theta^2 + \underbrace{r^2 \sin^2\theta}\; d\phi^2 \qquad \text{(vii)}$$

"metric coefficients" are what multiply the infinitesimals
(so the metric coeff of dr^2 is 1)

** Because coordinates change direction in general, we'll need to use infinitesimals dx etc instead of δx etc from now on.*

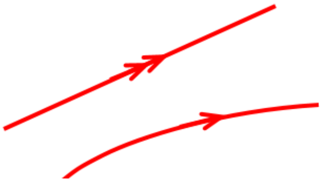
Curved space

First explored as *differential geometry* in the 19th century, but familiar to cartographers mapping the Earth's spherical surface:



some parallel lines meet (positive, or spherical, curvature)

or



some non-parallel lines don't meet (negative, or hyperbolic, curvature)

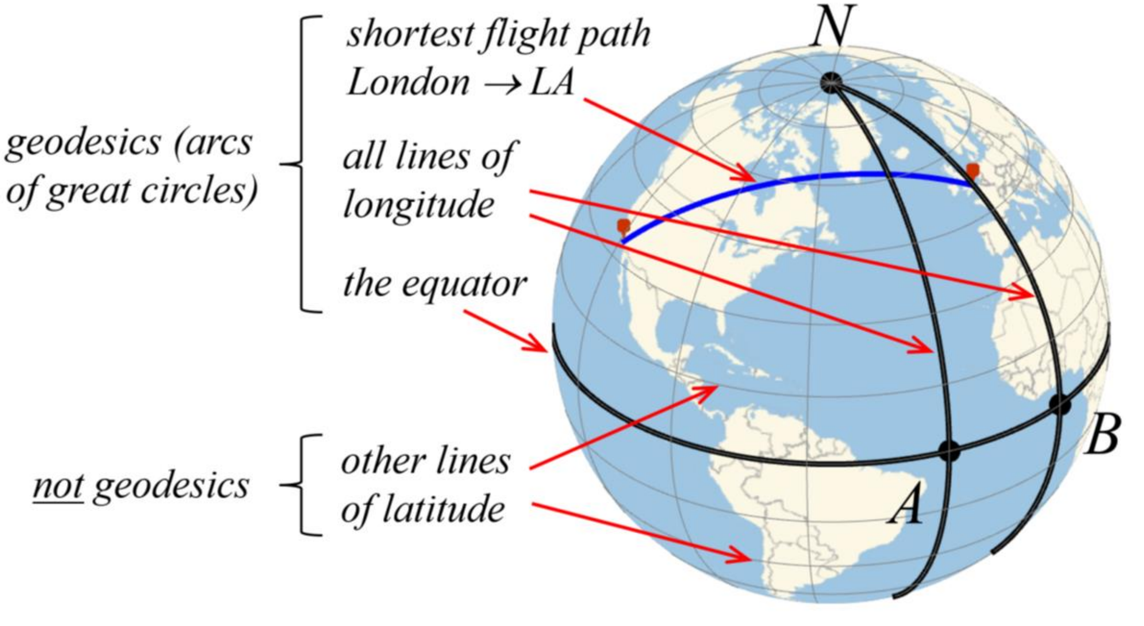
circumference: $C \neq 2\,\pi\,r$

interior angles: $A + B + C \neq 180^{\circ}$

not the geometry of Euclid & Co: "non-Euclidean geometry"

• Straight lines → *geodesics* in the language of curved space, but still defined by "don't deviate" and "shortest distance*".

eg, on a spherical surface, the geodesics are arcs of *great circles*



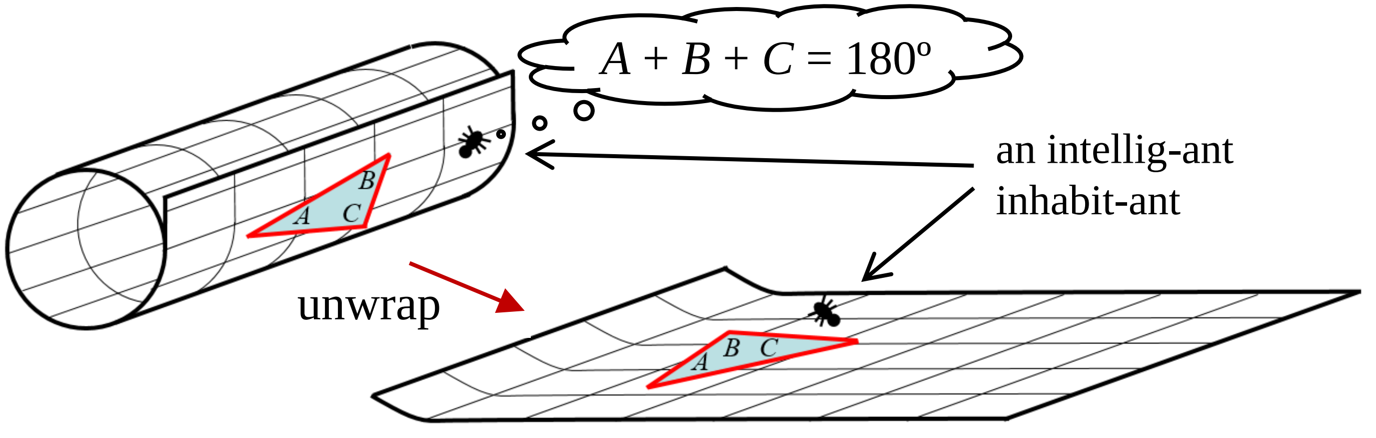
The lines of longitude at A and B are "parallel" (both ⊥ equator) and "straight", but they meet at the north pole N.

* *Strictly-speaking, that all neighbouring paths are longer: it's a local-minimum thing.*

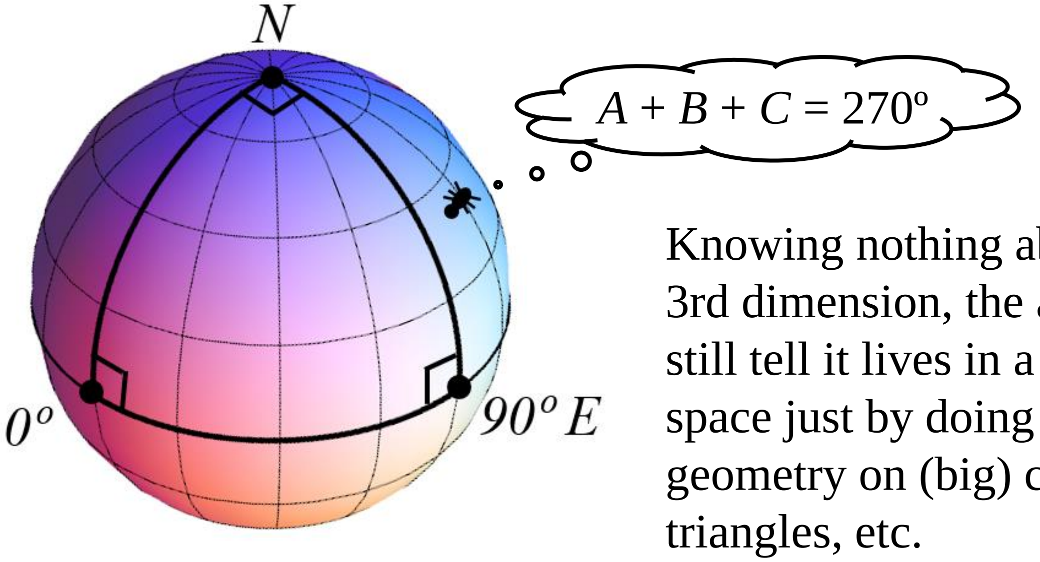
- *Extrinsic curvature* is about how a space is embedded within a higher-dimensional space. For example, 2-D cylindrical and spherical surfaces are both extrinsically curved in 3-D space.

- But in GR we only care about *intrinsic curvature*, measured *within* the space itself without reference to higher dimensions*.

The "curved" surface of a cylinder is actually intrinsically flat, as measured by a 2-D inhabitant. It can be unwrapped to a flat sheet without tearing, crumpling, or distorting 2-D geometric figures:



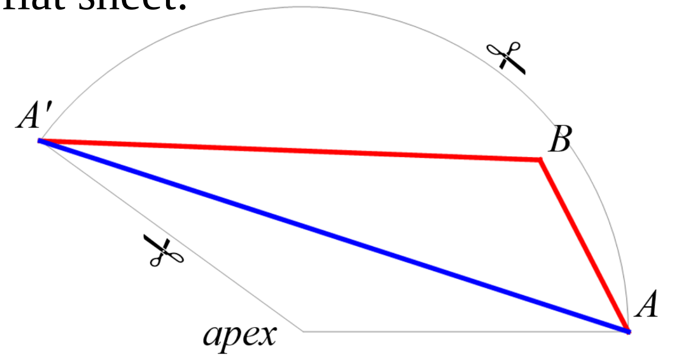
In contrast, the surface of a sphere really is intrinsically curved. It can't be flattened without distortion (eg there are no perfect map projections for the whole world).



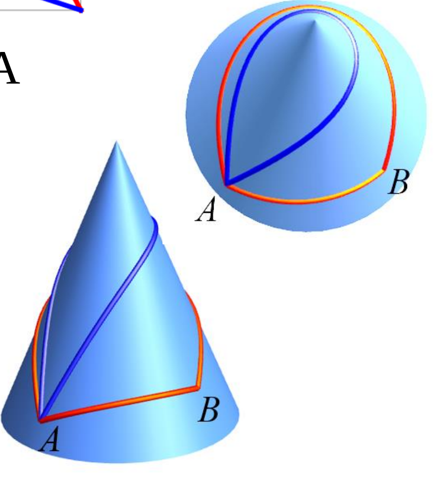
Knowing nothing about the 3rd dimension, the ant can still tell it lives in a curved space just by doing 2-D geometry on (big) circles, triangles, etc.

* *Why? Because spacetime doesn't have an outside to look at it from!*

• Spheres are *positively* curved* (circumferences of circles are less than $2\pi r$, see problem B1). So are cones (a circle round the apex has a short circumference), but all the curvature is at the apex. Our ant knows it's flat everywhere else, and we can make a cone from a flat sheet:

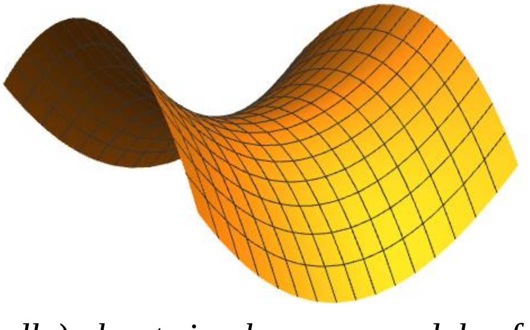


On a cone made from the diagram, points A and A' coincide. There are *two* straight lines joining A and B so A→B→A' is the edge of a *digon*: a polygon with 2 straight sides and 2 corners. Even better, the straight line A→A' joins A to itself and is the edge of a *monogon*.



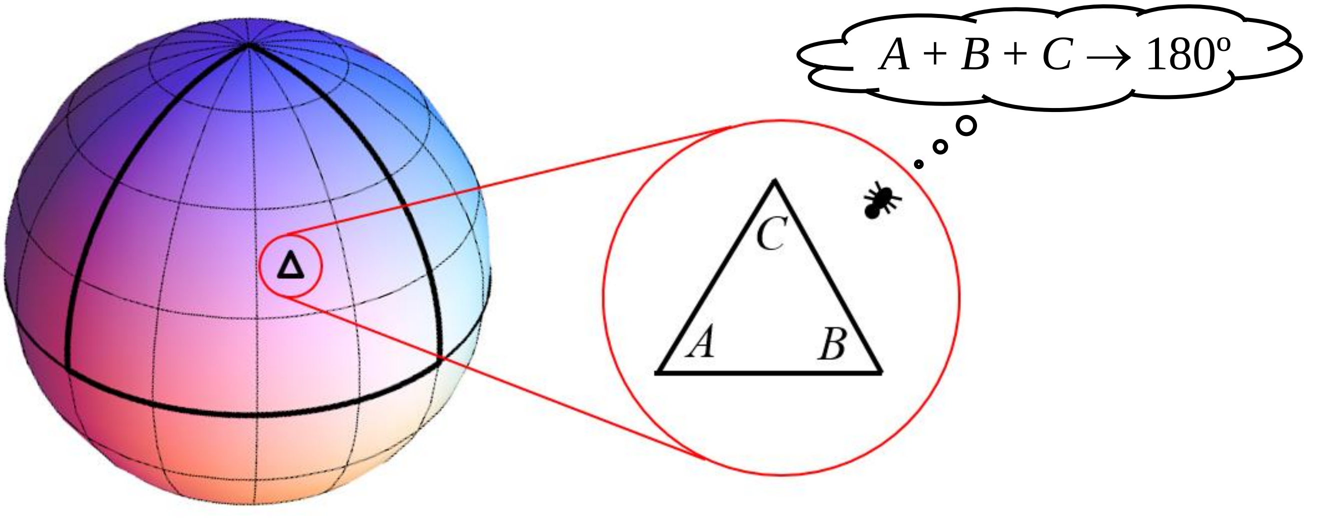
I mention polygons just to show how weird non-Euclidean geometry gets, even on simple familiar surfaces. They're not otherwise very important in GR ...

• *Negatively* curved surfaces* are like saddles (or Pringles). Circumferences of circles are more than $2\pi r$ (problem B1), and the interior angles of triangles add up to less than 180º. (For weird polygons in hyperbolic space, look up *apeirogon*.)



* *Also see the handout (Moodle) about simple paper models of non-Euclidean surfaces.*

• Local flatness: Even in a curved space*, a small-enough region is approximately flat and Euclidean (eg city maps don't need complicated projections).



• Curved metrics: eg for the 2-D surface of a sphere, take the metric of *flat 3-D space* in spherical polars

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \qquad \text{[(vii) on p. 42]}$$

and set $r = R$ (constant) to yield the metric of a *curved 2-D space* mapped by coordinates θ and ϕ:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

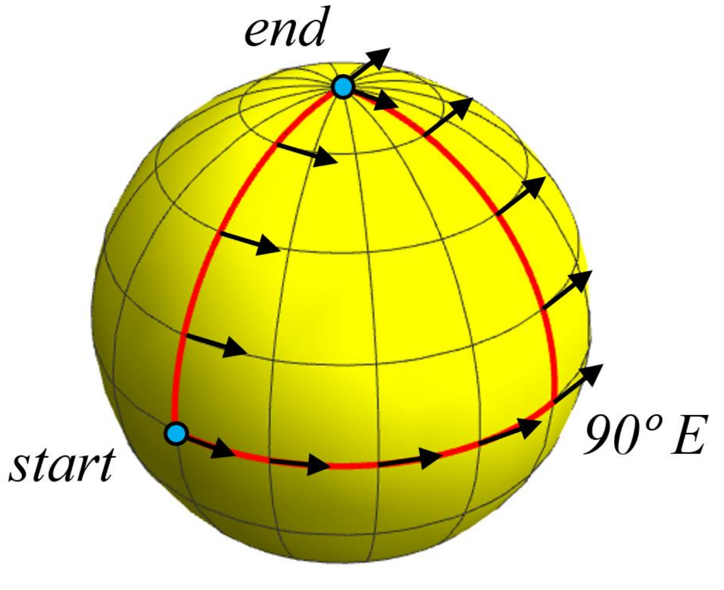
The metric relates the (maths) coordinates $d\theta$ and $d\phi$ to the (physics) distance *ds*. In doing so, it defines the geometry of the space. However, you can't usually tell just by looking - a complicated metric could just be flat space in weird coordinates. The definitive test is to compute the *Riemann curvature tensor* from the metric but, even without tensor analysis, we can still explore curvature by measuring geometric shapes. (See problem B1 for an example using the above metric.)

* *If it's well-enough behaved, unlike the apex of a cone! Even that can be tamed by rounding it off a bit - but then the rounded-off bit can't be made from a flat sheet.*

• Comparing vectors in different places: For example, "How fast is that thing *over there*, relative to me *over here*?"

This is a challenging (even meaningless) question in curved space or spacetime, eg distant things can seem to go faster than light, without breaking the $c = 1$ speed limit where they are.

To compare vectors in different places, you need to "parallel transport" a vector from one place to the other. On each step of the path, make sure the vector is parallel to its copy in the previous step. However, in curved space the final vector *depends on the path*, eg start on the equator of a sphere, and transport a vector to the north pole via two different routes:



Going directly in 3-D space in this example doesn't solve the problem (the ant can't do that), nor does insisting on a geodesic path (try travelling to the antipodes on a great circle via the equator versus via the pole).

In fact the changes to such vectors parallel-transported around small closed loops, ending back where they started, define the Riemann curvature tensor I mentioned a moment ago.

Flat spacetime

space dimensions

time dimensions

This is the 4-D, or 3+1-D, spacetime of SR (without gravity), originated by Einstein and Minkowski.

In SR, inertial worldlines maximise the proper time $\delta\tau$.

In spatial geometry, geodesics minimise distance δs and hence distance squared δs^2. Since $\delta s^2 = -\delta\tau^2$ in SR is negative, spacetime geodesics must maximise the proper time. Therefore:

Inertial worldlines are geodesics in spacetime

(This is a generalisation of Newton's first law of motion.)

- Geometry: Parallel worldlines stay parallel and never meet (they are objects with the same velocity) so spacetime in SR is flat. However, the minus sign in the metric makes it "Minkowskian" not "Euclidean" (eg the ancient Greeks would not recognise a 45° line as having zero length). Still, the spatial part (a "slice" at constant t) is ordinary 3-D Euclidean space.

- The Minkowski metric of flat spacetime (p. 30) is:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{[cartesian coords]}$$

or

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \qquad \text{[spherical polars]}$$

by substituting for the spatial part of the metric (p. 42). See problem B3.

Curved spacetime

This is the spacetime of GR, generalising features of both curved space and flat spacetime.

- Geometry: Parallel worldlines don't stay parallel (due to tidal gravity), so spacetime with gravity is curved: non-Euclidean and non-Minkowskian.

- Metric: "$ds^2 = \ldots$" with 3 space coords and 1 time coord (it may not be easy to tell which is which). It relates mathematical coordinates to the physical interval ds^2, and defines the geometry of the spacetime.

- Local flatness: A small-enough region is approximately flat and Minkowskian $\Rightarrow$ SR is always valid *locally*. Therefore it is still the case that:

$$\boxed{\text{Inertial worldlines are geodesics in spacetime}}$$

- Space-like: $ds^2 > 0$, *proper distance* $d\sigma^2 = ds^2$
- Time-like: $ds^2 < 0$, *proper time* $d\tau^2 = -ds^2$

(as in SR)

- Mechanics: The principle of maximal proper time, together with $d\tau^2 = -ds^2$ from the appropriate metric, leads to the *geodesic equation of motion* (see L9), which can be solved to find the worldlines of free-falling (including orbiting) particles and light.

We will be using this relation, with a negative ds^2 (certainly not an ordinary square) *all the time*. Recall my advice at the bottom of p. 33!

5. The Schwarzschild metric

The solution of Einstein's field equation *outside* an isolated, non-rotating, *spherically-symmetric* source of gravity of mass M (eg outside a planet or star, or everywhere around a black hole):

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2}$$

- My shorthand: "Sch." = Schwarzschild.
- Singularities: Infinite ds^2 for certain coordinate values, at which we can't use the metric to do geometry. There are two:

at	$r = 2GM \equiv r_s$	(the *Schwarzschild radius*)
and	$r = 0$	(the centre)

Both are well inside planets and ordinary stars, so for such objects $r > r_s$ everywhere the metric is valid. This means we can forget the singularities until we study black holes.

<u>What do the Sch. coordinates* mean?</u>

A good coordinate system should uniquely label events, but it doesn't have to mean anything. It so happens that the Sch. coords do have meanings - but they're not quite what they appear.

- $r >> r_s$: Far from M (or when $M \to 0$ since $r_s = 2GM$), the Sch. metric becomes flat spacetime in recognisable coords:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$ [see p. 48]

$\Rightarrow$ (t, r, θ, ϕ) are spherical polar coords with time *in this limit*. Sch. spacetime is *asymptotically flat*: SR is valid far from M, and hence Newton's laws as well for speeds slow compared to c.

* *Sch. coords = the coords (t, r, θ, ϕ) used above. (For other coords see L14 and L16.)*

- Meaning of t:

The square of the proper time $d\tau$ between events *in the same place* is $-ds^2$ with $dr = d\theta = d\phi = 0$ (p. 49):

$r_s \equiv 2GM$

$$d\tau = \left(1 - \frac{r_s}{r}\right)^{1/2} dt \qquad \text{[from the Sch. metric]}$$

For large $r >> r_s$ (where gravity is weak), $d\tau = dt$

$\Rightarrow$ coordinate time t is the proper time experienced by an observer at rest at large r (the *observer at infinity*).

However, for smaller r, proper time τ *does not match* coordinate time $t \Rightarrow$ *gravitational time dilation*. The time τ experienced by an observer at rest at finite r is less than the time t experienced by the observer at infinity*.

- Meanings of θ and ϕ:

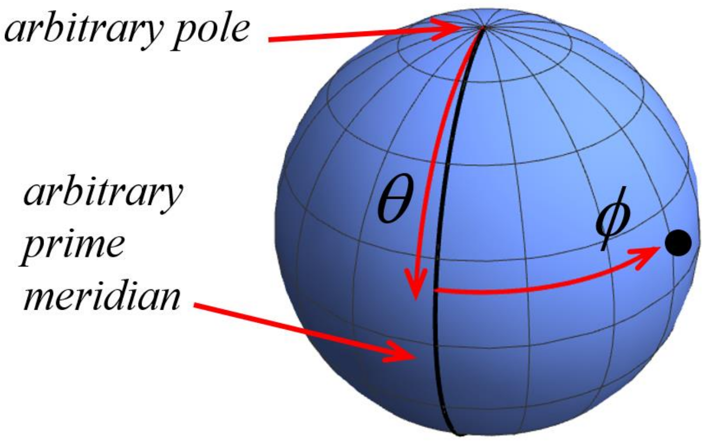
The square of the proper distance $d\sigma$ between neighbouring events is ds^2 when the time displacement $dt = 0$. On a shell of constant r (ie, $dr = 0$):

$$d\sigma^2 = r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2 \qquad \text{[from the Sch. metric]}$$

which is the same as the metric of a spherical surface of radius r (see p. 46).

$\Rightarrow$ θ and ϕ are the angles of ordinary spherical polars for any value of r, not just $r >> r_s$. θ is the angle down from an arbitrarily-defined pole (co-latitude) and ϕ is the angle around from an arbitrarily-defined prime meridian (longitude).

** The expression on p. 13 from the equivalence principle used approximations. The one here is exact.*



Because the constant-r shell has the same metric as a spherical surface of radius r, it has the same geometry: its circumference is $C = 2\pi r$, its surface area is $A = 4\pi r^2$, etc.

There will be times when we don't care much about θ and ϕ, especially since they behave so normally. Then the θ and ϕ parts of the metric can be abbreviated to

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2 \qquad \text{["solid angle form"]}$$

and the Sch. metric looks like

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Meaning of r:

This is the trickiest. It is *not* the distance from the origin - we can't measure distances through the singularities! But we *can* measure the circumference C of a constant-r shell: we find that it's $C = 2\pi r$, because (see above) the geometry of the shell is the same as that of a spherical surface of radius r.

$\Rightarrow$ r is the *reduced circumference* or *circumferential radius* of the surface of constant r

$$r = \frac{C}{2\pi}$$

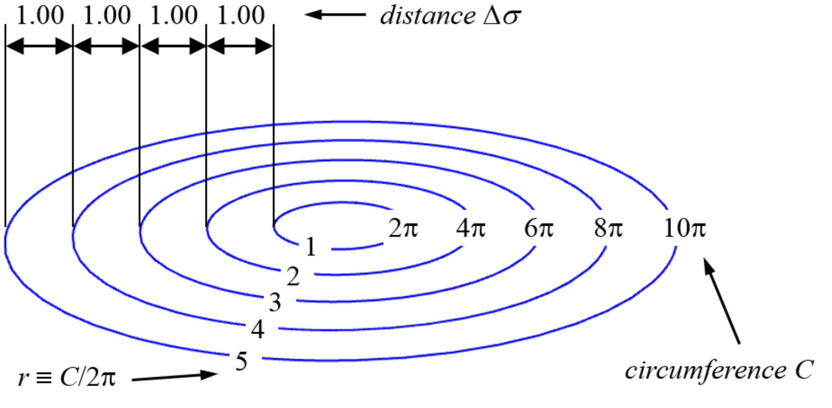
The preceding point is not merely pedantic. The square of the *radial* proper distance $d\sigma$ between neighbouring values of r is ds^2 when $dt = d\theta = d\phi = 0$, so

$$d\sigma = \frac{dr}{\left(1 - \frac{r_s}{r}\right)^{1/2}}$$

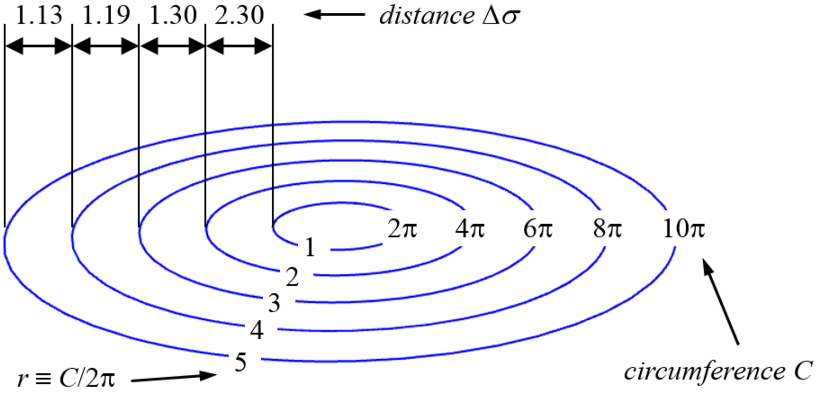
For finite r, $d\sigma > dr$: there's more distance between concentric shells than you'd expect from their circumferences. This is a non-Euclidean result: an example of gravity warping spacetime.

We can still loosely call r "the radius", but don't forget it isn't!

Here's an oblique view of some concentric circles on a flat Euclidean plane, with their separations $\Delta\sigma = r_2 - r_1$ marked:



And here are constant-r shells in Sch. spacetime (in units of r_s), with their proper-distance separations $\Delta\sigma$ marked*:

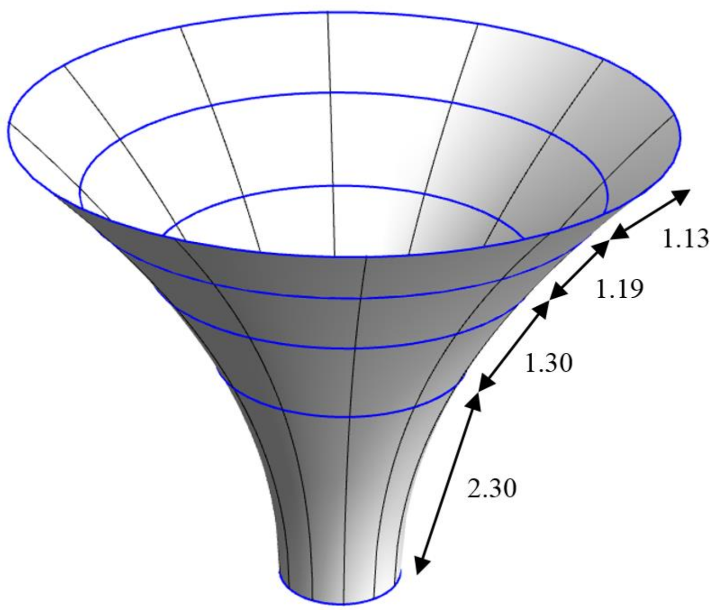


* *You'll derive the formula to calculate extended values of* $\Delta\sigma$ *in problem B6.*

How can we visualise this excess space between shells of given circumferences?

(a) Don't try: it's non-Euclidean curved spacetime!

(b) OK, if you must: *imagine* a (Euclidean) *hyperspace*, with an artificial fake not-really-there extra z dimension into which we "push" the shells until their separations are right:

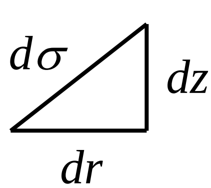


This is an example of an

- Embedding diagram: An imagined surface $z(r, \phi)$ in cylindrical polars, with the same relationship between the arc length σ and coords r and ϕ as the Sch. metric at fixed time t. (Spherical symmetry, so fix $\theta = \pi/2$ without loss of generality.)

For a given ϕ, our Sch. proper distance $d\sigma$ should match the arc length $d\sigma$ for increments dr and dz:

Euclidean arc length:



$$d\sigma^2 = dr^2 + dz^2$$

$$\Leftrightarrow$$

actual proper distance:

$$d\sigma = \frac{dr}{\left(1 - \frac{r_s}{r}\right)^{1/2}}$$

Equate $d\sigma$'s: $$\left(1-\frac{r_s}{r}\right)^{-1} dr^2 = dr^2 + dz^2$$

$$\Rightarrow \quad \left(\frac{dz}{dr}\right)^2 = \frac{r_s}{r-r_s}$$ [÷ both sides by dr^2]

This is a separable differential equation, which is easy to integrate to yield the embedding diagram $z(r, \phi)$:

$$z(r,\phi) = \int dz = \pm r_s^{1/2} \int \frac{dr}{\sqrt{r-r_s}} = \pm 2 r_s^{1/2} \sqrt{r-r_s} + k$$

from the √ (arc lengths are the same with the surface either way up)

constant of integration (just moves the whole surface up or down)

This surface (shown on the previous page) is known as *Flamm's paraboloid*, and I'm sure you've seen it before. Indeed there was probably a lump of something in the middle, pulling down a rubber sheet across which you roll a marble to illustrate curved spacetime acting like a gravitational force blah blah.

So, what exactly is pulling the lump down so that it deforms the sheet??

In fact an embedding diagram is just a picture of how proper distances relate to changes in r. Our z axis has no physical existence - there is no conceptual need for a hyperspace in which to embed the surface. And there are *several reasons* (see problem B7) why rolling a marble across the surface does not relate to particle motion in GR!

Gravitational time dilation

We saw in the last lecture that the Sch. metric predicts time dilation. This is of course one of the classical tests of GR. Now we can compare Einstein's answer to Newton's:

Classical test #1, gravitational time dilation

Newton: There is no time dilation - time is absolute.

Einstein: $$d\tau = \left(1 - \frac{2GM}{r}\right)^{1/2} dt$$

Time slows down in a gravitational field.

The "slow down" factor is close to 1 in most cases:

$$\left(1 - \frac{2GM}{r}\right)^{1/2} \approx 1 - 10^{-9} \qquad \text{[Earth's surface due to Earth]}$$

$$\approx 1 - 10^{-8} \qquad \text{[Earth's orbit due to the Sun]}$$

$$\approx 1 - 10^{-5} \qquad \text{[Sun's surface due to the Sun]}$$

$$\approx 1 - 10^{-3} \qquad \text{[white dwarf's surface]}$$

$$\approx 1 - 0.5 \qquad \text{[neutron star's surface]}$$

$$= 1 - 1 \qquad \text{[black hole event horizon]}$$

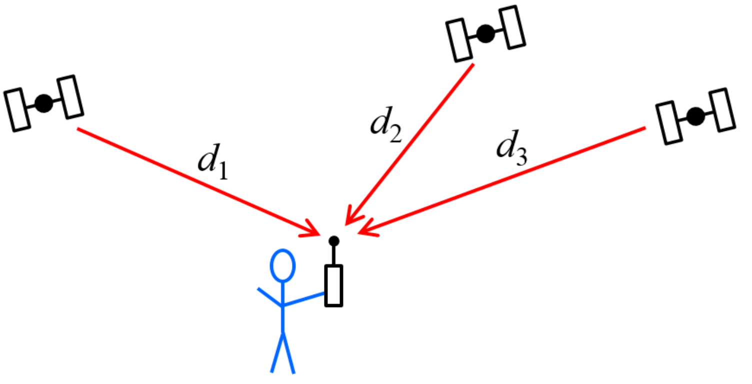
- Approximate expression: if $r >> r_s$, use a binomial approx

$$d\tau \approx \left(1 - \frac{GM}{r}\right) dt \qquad [(1 + x)^n \approx 1 + nx]$$

$$= (1 - gr)dt \qquad \left[\text{Newton:} \quad F = mg = G\frac{Mm}{r^2}\right]$$

(matches the result on p. 13 from the equivalence principle)

- Global Positioning System (GPS)



Radio signals from 3 (out of 24) satellites specify their positions and times. Comparison with the receiver's own clock ⇒ time travelled by each signal ⇒ the distance d_i to the satellite ⇒ the position of the receiver is at the intersection of 3 spheres of radii d_i centred on the satellites. (The satellites carry precise atomic clocks, but obviously a cheap receiver doesn't. A fourth satellite is therefore required to provide accurate time information.)

The timing calculations must include the effect of gravitational time dilation between the satellite's orbit and the ground. There will also be some SR time dilation due to the motion. Use the Sch. metric to find the time dilation for the satellite (in orbit "up there") and the receiver (on a rotating Earth "down here").

Sch. metric for equatorial motion ($\theta = \pi/2$) at fixed r ($dr = 0$):

$$d\tau^2 = -ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - r^2 d\phi^2$$

$$= dt^2\left\{1 - \frac{2GM}{r} - r^2\omega^2\right\} \qquad \left[d\phi = \frac{d\phi}{dt}dt = \omega dt\right]$$

Divide $d\tau$ for the satellite by $d\tau$ for the receiver:

$$\frac{d\tau_{sat}}{d\tau_{rec}} = \left(\frac{1 - \dfrac{2GM}{r_{sat}} - r_{sat}^2 \omega_{sat}^2}{1 - \dfrac{2GM}{r_{rec}} - r_{rec}^2 \omega_{rec}^2} \right)^{1/2}$$

Use a binomial approx to dispose of the square root (both numerator and denominator are close to 1), and subtract 1 to write the answer as a relative discrepancy in time:

$$\frac{\Delta t}{t} = \frac{d\tau_{sat} - d\tau_{rec}}{d\tau_{rec}} = \frac{GM}{r_{rec}} + \frac{r_{rec}^2 \omega_{rec}^2}{2} - \frac{GM}{r_{sat}} - \frac{r_{sat}^2 \omega_{sat}^2}{2}$$

For GPS satellites in 12-hour orbits, and a receiver at the equator (for simplicity), the discrepancy is 38.6 μs / day, of which 45.7 μs / day is due to gravitational time dilation and the rest is due to the satellite's relative velocity and SR.

- Gravitational redshift

Use the period T of an e.m. wave as a clock.

$$\omega = \frac{2\pi}{T} \qquad \Rightarrow \qquad \omega_\infty = \left(1 - \frac{2GM}{r} \right)^{1/2} \omega_r$$

frequency at ∞ frequency at r

Light of frequency ω_r emitted at r is redshifted to ω_∞ as it travels to ∞. (Or, light emitted at ∞ is blueshifted as it travels to r.)

WS Adams' 1925 redshift measurement in light from Sirius B (a white dwarf) seemed to confirm Einstein's classical test, but was later shown to be contaminated by light from Sirius A. The first reliable astronomical redshift was measured in 1962 by JW Brault in light from the Sun, confirming GR to 5% accuracy.

- Pound and Rebka experiment

The first successful measurement of gravitational time dilation (confirming the classical test) was not astronomical but an experiment on the Earth by R Pound and GA Rebka in 1959, confirming GR to 10% accuracy. γ rays from a source at the bottom of a tower travelled to the top, 22.5 m above. Their frequency was measured using the Mössbauer effect, a very precise solid-state method, tuned using the Doppler effect by oscillating the source.

- Shapiro delay

Light appears to slow down in gravitational fields, eg consider radial ($d\theta = d\phi = 0$) motion of light ($ds^2 = 0$), Sch. metric:

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

$$\Rightarrow \quad \frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right)$$ ["coordinate velocity" dr/dt]

$|dr/dt| < 1$ doesn't mean light has actually slowed down. r and t are just coordinates, not real distance and time except at ∞. However, it does represent a time delay as seen from large r. This was measured by II Shapiro by bouncing radar signals off Venus and Mercury in 1966-7, confirming GR to 20% accuracy.

- Atomic clocks

State-of-the-art strontium clocks are so accurate that the effects of gravitational time dilation over height differences of 1 mm are noticeable in the lab.

6. The geodesic equation of motion

General metrics

Say the spacetime coords are $x^\mu = x^0, x^1, x^2, x^3$. The superscripts ($\mu = 0, 1, 2, 3$) are index labels not exponents, with $\mu = 0$ as time if possible. A general metric is written as a sum:

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu dx^\nu$$

[eg 2-D: $ds^2 = g_{00}dx^0dx^0 + g_{01}dx^0dx^1 + g_{10}dx^1dx^0 + g_{11}dx^1dx^1$]

The $g_{\mu\nu}$ are the *metric coefficients* (and form the *metric tensor* when collected together, perhaps as a 4×4 matrix). Writing it out in full, the coeff multiplying a *diagonal term* like $(dx^1)^2$ is g_{11}.

But, *mixed* / *off-diagonal* / *cross terms* like dx^0dx^1 appear twice, as $g_{01}dx^0dx^1$ and $g_{10}dx^1dx^0$, even though dx^0dx^1 and dx^1dx^0 are the same. We could subtract an amount from g_{01} and add it to g_{10} without changing the overall sum. To remove this unwanted freedom we define $g_{\mu\nu}$ to be *symmetric* ($g_{\nu\mu} = g_{\mu\nu}$), so the term containing dx^0dx^1 in the overall sum is $2g_{01}dx^0dx^1$, etc.

[eg 2-D: $ds^2 = g_{00}(dx^0)^2 + 2g_{01}dx^0dx^1 + g_{11}(dx^1)^2$]

This reduces the 16 independent components of $g_{\mu\nu}$ to 10 in actual (4-D) spacetime.

A *diagonal metric* is one where the cross terms are all zero.

$$g_{\mu\nu} = 0 \qquad \text{if } \mu \neq \nu \quad \text{(diagonal metric)}$$

In a diagonal metric, the time coord can be recognised by having a negative metric coefficient $g_{00} < 0$,

because we need a time-like ds^2 when the other coords are fixed.

Example: the Sch. metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

In the above notation: $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ (though we could shuffle the numbers around for the spatial coords).

* $$g_{tt} = g_{00} = -\left(1 - \frac{2GM}{r}\right) \qquad g_{rr} = g_{11} = \left(1 - \frac{2GM}{r}\right)^{-1}$$

$$g_{\theta\theta} = g_{22} = r^2 \qquad g_{\phi\phi} = g_{33} = r^2 \sin^2\theta$$

There are no cross terms like $drd\theta$ so the metric is diagonal, and we identify t as the time coord because $g_{tt} < 0$ (if $r > 2GM$).

The geodesic equation

As outlined on p. 49, to study the motion of free-falling particles we derive a new equation of motion (replacing $F = ma$) using:

The principle of maximal proper time: the inertial worldline between two events maximises proper time. [SR, p. 29]

The proper time between two events on a worldline is given by integrating the metric along the worldline with $ds^2 = -d\tau^2$. The *calculus of variations*, which most of you won't know, can then be used to maximise this proper time. (There's a derivation on Moodle if you're interested, but it won't be in the exam.) The result is the *geodesic equation*, a differential equation for each coord $x^\nu(\tau)$ as τ (acting as a parameter) varies. This is our new equation of motion, which we solve to find the inertial / free-fall / geodesic worldline between two events:

* *We can informally write g_{00} as g_{tt} because x^0 is t, etc.*

$$\boxed{0 = \frac{d}{d\tau}\left\{\sum_{\beta=0}^{3} g_{\alpha\beta}\frac{dx^\beta}{d\tau}\right\} - \frac{1}{2}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \quad \text{[geodesic equation]}$$

These are 4 equations, one for each coord x^α ($\alpha = 0, 1, 2, 3$), like $F = ma$ is really a vector equation with 3 components. (In contrast, β, μ and ν are dummy indices that get summed over.)

This is a fundamentally-important equation for GR, but as a replacement for $F = ma$ it's not pretty! Further tensor notation improves its appearance a bit (p. 64) but not much. However, often some useful simplifications will apply.

- Simplifying the geodesic equation:

(#1) If there's a coord x^α which none of the $g_{\mu\nu}$ depends on, then $\partial g_{\mu\nu}/\partial x^\alpha = 0$ and the double sum in that x^α's equation vanishes

$$\Rightarrow \quad \sum_{\beta=0}^{3} g_{\alpha\beta}\frac{dx^\beta}{d\tau}$$

is a *constant of the motion*, ie it is conserved (because its τ derivative is zero).

(#2) Many metrics (including the Sch. metric) are diagonal:

$$g_{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu$$

If there's a coord x^α which none of the $g_{\mu\nu}$ depends on (as in #1), only the term with $\beta = \alpha$ survives:

$$\Rightarrow \quad g_{\alpha\alpha}\frac{dx^\alpha}{d\tau} \qquad \text{(viii)}$$

is conserved.

(#3) If all the $g_{\mu\nu}$ only depend on one of the coords x^α, the equations of motion for the other x^μ can be found as in #1 and #2 above. Then we can get the remaining x^α equation of motion directly from the metric:

$$-ds^2 = d\tau^2 = -\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$$

$$\Rightarrow \qquad 1 = -\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \qquad [\div\, d\tau^2]$$

This can be solved for the $dx^\alpha/d\tau$ that we seek, given that the other $dx^\mu/d\tau$ are already known.

If the metric is diagonal, the equation further simplifies to

$$1 = -\sum_{\mu=0}^{3} g_{\mu\mu}\left(\frac{dx^\mu}{d\tau}\right)^2 \qquad \text{[diagonal metric]}$$

All three of these simplifications can be used when considering motion under the Sch. metric.

<u>Trivial example</u>: free-fall motion in the absence of gravity, use the Minkowski metric in cartesian coords

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{[p. 48]}$$

A diagonal metric with coeffs: $g_{tt} = -1$, $g_{xx} = g_{yy} = g_{zz} = 1$

None of the $g_{\mu\nu}$ depends on any of the coords, so (viii) $\Rightarrow$

$$\frac{dt}{d\tau} = k_0 \qquad \frac{dx}{d\tau} = k_1 \quad \frac{dy}{d\tau} = k_2 \quad \frac{dz}{d\tau} = k_3$$

constant time dilation — constant velocity (magnitude and direction)

- Aside: the Einstein summation convention (not used in this unit, and certainly not examined)

Why write indices as superscripts, risking confusion with exponents? It's the *Einstein summation convention* of tensor analysis: if an index is repeated in a product, once as a subscript and once as a superscript, summation over that index is implied and the Σ signs are dropped. Thus the general metric on p. 60 becomes

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

μ and ν appear as super/sub pairs, so each is a dummy index summed from 0 to 3. The geodesic equation becomes

$$0 = \frac{d}{d\tau}\left\{ g_{\alpha\beta} \frac{dx^\beta}{d\tau} \right\} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

Again μ, ν and β are summed, but α is not repeated: it is not a dummy index to be summed over, but simply takes the four values 0, 1, 2 or 3 in turn, giving four separate equations.

This upstairs-downstairs feature represents an important distinction within tensor analysis, but it won't be important for us in PH30101. Because we don't use tensors in this unit, we'll just keep the Σ signs. However, we will use the superscripts in x^α etc for consistency, to help those who are going to study GR further.

The geodesic equation can be (and usually is) written as

$$0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

by those who know tensor analysis (and know what $\Gamma^\alpha_{\mu\nu}$ is).

7. Orbits in Schwarzschild spacetime

Now we can use the Sch. metric and the geodesic equation to study the motion of free-falling particles - ie, their orbits.

Equations of motion

- Symmetry

The orbit must lie in a plane through the origin: the particle can't leave the plane in any preferred direction (left/right) because, by symmetry, there is none. Since the origins of θ and ϕ (pole and prime meridian) are arbitrary, *without loss of generality* we can orient the coords so that the $\theta = \pi/2$ plane coincides with the orbit. This means $\sin\theta = 1$ and $d\theta = 0$, and yields the *equatorial Sch. metric* for the remaining 3 coords:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

- ϕ equation

The metric is diagonal and none of its coefficients depends on ϕ, so

$$g_{\phi\phi}\frac{d\phi}{d\tau} = r^2\frac{d\phi}{d\tau} \qquad \text{[(viii) on p. 62]}$$

is constant. Since this matches specific angular momentum l in the Newtonian limit [(i) on p. 19] and is conserved, we will call it the *relativistic specific angular momentum*:

$$l = r^2\frac{d\phi}{d\tau}$$

and our "angular velocity" ϕ equation of motion is

$$\boxed{\frac{d\phi}{d\tau} = \frac{l}{r^2}} \qquad [l \text{ constant}]$$

- t equation

Motion through space*time* $\Rightarrow$ an equation of "motion" for $t(\tau)$. None of the metric coefficients depends on t, so

$$g_{tt}\frac{dt}{d\tau} = -\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} \qquad \text{[(viii) on p. 62]}$$

is constant. For large r, only SR time dilation makes the particle's τ differ from the static observer's t, so $dt/d\tau = \gamma$ (p. 29). Now, in SR, energy per unit mass is $e = \gamma$ (p. 30). Since minus our constant is specific energy at $r \to \infty$ but conserved everywhere, we will call it the *relativistic specific energy*:

$$e = \left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}$$

and our "time dilation" t equation of motion is

$$\boxed{\frac{dt}{d\tau} = \frac{e}{\left(1-\frac{2GM}{r}\right)}} \qquad [e \text{ constant}]$$

- r equation

The metric coeffs do depend on r so there's no constant for r. But since r is the only such coord, we can use simplification #3 of the geodesic equation (p. 63) and get "radial velocity" $dr/d\tau$ directly from the metric and the other equations of motion:

$$1 = -\sum_{\mu=0}^{3} g_{\mu\mu}\left(\frac{dx^\mu}{d\tau}\right)^2 \qquad \text{[diagonal metric]}$$

$$= \left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{\left(1-\frac{2GM}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$$= \cancel{\left(1-\frac{2GM}{r}\right)} \frac{e^2}{\left(1-\frac{2GM}{r}\right)^{\cancel{2}}} - \frac{1}{\left(1-\frac{2GM}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 - \cancel{r^2}\,\frac{l^2}{r^{\cancel{4}}}{}_{2}$$

Solve to obtain:

$$\boxed{\frac{dr}{d\tau} = \pm\left\{e^2 - \left(1-\frac{2GM}{r}\right)\left(1+\frac{l^2}{r^2}\right)\right\}^{1/2}}$$

These 3 equations of motion can in principle be solved for the spacetime coords t, r, θ (= π/2) and ϕ as parametric functions of τ, tracing out an orbital path.

The effective potential

For now, though, we get more insight by defining an *effective (specific) energy E* from the true energy e:

$$E = \frac{e^2 - 1}{2} \qquad \text{[then subst } e^2 \text{ from here]}$$

The reason is that, far from M (where $e = \gamma$) and at non-relativistic speeds (where $\gamma^2 = (1 - v^2)^{-1} \approx 1 + v^2$), E tends to the Newtonian kinetic energy per unit mass $\frac{1}{2}v^2$. We can therefore compare E to the energy E_N we used to study Newtonian orbits in L3, and see that the expressions are *almost* the same.

The only difference is an extra *Einstein term* in the effective potential $V(r)$, compared to (iii) on p. 20. It's negligible for $r >> r_s$, where GR and Newtonian gravity agree. But it completely changes the behaviour of $V(r)$ for small r, going to $-\infty$ instead of $+\infty$ as $r \to 0$.

GR: $$E = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}$$

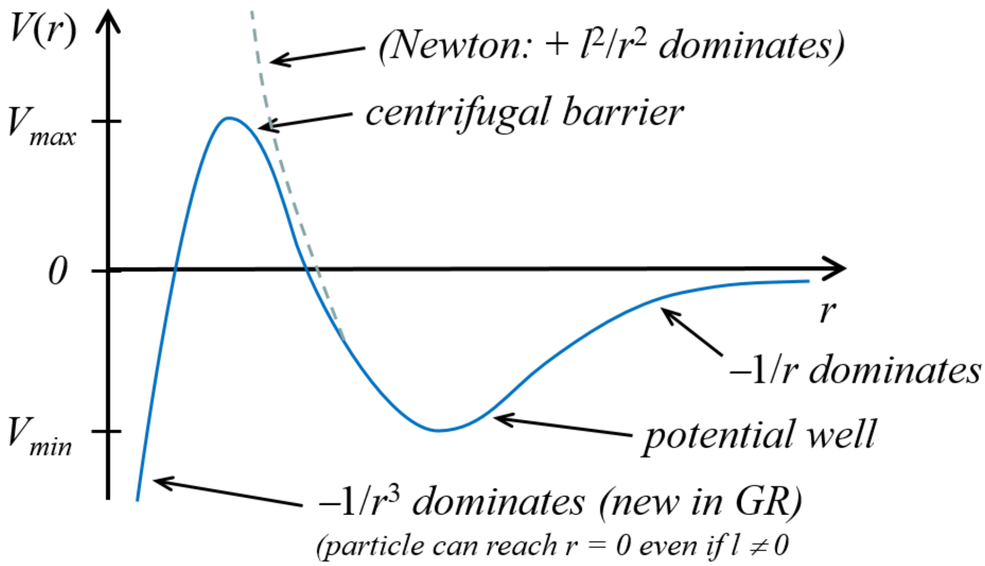
cf Newton: $$E_N = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 \underbrace{- \frac{GM}{r} + \frac{l^2}{2r^2}}_{\text{effective potential } V(r)}$$ [(ii) on p. 20]

effective energy / m — effective KE / m — effective potential $V(r)$

$$\Rightarrow \qquad \boxed{V(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}}$$

gravitational potential — centrifugal term (from angular KE) — extra "Einstein" term

Plot $V(r)$ for big-enough* fixed l: the centrifugal barrier rises to a peak before falling to $-\infty$ as $r \to 0$, allowing *new types of orbits*:



* *If l is too small, the peak subsides and V(r) becomes monotonic.*

Recall that allowed orbits are where $E \geq V(r)$:

$E = V_{min}$ and $r >$ point F $\Rightarrow$ stable circular bound orbit at A. This is the only allowed value of r (beyond F) for this energy.

$V_{min} < E < 0$ and $r >$ point F $\Rightarrow$ ellipse-like bound orbit between perihelion B and aphelion C.

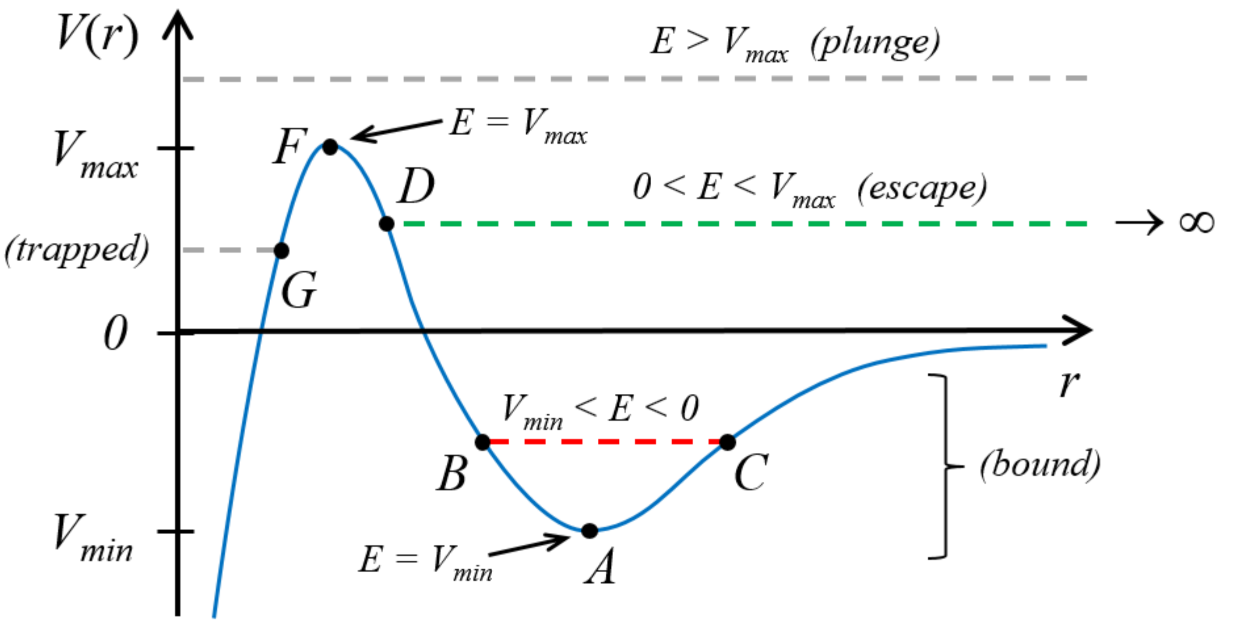
$0 \leq E < V_{max}$ and $r >$ point F $\Rightarrow$ hyperbola- or parabola-like escape orbit with perihelion D and no aphelion.

The above are analogous to the Newtonian cases, but now we also have:

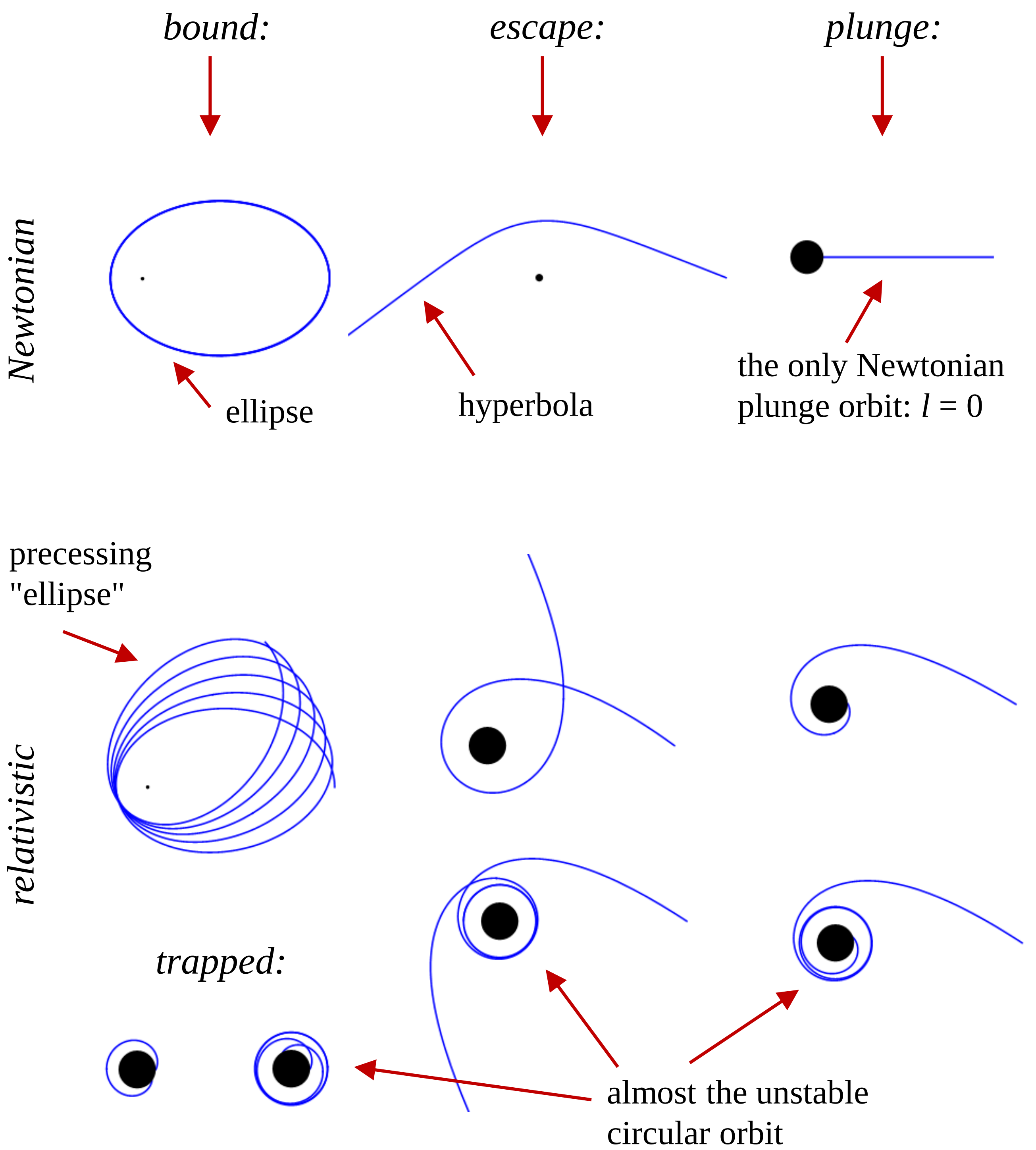
$E = V_{max} \Rightarrow$ *unstable* circular "knife-edge" orbit at F.

$E > V_{max} \Rightarrow$ plunge orbit, the particle falls all the way to $r = 0$ (if it's moving inwards) even though $l \neq 0$, or it escapes (if it's moving outwards).

$E < V_{max}$ and $r <$ point F $\Rightarrow$ trapped orbit with aphelion at G before falling to $r = 0$.



We'll need more space to draw possible orbits than we did in the Newtonian case! The $V(r)$ curves were schematic sketches, but here are some actual computer-generated orbital paths. The Newtonian curves are of course also GR curves in the appropriate limit:



Bound orbits

Now we will study bound orbits in a little more detail. Follow the procedure from L3 on p. 22, but with GR's *l* and *E*:

$$\left.\begin{aligned} l &= r^2 \frac{d\phi}{d\tau} \\ E &= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} \end{aligned}\right\}$$

like (i) and (ii) on pp. 19-20 but with the Einstein term

Work with $u = 1/r$ and the chain rule as before to get

$$\frac{E}{l^2} = \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 - \frac{GM}{l^2}u + \frac{1}{2}u^2 - GMu^3$$

then differentiate w.r.t. ϕ to get an orbit equation:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{l^2} + 3GMu^2$$

just like the Newtonian one but with an extra term at the end.

- Circular orbits

The orbit equation has exact solutions for $u = u_0$, constant:

$$u_0 = \frac{GM}{l^2} + 3GMu_0^2$$

Solve this quadratic* for u_0 and use $r = 1/u$ to get the radius:

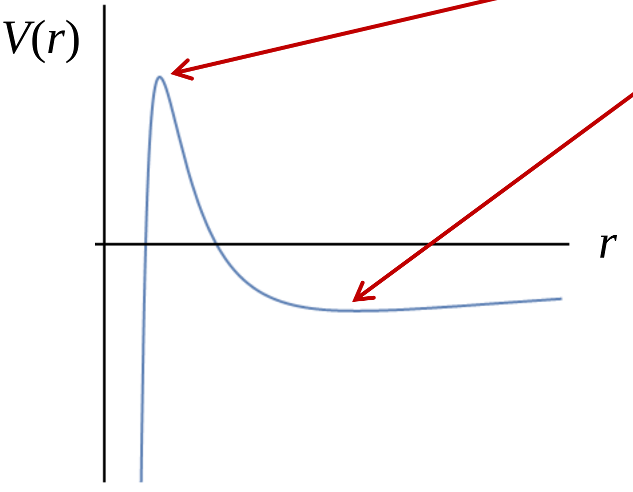
$$r_0 = \frac{6GM}{1 \pm \left[1 - 12\left(\frac{GM}{l}\right)^2\right]^{1/2}}$$

* *Or, derive more directly by seeking the maximum and minimum of V(r) - problem C3.*

There are real solutions r if the square root is non-negative:

$$l \geq \sqrt{12}\ GM$$

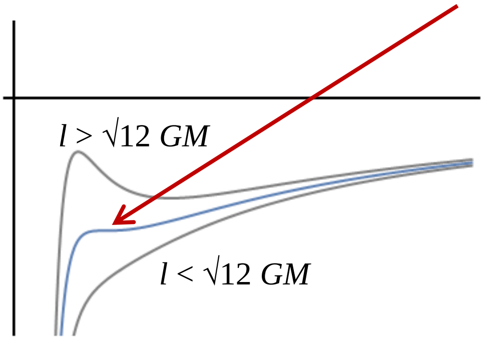
which are a stable (minimum V) circular orbit with $r_0 > 6GM$ and an unstable (maximum V) circular orbit with $r_0 < 6GM$.



As l is reduced, the peak in $V(r)$ declines until at

$$l = \sqrt{12}\ GM$$

the two orbits coincide (at a point of inflection in V) to become the *innermost stable circular orbit* or ISCO with $r_0 = 6GM$:



For smaller l, no circular orbits are possible. This contrasts with the Newtonian case, where circular orbits exist for any l or r.

- Perihelion shift

The extra term in the orbit equation prevents an exact solution for non-circular orbits, but we can find an approximation for orbits that are well outside the Sch. radius. First consider again a circular orbit (p. 71), for which

$$u_0 = \frac{GM}{l^2} + 3GMu_0^2$$

Now consider an *almost-circular orbit* that differs from the circular orbit by a small ϕ-dependent *perturbation* $f(\phi)$:

$$u(\phi) = u_0 + u_0 f(\phi)$$

Substitute into the orbit equation on p. 71:

$$u_0 \frac{d^2 f}{d\phi^2} + u_0 + u_0 f = \frac{GM}{l^2} + 3GMu_0^2 + 6GMu_0^2 f + 3GMu_0^2 f^2$$

equal $\Rightarrow$ cancel (small)^2 $\Rightarrow$ neglect

$$\Rightarrow \qquad \frac{d^2 f}{d\phi^2} + [1 - 6GMu_0] f = 0$$

This is the s.h.m. equation, and has the solution

$$f(\phi) = A\cos\left\{[1 - 6GMu_0]^{1/2}\phi\right\}$$

where the origin of ϕ is chosen so that there is a perihelion (max $f \Rightarrow$ max $u \Rightarrow$ min r) at $\phi = 0$. The period of the perturbation (ie ϕ at the next perihelion) is

$$\phi = \frac{2\pi}{[1 - 6GMu_0]^{1/2}} \qquad ["2\pi/\omega"]$$

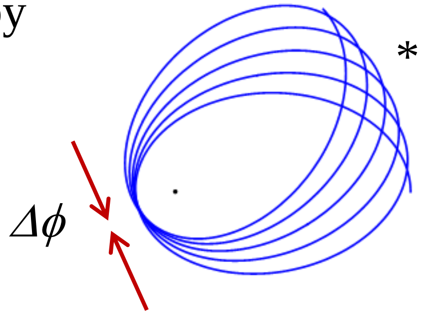
Since we're assuming $r >> r_s$, $GMu_0 = GM/r_0$ is small compared to 1 and we can safely use a binomial approx:

$$\phi = 2\pi\left(1 - 6GMu_0\right)^{-1/2} \approx 2\pi\left(1 + 3GMu_0\right)$$

$$\approx 2\pi + 6\pi\left(\frac{GM}{l}\right)^2 \qquad [u_0 = GM/l^2 + 3GMu_0^2 \approx GM/l^2]$$

This period exceeds 2π (one revolution) by

$$\Delta\phi = 6\pi\left(\frac{GM}{l}\right)^2$$



Subst. for l using the (approximately valid) Newtonian result (v) from p. 23 for the semi-major axis a in terms of l:

$$a = \frac{l^2}{GM(1-\varepsilon^2)} \qquad \Rightarrow \qquad \Delta\phi = \frac{6\pi GM}{(1-\varepsilon^2)a}$$

Unlike Newtonian orbits, GR orbits don't quite close, which is one of the classical tests:

Classical test #2, perihelion shift

Newton: In the absence of other influences, a bound orbit is closed and its perihelion does not shift, $\Delta\phi = 0$.

Einstein: In the absence of other influences, the perihelion of a bound orbit precesses by angle

$$\Delta\phi = \frac{6\pi GM}{(1-\varepsilon^2)a} \quad \text{per orbit.}$$

* *Aphelion shift is easier to see because it's further out, but the angle is the same.*

- Mercury

Careful astronomical observations in the late 19th century found that the perihelion of the planet Mercury shifts at the rate of 574" (arc seconds) per century. Newtonian theory successfully explained most of it (due to known "other influences" like the effects of Venus, Jupiter etc), but a residual shift of 43 "/century could not be accounted for. This value was large compared with measurement uncertainty, so could not be ignored.

Previously, a 300"/century irregularity in the orbit of Uranus led UJJ Le Verrier to propose the existence of an unknown planet further out - Neptune, discovered in 1846. Making the most of a good idea, he then proposed another unknown planet to explain the discrepancy in Mercury's orbit. He called it Vulcan, and expected it to orbit closer to the Sun than Mercury. However, it was never found.

Then Einstein calculated the perihelion shift for Mercury due to GR (using the preceding equation) and got the answer:
43 "/century! Hence the anomaly was explained by a new theory of gravity, rather than a new planet.

This was perhaps not as impressive as the other two classical tests. Here Einstein explained an old observation, rather than predicting the outcomes of new ones. (If it had failed the test, maybe Einstein wouldn't have published his theory and none of us would ever have known of the failure.)

A much bigger perihelion shift >4°/year has since been measured in the binary neutron stars PSR 1913+16, also matching the predictions of GR.

Radial (vertical) motion

For a vertical drop or rise, ϕ does not change at all, so

$$l = r^2 \frac{d\phi}{d\tau} = 0 \qquad [\phi \text{ equation of motion}]$$

We therefore find r as a function of proper time from:

$$\frac{dr}{d\tau} = \pm\left\{e^2 - \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{l^2}{r^2}\right)\right\}^{1/2} \qquad [r \text{ equation of motion, p. 67}]$$

$$= \pm\left(e^2 - 1 + \frac{r_s}{r}\right)^{1/2} \qquad \text{(ix)} \qquad [l = 0]$$

For r in terms of coordinate time t rather than proper time τ, use the t equation of motion (p. 66) and the chain rule:

$$\frac{dt}{d\tau} = e \bigg/ \left(1 - \frac{r_s}{r}\right) \qquad \frac{dr}{dt} = \frac{dr}{d\tau} \bigg/ \frac{dt}{d\tau}$$

If the particle falls *from rest* at a given $r = r_0$, its energy e is given by (ix) with $dr/d\tau = 0$ at r_0:

$$e^2 = 1 - \frac{r_s}{r_0} \qquad \Rightarrow \qquad \frac{dr}{d\tau} = \pm\left(\frac{r_s}{r} - \frac{r_s}{r_0}\right)^{1/2}$$

An important example is free fall from rest at large r ($r_0 \to \infty$), so $e^2 = 1$ and:

$$\frac{dr}{d\tau} = \pm\left(\frac{r_s}{r}\right)^{1/2} \qquad \text{(x)}$$

$$\frac{dr}{dt} = \pm\left(\frac{r_s}{r}\right)^{1/2}\left(1 - \frac{r_s}{r}\right) \qquad \text{(xi)}$$

[free fall from rest at large r]

All these equations can be integrated to give $r(t)$ or $r(\tau)$, but in most cases the exercise is no fun at all. We are going to need them later on though.

Photon orbits

We'd like to use our equations of motion from pp. 65-66

$$\frac{d\phi}{d\tau} = \frac{l}{r^2} \qquad\qquad \frac{dt}{d\tau} = \frac{e}{\left(1-\frac{r_s}{r}\right)}$$

to study light paths near gravitating masses. But, for light-like worldlines $ds^2 = -d\tau^2 = 0$ so we can't use proper time τ! (Recall from SR that light does not experience proper time.) Instead we divide the equations and use t instead of τ as the parameter:

$$\frac{d\phi}{dt} = \frac{l}{e} \times \frac{1}{r^2}\left(1-\frac{2GM}{r}\right) \qquad \text{[chain rule]}$$

As we did with particle motion, we get the r equation directly from the Sch. metric. Set $ds^2 = 0$ (light-like) and divide by dt^2:

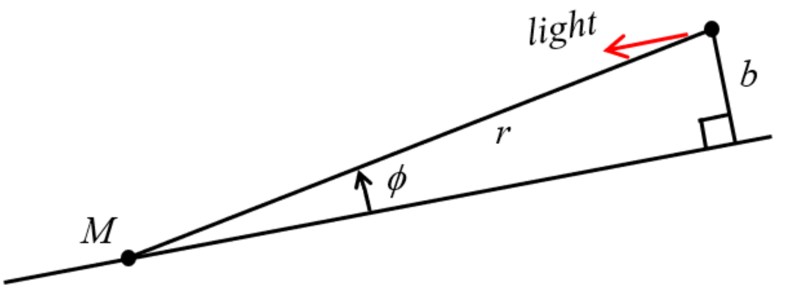
$$1 = \frac{1}{\left(1-\frac{2GM}{r}\right)^2}\left(\frac{dr}{dt}\right)^2 + \frac{r^2}{\left(1-\frac{2GM}{r}\right)}\left(\frac{d\phi}{dt}\right)^2$$

$$= \frac{1}{\left(1-\frac{2GM}{r}\right)^2}\left(\frac{dr}{dt}\right)^2 + \left(\frac{l}{e}\right)^2 \times \frac{1}{r^2}\left(1-\frac{2GM}{r}\right) \qquad \text{[subst. } d\phi/dt\text{]}$$

We now have equations of motion for $\phi(t)$ and $r(t)$.

- The meaning of l/e (angular momentum per unit energy?)

Consider a light beam approaching M with impact parameter b. Its path is straight while it is far away:



$$\phi \approx \sin\phi = \frac{b}{r} \qquad [r >> b \Rightarrow \text{small angle } \phi]$$

$$\Rightarrow \qquad \frac{d\phi}{dr} = -\frac{b}{r^2} \qquad [\text{differentiate}]$$

Also the light is heading almost directly towards M, so

$$\frac{dr}{dt} \to -1 \qquad [\text{speed of light, towards } M]$$

$$\Rightarrow \qquad \frac{d\phi}{dt} = \frac{d\phi}{dr}\frac{dr}{dt} \to \frac{b}{r^2} \qquad [\text{chain rule}]$$

Compare with our ϕ equation of motion for $r \to \infty$:

$$\frac{d\phi}{dt} = \frac{l}{e} \times \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) \to \frac{l}{e} \times \frac{1}{r^2}$$

$\Rightarrow l / e$ is the *impact parameter* of the light path far from M:

$$\frac{l}{e} = b$$

- Effective potential

Our r equation (previous page) can now be written as

$$\frac{1}{b^2} = \left[\frac{1}{b}\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt}\right]^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

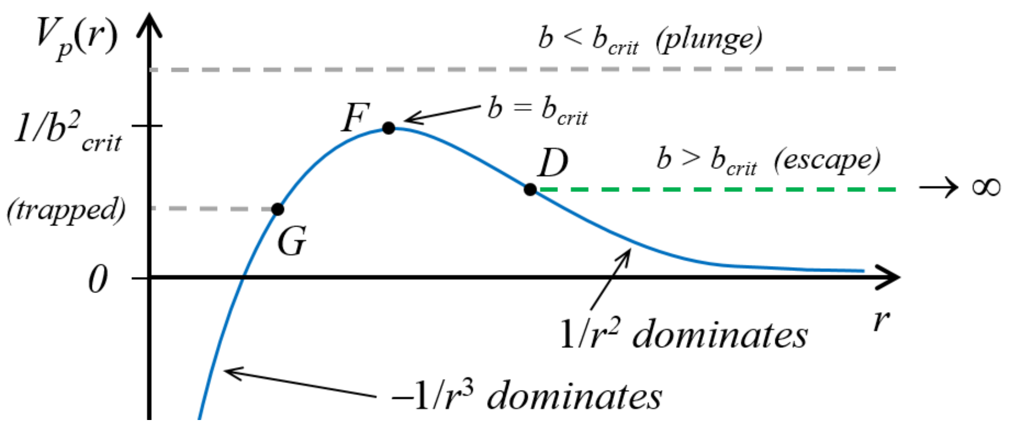
which is of the form:

constant = (*function of velocity*)^2 + *function of r*

just like an energy equation! The light is only allowed where the "pretend energy" $1/b^2$ exceeds a *photon effective potential*

$$V_p(r) = \frac{1}{r^2} - \frac{2GM}{r^3}$$

Unlike $V(r)$ for particles there's no local minimum so no bound orbits, but there is a maximum where $V_p(r) = 1/b^2_{crit}$:



Allowed photon orbits are where $V_p(r) \leq 1/b^2$:

$b > b_{crit}$ and $r >$ point F $\Rightarrow$ hyperbola-like escape orbit with perihelion D and no aphelion.

$b = b_{crit} \Rightarrow$ unstable circular orbit at F.

$b < b_{crit} \Rightarrow$ plunge orbit, the light falls all the way to $r = 0$ (if it's moving inwards), or it escapes (if it's moving outwards).

$b > b_{crit}$ and $r <$ point F $\Rightarrow$ trapped orbit with aphelion at G before falling to $r = 0$.

- Orbit equation for light

To find the photon orbits, follow the steps we did on p. 22 and p. 71 but with the ϕ and energy-like equations:

$$\frac{d\phi}{dt} = \frac{b}{r^2}\left(1 - \frac{2GM}{r}\right)$$

$$\frac{1}{b^2} = \left[\frac{1}{b}\left(1 - \frac{2GM}{r}\right)^{-1}\frac{dr}{dt}\right]^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

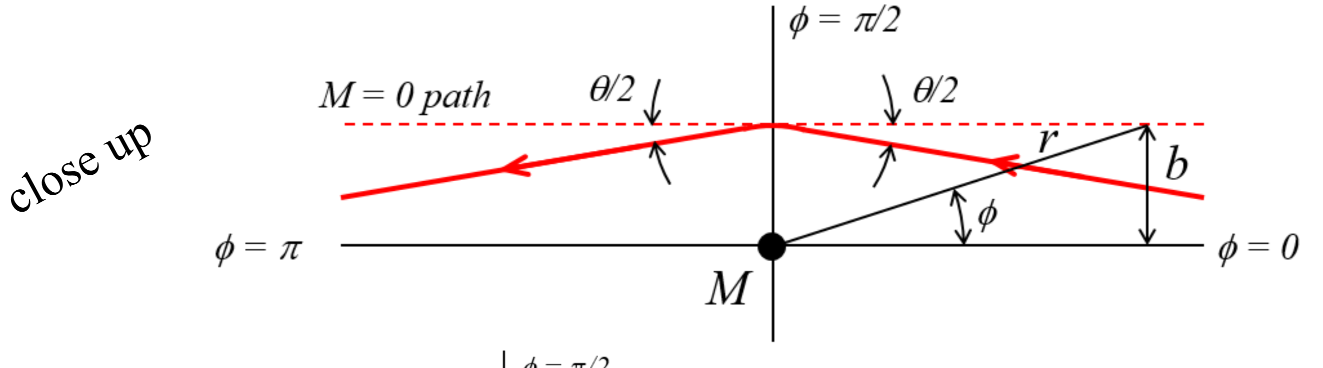
Use $u = 1/r$ and the chain rule (as before) to write the 2nd equation with $du/d\phi$ instead of dr/dt, then differentiate w.r.t. ϕ:

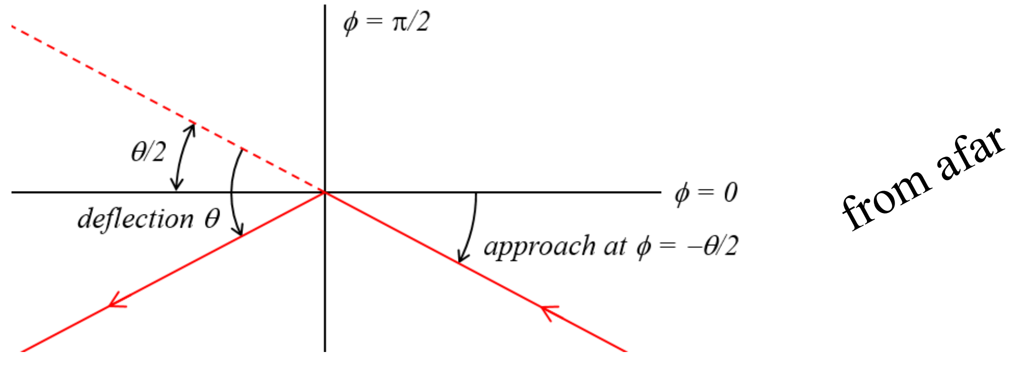
$$\frac{d^2u}{d\phi^2} + u = 3GMu^2$$

an orbit equation like the one for particles but with a bit missing.

- Deflection of light

We'll use the equation to find out how much light is deflected by a massive object. The geometry is like that on p. 24:





First consider the straight path followed when $M = 0$:

$$b = r\sin\phi \qquad \Rightarrow \qquad u(\phi) \equiv \frac{1}{r} = \frac{\sin\phi}{b}$$

is the equation of an undeflected path with impact parameter b.

For weak deflection, where the impact parameter is large compared to the Sch. radius:

$$\frac{2GM}{b} << 1$$

consider an *almost-straight* path that differs from the straight path by a small ϕ-dependent *perturbation* $f(\phi)$:

$$u(\phi) \equiv \frac{\sin\phi}{b} + \frac{f(\phi)}{b}$$

Subst. into the orbit equation, neglect second-order terms with $f \times 2GM/b$ and f^2, and use a trig identity to write $\sin^2\phi$ in terms of $\cos 2\phi$:

$$\frac{d^2 f}{d\phi^2} + f = \frac{3GM}{2b}\left(1 - \cos 2\phi\right)$$

This is the equation for forced s.h.m., which can be solved using YR1 methods:

$$f(\phi) = \frac{GM}{2b}\left(3 + \cos 2\phi\right)$$

$$\Rightarrow \qquad u(\phi) = \frac{1}{b}\left[\sin\phi + \frac{GM}{2b}\left(3 + \cos 2\phi\right)\right]$$

To get the overall deflection angle θ, use the fact that as $r \to \infty$ ($u = 0$), incoming rays approach from $\phi = -\theta/2$:

$$0 = -\sin\left(\theta/2\right) + \frac{GM}{2b}\left(3 + \cos\theta\right)$$

Small deflection angle: $\sin(\theta/2) \approx \theta/2 \qquad \cos\theta \approx 1$

$$\Rightarrow \qquad \theta = \frac{4GM}{b}$$

So this is our final classical test for GR:

Classical test #3, deflection of light

Newton: Light approaching mass M with (large) impact parameter b is deflected by an angle* of

$$\theta = \frac{2GM}{b}$$

Einstein: Light approaching mass M with (large) impact parameter b is deflected by twice that angle

$$\theta = \frac{4GM}{b}$$

- Observing deflection of light

The deflection of starlight by the Sun was measured in 1919 by A Eddington during a total eclipse, when stars near the Sun in the sky could be observed. His expedition measured a deflection of 1.98" for a certain star, compared with the GR value of 1.74" and a Newtonian value of 0.87".

This was the first successful prediction of a new observation by GR (the perihelion shift of Mercury was an explanation of an old observation), and it made Einstein world-famous.

The deflection of light has since been observed on a much bigger scale by the imaging of distant galaxies by closer ones: *gravitational lensing*. See problem C6.

** I've put $c = 1$ into the expression from p. 25.*

8. Schwarzschild black holes

black hole = massive object with an event horizon

event horizon = surface that can only be crossed in one direction

Sch. black hole = spherically-symmetric black hole, without spin

Recall that the Sch. metric has singularities (infinities) at $r = 0$ and $r = r_s$. Because the metric is only valid outside an object, we ignored them for planets and ordinary stars that are bigger than their own Sch. radius. For black holes, where the mass is entirely within r_s, the $r = r_s$ singularity lies in a valid part of the metric and can't be ignored any more.

The singularity at the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad \text{[from p. 52]}$$

- Sch. metric $\rightarrow \infty$ at $r = r_s \Rightarrow$ apparently weird things:

Time dilation:

$$d\tau = \left(1 - \frac{r_s}{r}\right)^{1/2} dt \qquad \text{[from p. 56]}$$

$\Rightarrow$ time stops (and infinite redshift) at r_s?

Vertical drop, eg from rest at large r:

"coord velocity" $$\frac{dr}{dt} = -\left(\frac{r_s}{r}\right)^{1/2}\left(1 - \frac{r_s}{r}\right) \qquad \text{[(xi) on p. 76]}$$

$\Rightarrow$ falling objects stop at r_s?

Inside $r < r_s$: coord velocity: $\dfrac{dr}{dt} > 0$ [previous with $r < r_s$]

$\Rightarrow$ free-falling objects *rise upwards* to r_s?

or, time goes backwards?

- Types of singularity:

A metric relates the geometry of spacetime (ie physics) to the coords used to pinpoint events (ie maths). Either can blow up:

physical singularity: spacetime becomes infinitely curved?

For example, the tip of a cone is an infinitely-curved physical / geometric singularity.

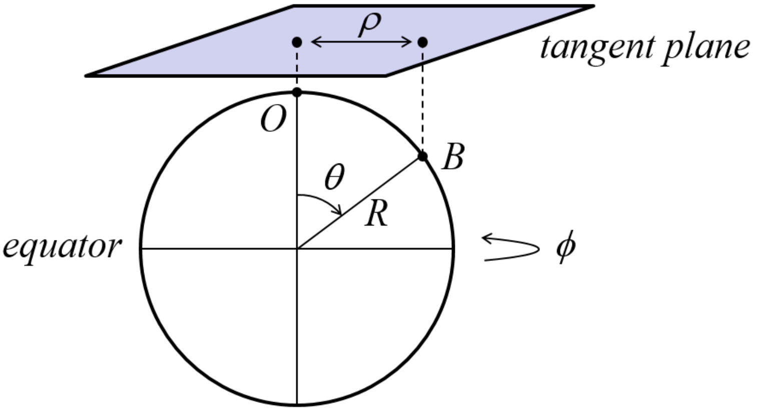
coordinate singularity: the coord system fails but the physics is well-behaved?

For example, a 2-D spherical surface of radius R has no physical singularities - it's smoothly curved. A sensible way to map it is to use the spherical polar angles θ and ϕ:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \; d\phi^2$$ [from p. 46]

This is a well-behaved metric, with no singularities for any values of θ and ϕ, which cover the whole surface.

An alternative (and less intelligent) way to measure from the pole is to use the projected distance $\rho = R \sin\theta$ on a tangent plane instead of the angle θ. It certainly acts as a coord in the northern hemisphere: pairs (ρ, ϕ) uniquely specify points, which is all a coordinate system has to do.



ρ can be substituted for θ to give the metric in (ρ, ϕ) coords:

$$ds^2 = \frac{R^2}{R^2 - \rho^2} d\rho^2 + \rho^2 \, d\phi^2$$

This metric has a singularity at $\rho = R$. The physical surface remains well-behaved, so it is only a coordinate singularity.

Because we constructed that metric ourselves we can see why it goes wrong: the coords just stop working at the equator. But if we're given the metric without explanation, it's not so obvious.

- Which type is r_s?

Re-visit the vertical drop, but with proper time instead:

$$\frac{dr}{d\tau} = -\left(\frac{r_s}{r}\right)^{1/2} \qquad \text{[(x) on p. 76]}$$

This describes how r changes with time τ experienced by the falling object, rather than the time t of an observer at infinity. There is no strange behaviour at or within r_s. The object keeps falling and reaches $r = 0$ in finite proper time (the equation is easily integrated, see p. 93). It seems perfectly well-behaved, and indicates that r_s *is merely a coordinate singularity*, due to a failure of the t coord. (We'll prove it in the next lecture by finding alternative coords that eliminate the singularity at r_s.)

- Light cones: the set of time-like worldlines through a given event, with edges defined by light-like worldlines (p. 33). They depict causality: past and future. What do they look like in Sch. spacetime, on a t versus r spacetime diagram?

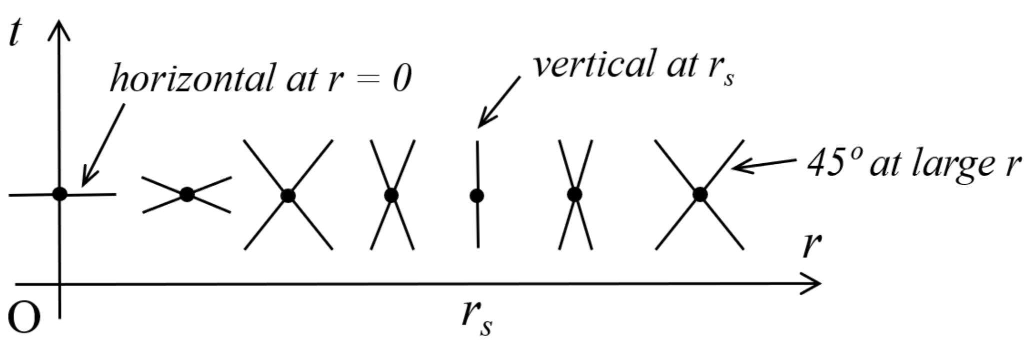
Find the bounding worldlines, for light going radially in or out. Light means $ds^2 = 0$, and radial motion means $d\Omega^2 = 0$ since

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2 \qquad \text{[from p. 52]}$$

Take the Sch. metric of p. 83, set $ds^2 = 0$ and $d\Omega^2 = 0$, divide by dr^2, and solve for the slopes dt/dr of the light-like worldlines:

$$0 = -\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{dr}\right)^2 + \left(1-\frac{r_s}{r}\right)^{-1} \qquad \Rightarrow \qquad \frac{dt}{dr} = \pm\frac{1}{\left|1-\frac{r_s}{r}\right|}$$

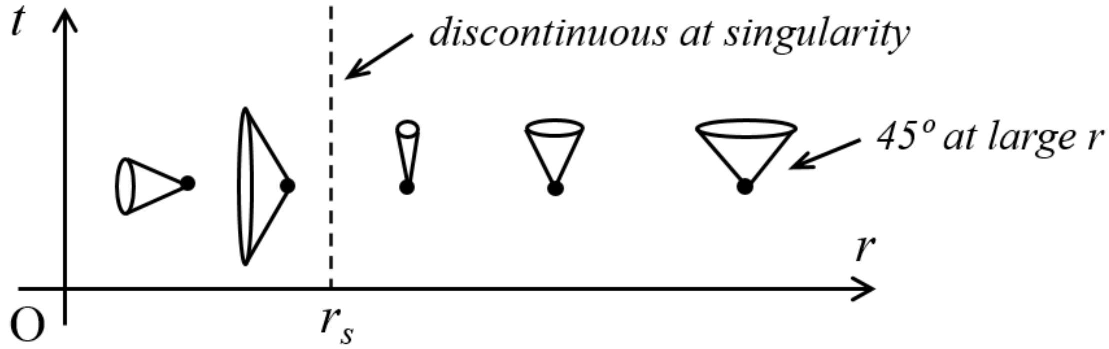
Plotting these slopes for r values from 0 to ∞ gives:



But which way do the light cones open? In other words, are time-like worldlines more vertical or more horizontal (on the diagram) than light-like ones?

At large r the Sch. metric looks like the Minkowski metric of flat spacetime (p. 50), so light cones open vertically here (p. 33). This behaviour is continuous down to r_s. However, the coord singularity at r_s means we can't assume it continues for $r < r_s$. A more-careful analysis is needed, which you'll do as problem C10. The result is that light cones open *horizontally* for $r < r_s$!

*



For large r the light cone is *upright* with 45° sides, as in the Minkowski spacetime of SR (p. 33). As r decreases the cone narrows, to become infinitely thin vertically at r_s. Then there's a discontinuity (because of the singularity) to a wide-open light cone *on its side*, facing towards $r = 0$. As r decreases further the light cone closes to become infinitely thin horizontally at $r = 0$.

Why do the light cones flip at r_s? Look again at the Sch. metric:

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

When $r < r_s$, the coeffs of dt^2 and dr^2 swap signs. Recall that for a diagonal metric (like this one) the negative coeff identifies the time coordinate. So, for $r < r_s$, r becomes the time coordinate and t becomes a space coordinate!

The Sch. radius r_s is an *event horizon* passable only inwards. Just as t inevitably increases outside the horizon, an object's r inevitably decreases once inside. Then the object's unavoidable future is $r = 0$ where (or rather, when!) its worldline *terminates*.

On the other hand, t can go in either direction inside. It follows from the t equation of motion on p. 66: $dt/d\tau = e/(1 - r_s/r)$ that both positive and negative energies e are possible inside the horizon, since $dt/d\tau$ can take either sign. Negative e is not possible outside the horizon, where t is always increasing.

* *I've shown only the future light cones here: the past light cones just go the other way.*

Painlevé-Gullstrand (PG) coordinates

The Sch. time coord t (time at rest at large r) behaves badly at $r = r_s$ but the proper time τ of a free-falling object behaves well. It is therefore just a coordinate singularity that can be eliminated by replacing t with a different time coord. There are several ways to do this. Here is P Painlevé and A Gullstrand's:

- The *PG time* t' (of an event): the time read from a free-falling clock that was dropped from rest at large r and that happens to fall past the event as it occurs. The *PG coords* are t' together with the three spatial coords (r, θ, ϕ) of the Sch. metric.

Note that t' is well-defined whatever the observer's location and motion, including inside the event horizon. Most importantly it's measured locally to the event, unlike t. Now we'll relate t to t'.

If we start with the clock's reading t' at the event and subtract the proper time $\tau_{journey}$ of the clock's fall from its release at large r, we get the clock's reading when it was dropped. This is also the Sch. time when it was dropped, because until then it was at rest at large r and so read Sch. time. If we then add the Sch. time $t_{journey}$ of the clock's fall, we get the Sch. time t of the event:

$$t = t' - \tau_{journey} + t_{journey}$$

and how it varies with the event's r coord:

$$\frac{dt}{dr} = \frac{dt'}{dr} - \frac{d\tau_{journey}}{dr} + \frac{dt_{journey}}{dr} \qquad \text{[differentiate]}$$

But we already know dr/dt and $dr/d\tau$ for the journey of an object falling from rest at large r, from (x) and (xi) on p. 76. Here they are again, with the "journey" subscript added and the appropriate choice of sign for an inward fall:

$$\frac{dr}{d\tau_{journey}} = -\left(\frac{r_s}{r}\right)^{1/2}$$

$$\frac{dr}{dt_{journey}} = -\left(\frac{r_s}{r}\right)^{1/2}\left(1-\frac{r_s}{r}\right)$$

Substituting these in, and multiplying by *dr*:

$$dt = dt' + \left(\frac{r}{r_s}\right)^{1/2} dr - \frac{\left(\frac{r}{r_s}\right)^{1/2}}{\left(1-\frac{r_s}{r}\right)} dr = dt' - \frac{\left(\frac{r_s}{r}\right)^{1/2}}{\left(1-\frac{r_s}{r}\right)} dr$$

This can (with difficulty) be integrated to give a transformation between t and t', but we won't bother because we can substitute it straight into the Sch. metric to yield the *PG metric*:

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r}\right)dt'^2 + 2\left(\frac{r_s}{r}\right)^{1/2} dt'dr + dr^2 + r^2 d\Omega^2}$$

The physics is unchanged. It's still Schwarzschild spacetime, just using different coords*. Notice:

No more singularity at r_s! (There is still one at $r = 0$.)

g_{tt} is positive for $r < r_s$ but this does not mean t' is a space coord inside the horizon. The "negative-g" rule applies to diagonal metrics, and this one has a cross term. In fact t' is *manifestly* a time coord for all r, because it's measured by a (local) clock.

For $r < r_s$ every term on the RHS except the cross term is positive, but time-like or light-like (ie allowed) worldlines require $ds^2 \le 0$. Therefore it is necessary that $dt'dr < 0$: future ($dt' > 0$) motion inside the horizon must be inwards ($dr < 0$).

* *It's Sch. spacetime in PG coords, instead of Sch. spacetime in Sch. coords.*

- Light cones

Repeat the exercise on p. 86 but with PG coords, to get light cones on a t' versus r spacetime diagram. Writing β for the mixed velocity* of the falling PG clock (just as a convenient proxy for r)

$$\beta = \left| \frac{dr}{d\tau_{journey}} \right| = \left(\frac{r_s}{r} \right)^{1/2}$$

the PG metric is

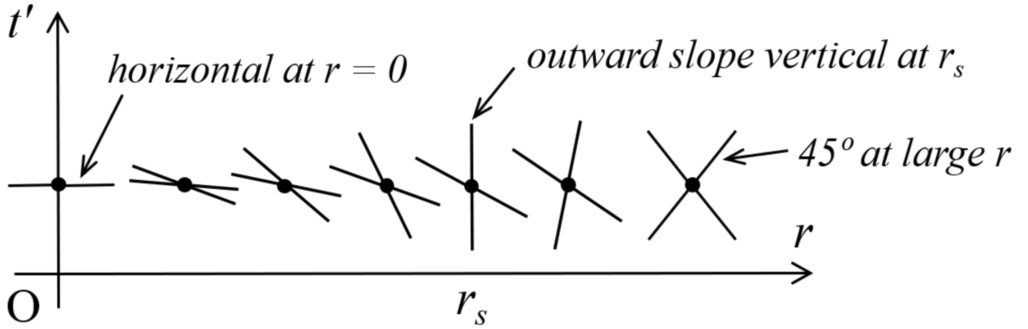
$$ds^2 = -\left(1-\beta^2\right)dt'^2 + 2\beta dt' dr + dr^2 + r^2 d\Omega^2$$

As before, find the bounding worldlines by setting $ds^2 = 0$ (light) and $d\Omega^2 = 0$ (radial motion), divide by dr^2, and solve the resulting quadratic equation for dt'/dr:

$$0 = -\left(1-\beta^2\right)\left(\frac{dt'}{dr}\right)^2 + 2\beta\left(\frac{dt'}{dr}\right) + 1 \qquad \Rightarrow \qquad \frac{dt'}{dr} = \frac{\beta \pm 1}{1-\beta^2}$$

$$\Rightarrow \qquad \frac{dt'}{dr} = -\frac{1}{1+\beta} \quad \text{and} \quad \frac{dt'}{dr} = \frac{1}{1-\beta} \qquad [1-\beta^2 \equiv (1-\beta)(1+\beta)]$$

Plotting both slopes for r values from 0 to ∞, with $\beta(r)$ defined at the top of the page, gives:



* *It's "mixed" because it's the change of coord distance r w.r.t. proper time τ.*

Like the Sch. metric, the PG metric looks like the Minkowski metric of flat spacetime at large r, so again the light cones open vertically here. But, unlike the Sch. metric, the PG metric has no coordinate singularity at r_s. Hence the openings can be tracked continuously all the way down to $r = 0$, without the more-careful analysis of problem C10: no singularity gets in the way.

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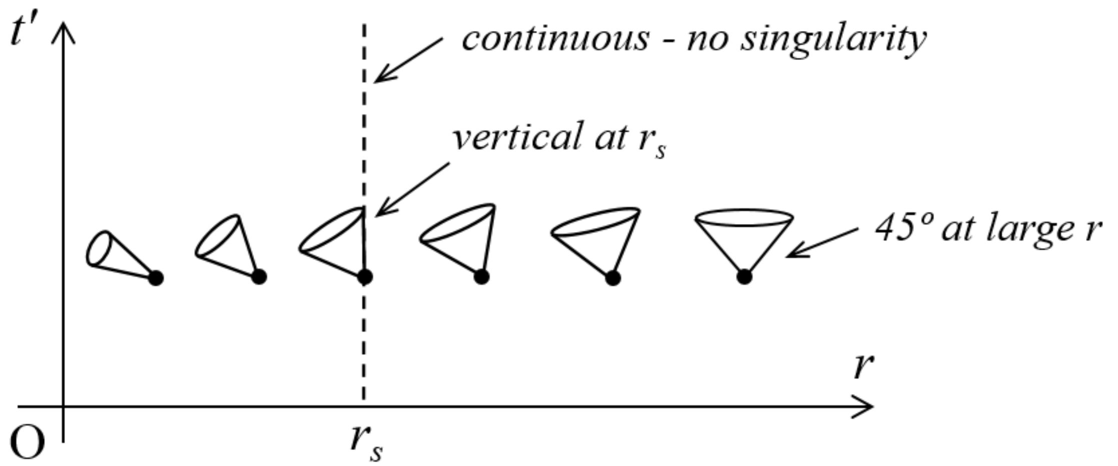


As r decreases, the initially-upright light cone tips over towards the left and narrows. At r_s the outward slope is vertical: this is where $\beta = 1$, so the $1 - \beta$ denominator in the second expression for dt'/dr (on the previous page) is zero.

Inside the horizon both edges have negative slope, so all future worldlines are inward. This confirms that the future light cones in Sch. coords on p. 87 should indeed open to the left (not to the right) inside the horizon.

Remember that the physics here and on p. 87 is the same - it's Schwarzschild spacetime in both cases. We've only changed the coordinate system used to describe it.

* *Again, I've shown only the future light cones here.*

- The river model

There are other ways to get rid of the singularity at r_s, but the PG metric in particular has an intuitively-appealing property. The metric can be written (prove by working backwards) as

$$ds^2 = -dt'^2 + \underbrace{(dr + \beta dt')^2}_{\equiv dr'^2} + r^2 d\Omega^2$$

This looks like the (flat) Minkowski metric of SR in spherical polars, but with an r' coord that "flows inwards" at the speed

$$\beta = \left(\frac{r_s}{r}\right)^{1/2}$$

of the falling clocks that defined the PG time t'. (A point of constant r', ie $dr' = 0$, moves as $dr/dt' = -\beta$.) It's as if *spacetime itself flows inwards* at this speed, and the PG clocks are carried along inertially in the flow. The flow gets faster at smaller r, and exceeds the speed of light* ($\beta = 1$) inside the horizon.

Particles and light move in this flowing spacetime like fish in a river. For $r > r_s$, fast-enough fish can overcome the flow and swim upstream. For $r < r_s$, even the fastest fish (light) cannot make headway against the flow, and are carried downstream.

Is this real physics? It's no more fanciful than the cosmological picture that "spacetime itself is expanding" after the Big Bang...

** Should we worry about the flow exceeding the speed of light? No. SR is always valid locally (p. 49) so local measurements "in the river" will always be limited by the speed of light. But attempts to measure non-locally (measuring "here" the speed of something "there") can give strange answers, see p. 47. It's like at the edge of the observable Universe, where galaxies recede at the speed of light: another non-local velocity.*

The central singularity ($r = 0$)

We now know that r_s is merely a coordinate singularity. There is no extreme physics at r_s, and the singularity can be removed by changing the coords. What about the other singularity, at $r = 0$?

Tensor analysis: The *Riemann curvature tensor* is infinite at $r = 0$, meaning that spacetime is infinitely curved. No change of coords can eliminate the singularity. It is a real / physical / geometric singularity, and is referred to as the singularity.

- A journey to the singularity

An observer falls from rest at large r (like a PG clock):

$$\frac{dr}{d\tau} = -\left(\frac{r_s}{r}\right)^{1/2}$$ [(x) from p. 76]

This equation is easily integrated to find the proper time to fall from arbitrary r to the singularity:

$$\tau = \frac{2r_s}{3}\left(\frac{r}{r_s}\right)^{3/2} \qquad \text{(xiii)}$$

so the time spent inside the horizon is

$$\tau = \frac{2r_s}{3} \qquad [r = r_s]$$

For a solar-mass black hole (with a mass of the order of the Sun's), τ is a few microseconds. You'll need a much bigger black hole to "enjoy" the experience of life inside.

- Tidal forces

A coordinate acceleration* α can be defined for our observer:

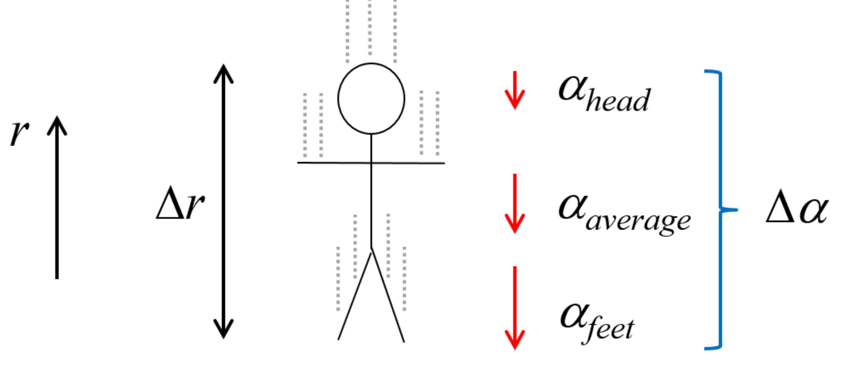
** This is an acceleration relative to the coordinate system, which is not inertial: static (fixed r) observers feel an acceleration due to gravity, ie a weight.*

$$\alpha(r) \equiv \frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) = \frac{d}{dr}\left(\frac{dr}{d\tau}\right) \times \frac{dr}{d\tau} \qquad \text{[chain rule]}$$

$$= -\frac{r_s}{2r^2} \quad * \qquad [dr/d\tau \text{ on previous page}]$$

The free-falling observer moves inertially, and so does not feel any acceleration. α is just an acceleration relative to the coordinate system (see problem D3).

But, because α varies with r, different parts of the observer's body try to fall with different accelerations, setting up *tidal forces*. We can use α to work these out because r is a proper distance for our observer (in the PG metric, keep t' and the other space coords fixed and get $ds^2 \equiv d\sigma^2 = dr^2$).



If there's an acceleration difference $\Delta\alpha$ along the distance Δr between the observer's extremities, a tensile tidal force (problem A2) of the order of

$$F \sim m\Delta\alpha = m\left(\frac{d\alpha}{dr}\right)\Delta r \qquad \text{[chain rule]}$$

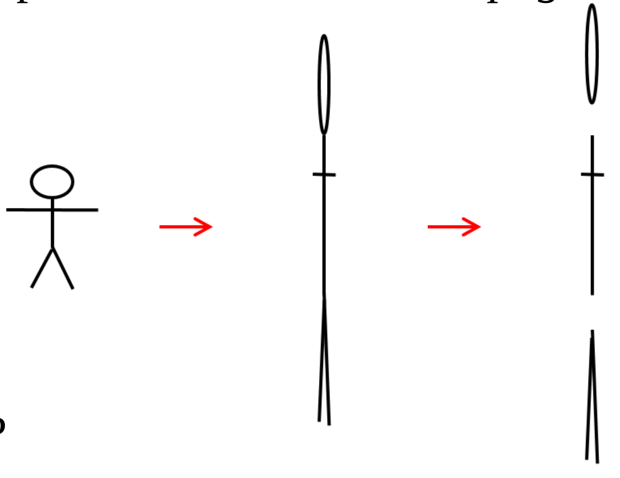
will be set up. The force depends on their height Δr, and on the *tidal acceleration gradient* $d\alpha/dr$. From $\alpha(r)$ above:

$$\frac{d\alpha}{dr} = \frac{r_s}{r^3}$$

* *Interestingly, gives the Newtonian expression if you substitute r_s.*

The tidal gradient $d\alpha/dr \to \infty$ as $r \to 0$: an example of an extreme physical effect, and strong evidence for a physical singularity at $r = 0$ even without knowledge of tensors.

The observer's body (and, soon afterwards, constituent atoms) will be irresistably stretched and torn apart as the singularity is approached, a phenomenon known as *spaghettification*.



- Will it hurt?

Assume we can endure a tidal acceleration gradient of

$$\frac{d\alpha}{dr} = q$$

The tidal gradient will reach this value at the "pain radius"

$$r_{pain} = \left(\frac{r_s}{q}\right)^{1/3}$$ [$d\alpha/dr$ on prev. page]

The pain will last at most until $r = 0$, for a "pain time" given by substituting r_{pain} into (xiii) on p. 93:

$$\tau_{pain} = \frac{2r_{pain}^{3/2}}{3r_s^{1/2}} = \frac{2}{3q^{1/2}}$$

Remarkably, this is independent of the mass of the black hole. It only depends on the observer's pain threshold. Reasonable values of q [see D3(d)] give τ_{pain} of the order of 0.1 s, similar to a typical reaction time. Spaghettification probably doesn't hurt!

- Maximising survival time

The "free-fall from large r" trajectory doesn't maximise the time experienced inside the black hole. From the Sch. metric:

$$d\tau^2 = -ds^2 = \frac{dr^2}{\left(\frac{r_s}{r}-1\right)} \underbrace{-\left(\frac{r_s}{r}-1\right)dt^2 - r^2 d\Omega^2}_{\leq 0 \text{ inside the horizon}}$$

To maximise τ we need (a) $d\Omega^2 = 0$ (ie radial motion) and (b) $dt = 0$. From the t equation of motion for free fall:

$$\frac{dt}{d\tau} = \frac{e}{\left(1-\frac{2GM}{r}\right)}$$

[see p. 66]

ie, $dt = 0 \Rightarrow$ free fall with $e = 0$. So, to maximise experienced time, get yourself into a radial $e = 0$ trajectory* asap then *switch your engines off*. The maximum proper time is

$$\tau = -\int_{r_s}^{0} \frac{dr}{\left(\frac{r_s}{r}-1\right)^{1/2}} = -2r_s\int_{\pi/2}^{0} \sin^2\theta \, d\theta \qquad [\text{subst } r = r_s \sin^2\theta]$$

$$= \frac{\pi}{2} r_s$$

which for all the fuss isn't much better than

$$\tau = \frac{2r_s}{3}$$

for free fall from rest at large r (p. 93).

* *An advanced optional thinking question: how* do *you get into an $e = 0$ trajectory?? A closely-related question, given the t equation of motion: how do you change your motion in the t direction, which is a space coord inside the black hole?*

- So, what happens at $r = 0$?

Nobody knows.

(GR doesn't say.)

The singularity cannot be observed: it is hidden from view, "clothed" by the event horizon.

All of the black hole's mass M (and anything that falls in) goes there.

There are infinite tidal forces and an infinite spacetime curvature.

It's worth remembering that, if a physical theory gives an infinite answer, it's either a somewhat-abstract concept (eg infinite temperature means that all of a system's microstates have equal occupancy while it is in thermal equilibrium) or it means *the theory has broken down* (eg the ultraviolet catastrophe of black-body radiation showed that the classical theory had failed, leading to Planck's founding of quantum physics).

It's therefore likely that GR ceases to accurately describe physics at the centre of a black hole. It is after all a classical theory: a theory of *quantum gravity* (see L20) may smooth out the all-at-once nature of the singularity and make it "fuzzy".

Nevertheless, this is speculation, and the only reliable answer so far is the one at the top of this page.

Kruskal-Szekeres (KS) coords

This is another way to visualise Sch. spacetime and avoid the singularity at r_s. Its advantages over PG are that the metric is diagonal, its light cones are simple, and it has great conceptual power. The disadvantages are that it is mathematically abstract and not much use for calculating orbits.

You are advised to focus on the concepts rather than the maths!

We replace both t and r this time, with new coords u and v:

$$t = r_s \ln\left|\frac{u+v}{u-v}\right|$$

$$\left(\frac{r}{r_s} - 1\right)e^{r/r_s} = u^2 - v^2$$

The expression for t is straightforward but note that r is only *implicitly* defined: it can't be written "$r = ...$"

Substitution into the Sch. metric and 2 pages of dedicated algebra (problem D4, only completists need attempt) yield the KS metric:

$$\boxed{ds^2 = -\frac{4r_s^3}{r}e^{-r/r_s}\left(dv^2 - du^2\right) + r^2 d\Omega^2}$$

There is no singularity at r_s. (There is still one at $r = 0$.)

It still contains r - we can't get rid of r, so instead treat it as an implicit function of u and v.

It's diagonal - no cross terms. The coefficient of dv^2 is always negative, so v is always time-like and u is always space-like.

- Light cones

As on p. 86 and p. 90, set $ds^2 = 0$ (light) and $d\Omega^2 = 0$ (radial motion), divide by du^2 and solve for dv/du:

$$0 = \left(\frac{dv}{du}\right)^2 - 1 \qquad \Rightarrow \qquad \frac{dv}{du} = \pm 1$$

The edges of light cones on a v versus u spacetime diagram are at ±45° *everywhere* - just like in flat Minkowski spacetime! The light cones always open vertically because v is always the timelike coord.

The price of this very-simple rule for light cones is that the other features of the spacetime diagram must be severely deformed to accommodate it. We need to consider how familiar values of r and t map onto a KS spacetime diagram.

To plot *lines of constant r*, the r transformation becomes

$$u^2 - v^2 = \left(\frac{r}{r_s} - 1\right) e^{r/r_s} = \text{constant}$$

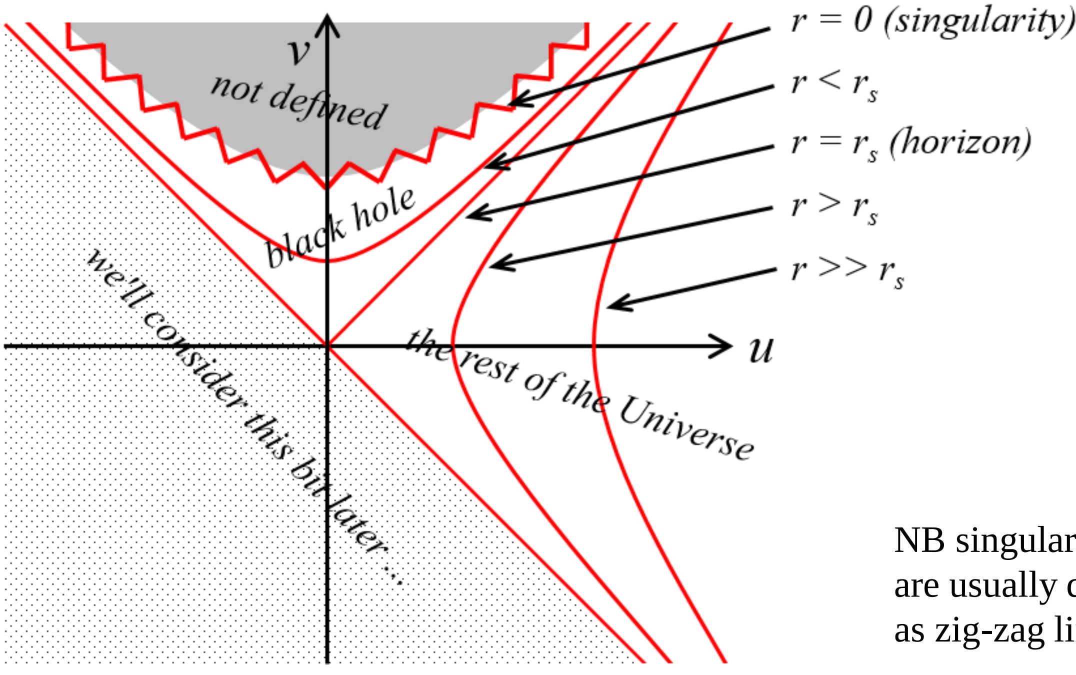
This is the equation of a hyperbola with ±45° asymptotes.

For $r > r_s$ (the outer Universe) the RHS is +ve. Set $v = 0$ to see that the curves cross the u axis vertically at real values of u.

For $r = r_s$ (the horizon) the RHS is zero. We get the asymptotes $v = \pm u$.

For $r < r_s$ (the black hole) the RHS is –ve. Set $u = 0$ to see that the curves cross the v axis horizontally at real values of v. The singularity $r = 0$ is $v^2 - u^2 = 1$, which crosses the v axis at $v = 1$.

The part of the diagram above the singularity corresponds to complex values of r, and so is not defined.

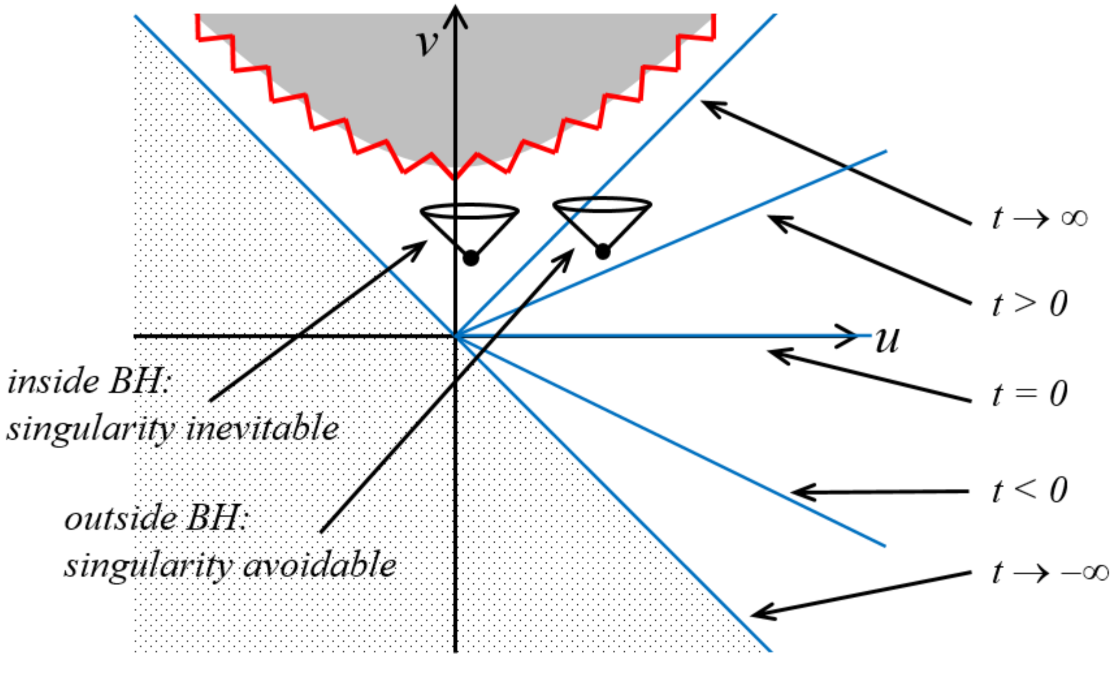


NB singularities are usually drawn as zig-zag lines

To plot *lines of constant t*, solve the t transformation on p. 98 for v assuming* $u > v$:

$$v = \left(\frac{e^{t/r_s} - 1}{e^{t/r_s} + 1} \right) u = \text{constant} \times u$$

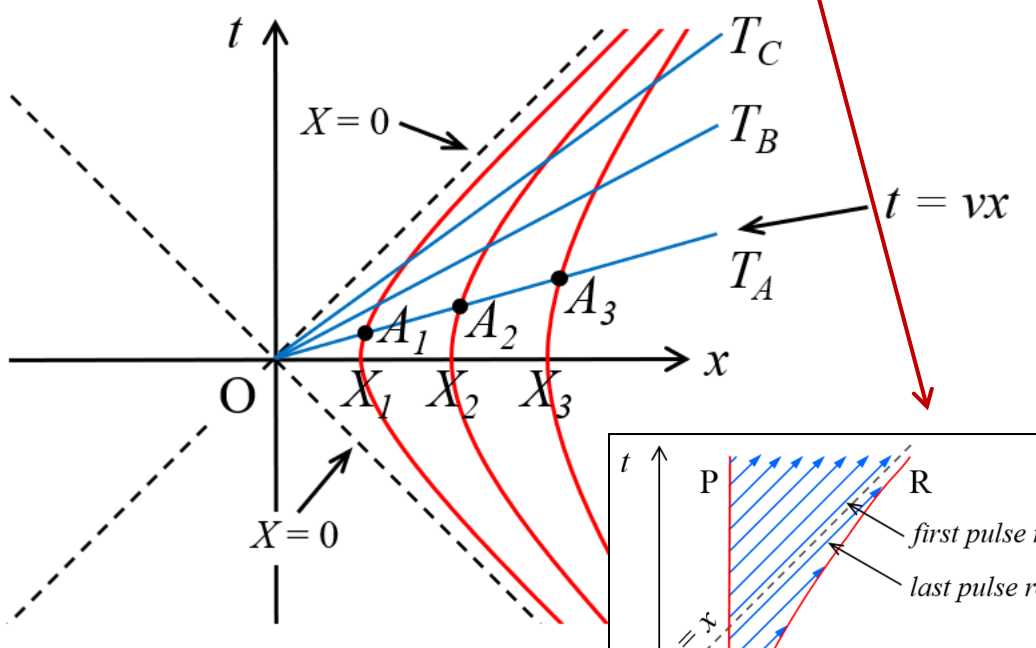
These are straight lines through the origin, with slopes from –1 for $t \to -\infty$ to +1 for $t \to \infty$.

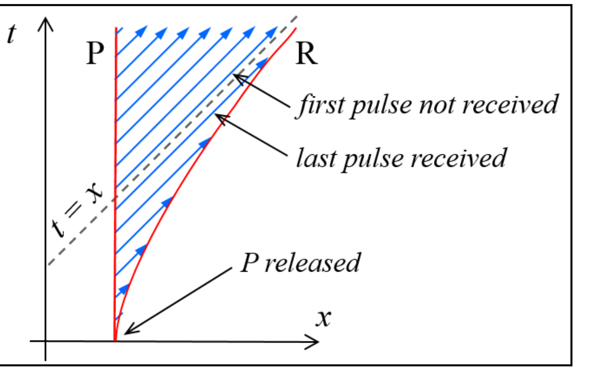


** Repeat but assuming $u < v$ instead to give lines of constant t inside the black hole - and help advanced thinkers to answer the question in the footnote on p. 96.*

• Relation to the Rindler frame

Our plots of Sch. coords (t, r) on KS spacetime diagrams (v, u) look a lot like plots of Rindler coords (T, X) on Minkowski spacetime diagrams (t, x) from p. 35-38. The light cones and event horizon (but not the singularity) are analogous. Both can be used, in similar ways, to qualitatively study causal relations between different observers, eg the probe P falling through the horizon, or what the rest of the Universe looks like to P, or whether P gets to see the singularity once inside the horizon.



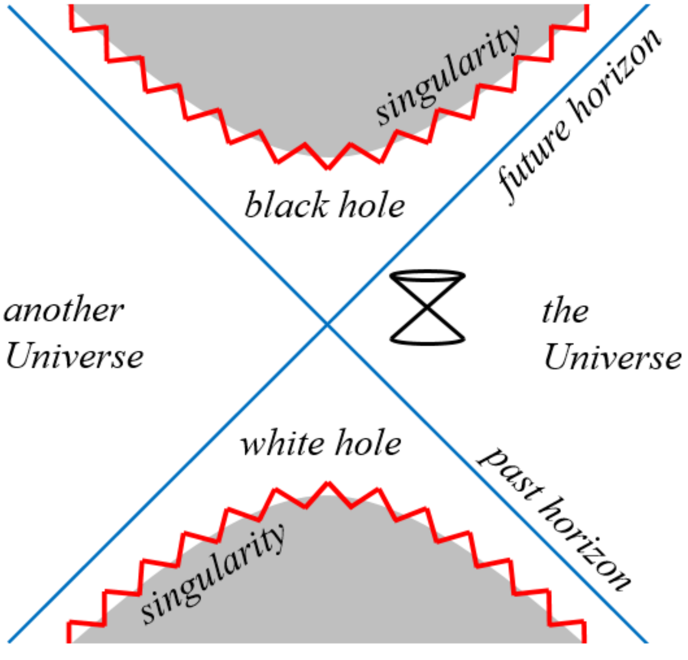


(figs copied from pp. 36 & 38)

The Minkowski coords were "natural" for inertial frames in SR, Rindler coords representing a more-complicated accelerating frame. So it looks like KS coords are natural for spacetime around a black hole, with Sch. coords representing a more-complicated accelerating frame! Indeed, an observer at rest in Sch. coords does feel an acceleration due to gravity.

• The Kruskal extension (*speculation alert!*)

The Sch. coords sit in the top-right half ("east and north") of the KS diagram. The bottom-left half ("west and south") is a theoretical extension. The Rindler analogy suggests what the extension represents: when prompted on p. 36, did you think about the line $t = -x$ in the Rindler frame ($t = +x$ being the event horizon)?



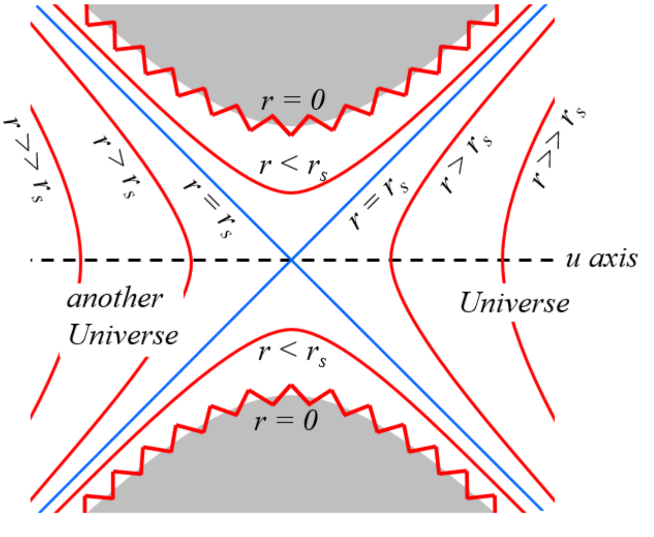
East quadrant: the rest of the Universe, ie normal space outside the black hole, $r > r_s$. Future worldlines (see the light cone...) lead only elsewhere in normal space or into the black hole, but past worldlines can lead *from* the south quadrant. There is no causal connection at all (past or future) with the west quadrant.

North quadrant: the black hole, $r < r_s$. Future worldlines lead only to the singularity, but past worldlines lead from any of the other three quadrants. Matter and light can enter through the horizon but never leave.

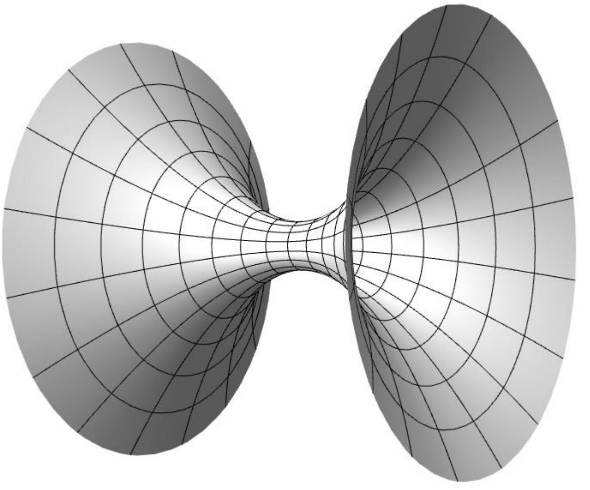
West quadrant: Exactly like the east quadrant (ie normal space) but causally disconnected from it ⇒ it's *another Universe*! Observers in the black hole can see both universes and meet travellers from the other universe (shortly before being spaghettified).

South quadrant: a time-reversed black hole, called a *white hole*. Past worldlines lead only from the singularity, but future worldlines lead to any of the other three quadrants. Matter and light can leave through the horizon but never enter. An object leaving the white hole follows the time-reverse of the trajectory of an object falling into the black hole. If you look at the black hole from outside, what you actually see is the white hole.

The Einstein-Rosen bridge: Consider the "slice" of spacetime represented by the u axis. On the KS diagram, it's the horizontal line through the diagram's origin where the horizons intersect. Here's a diagram with contours of r marked, now including both branches of each hyperbola:

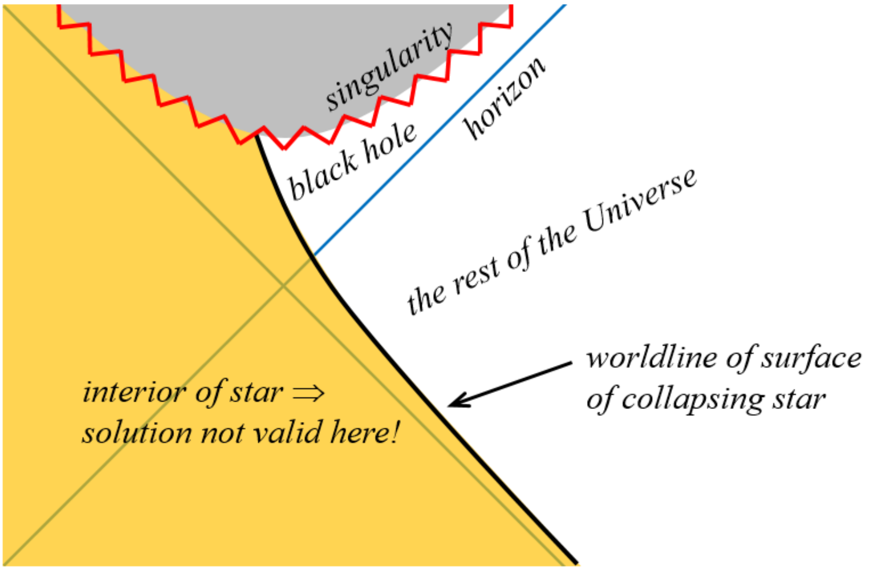


As you move from right to left, see how r decreases from very large values down to r_s at the origin, then increases again in the other universe. The embedding diagram along this path is just two Flamm's paraboloids (p. 54) joined at their throats:



It's called the Einstein-Rosen bridge, and it's the simplest example of a *wormhole* in spacetime. Unfortunately, it can't be used to travel between the two universes. As we've already seen, they are causally disconnected: all the worldlines through the wormhole are space-like.

To add to the sci-fi fan's disappointment: the extra universe and the white hole are valid solutions of Einstein's equation, but they only appear for *eternal* black holes with no beginning in time: the white-hole horizon is $t \to -\infty$. For *astrophysical* black holes formed by the collapse and/or merger of stars sometime in the finite past, the in-falling matter forms a boundary beyond which the "vaccum solution" of Sch. spacetime is not valid.



9. Kerr (rotating) black holes

In L7 - L16 we studied the Schwarzschild solution for spacetime outside a non-rotating, spherically-symmetric source of gravity. But real astrophysical objects (including black holes) rotate. As an example of how hard it is to find out how "matter tells spacetime how to curve", 45 years passed between K Schwarzschild's work and R Kerr's solution of Einstein's field equation outside a *rotating*, *axially-symmetric* point-source of gravity of mass M and angular momentum* (or *spin*) J.

The Kerr metric

In *Boyer-Lindquist coords* (t, r, θ, ϕ), the Kerr metric is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{\phi t}d\phi dt$$

where

$$g_{tt} = -\left(1 - \frac{r_s r}{\rho^2}\right) \qquad g_{\phi\phi} = \left\{r^2 + a^2 + \frac{a^2 r_s r \sin^2\theta}{\rho^2}\right\}\sin^2\theta$$

$$g_{rr} = \frac{\rho^2}{\Delta} \qquad g_{\theta\theta} = \rho^2 \qquad g_{\phi t} \equiv g_{t\phi} = -\frac{a r_s r \sin^2\theta}{\rho^2}$$

$$\Delta = r^2 + a^2 - r_s r \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

$a = J/M$ ⟵ the *angular momentum* parameter*

Positive ϕ is defined to be in the direction of spin, so $a \geq 0$.

$r \to \infty$ (far away) $\Rightarrow$ reduces to Minkowski (flat) metric

$a \to 0$ (no spin) $\Rightarrow$ reduces to Sch. metric

$r_s \to 0$ (no mass) but fixed $a \Rightarrow$ reduces to flat spacetime

There's a cross term $d\phi \times dt \Rightarrow$ the *sign* of $d\phi/dt$ matters

* *We'll use J or a for the spin angular momentum of the central mass, and L or l for the orbital angular momentum of a particle moving around it.*

- Singularities: infinite ds^2 when $\rho^2 = 0$ or $\Delta = 0$

$\underline{\rho^2 = 0}$ $\Rightarrow$ $r^2 + a^2\cos^2\theta = 0$ $\Rightarrow$ $r = 0$ and $\theta = \pi/2$

This corresponds to the central singularity of Sch. spacetime when $a = 0$, so it is a physical singularity. (Why does the value of θ matter if $r = 0$? Because in Boyer-Lindquist coords $r = 0$ is *not a point* when $a \neq 0$ - problem D8(c)).

$\underline{\Delta = 0}$ $\Rightarrow$

$$r^2 - r_s r + a^2 = 0$$

Quadratic:
$$r_\pm = \frac{r_s}{2} \pm \left[\left(\frac{r_s}{2}\right)^2 - a^2\right]^{1/2}$$

The bigger solution r_+ is r_s when $a = 0$ (the Sch. limit), so it is a coordinate singularity at the *event horizon* r_H:

$$\boxed{r_H = GM + \left[(GM)^2 - a^2\right]^{1/2}} \qquad \text{[event horizon, where } \Delta = 0\text{]}$$

Indeed, notice that g_{rr} changes sign at $r < r_H$, suggesting that r becomes time-like. (Although the rule about the sign of $g_{\mu\mu}$ indicating the time coord only works for diagonal metrics, the cross term in the Kerr metric doesn't involve dr.)

For a rotating black hole ($a \neq 0$), $r_H < r_s$.

The other solution r_- is another coordinate singularity called the Cauchy horizon. In the Sch. limit $a = 0$, $r_- \to 0$ and vanishes. The Cauchy horizon is not very relevant since it is inside the event horizon.

• The Cosmic Censorship Principle

If $a > GM$ there's no real value for r_H and so no horizon. Without an event horizon, there's no black hole. The $\rho^2 = 0$ physical singularity becomes a *naked singularity*, "un-clothed" by a horizon. We could observe it, or visit it and return.

The conjecture that physics does not allow naked singularities is called the *cosmic censorship principle*. For a Kerr black hole:

$$a < GM$$

This conjecture is not proven, but is very likely and widely believed. For example, evidence suggests that an $a > GM$ black hole cannot form. The limiting case of a Kerr black hole with $a = GM$ is called an *extremal black hole*.

• The static limit

The time dilation of an object at rest relative to the observer at infinity is given by $d\tau^2 = -ds^2$ with $dr = d\theta = d\phi = 0$:

$$d\tau = -g_{tt}^{1/2} dt = \left(1 - \frac{r_s r}{\rho^2}\right)^{1/2} dt$$

so there's infinite time dilation (and redshift) when $g_{tt} = 0$:

$$1 - \frac{r_s r}{\rho^2} = 0 \qquad \Rightarrow \qquad r^2 - r_s r + a^2 \cos^2\theta = 0$$

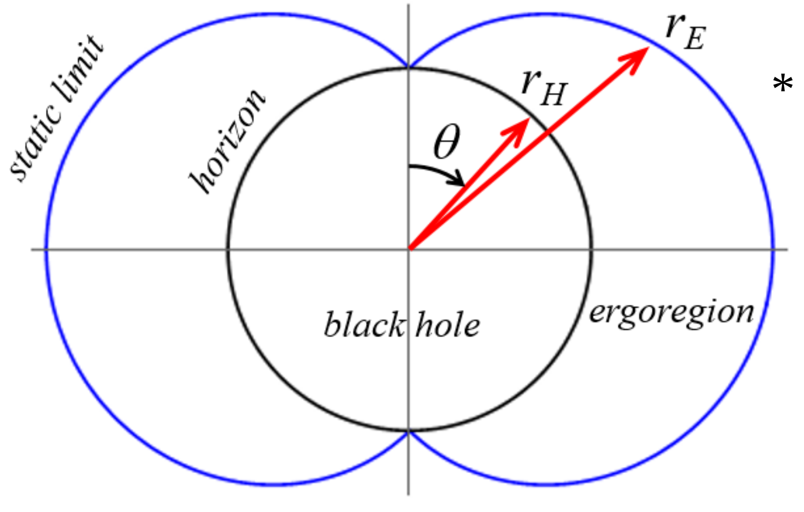
like the quadratic for the horizons but with $\cos^2\theta$, and solutions

$$r'_\pm = \frac{r_s}{2} \pm \left[\left(\frac{r_s}{2}\right)^2 - a^2\cos^2\theta\right]^{1/2}$$

The r'_- solution is inside the horizon, so $r'_+ = r_E$:

$$\boxed{r_E = GM + \left[(GM)^2 - a^2 \cos^2\theta \right]^{1/2}} \qquad \text{[static limit, where } g_{tt} = 0\text{]}$$

marks the *static limit* or *infinite redshift surface.* Unlike the horizon r_H it depends on θ. At the poles ($\theta = 0$ and π) it coincides with the horizon but otherwise is *outside the horizon*, reaching $r_E = 2GM = r_s$ at the equator. The space between the horizon and the static limit is called the *ergoregion*.



- The ergoregion

In Sch. black holes the event horizon and static limit coincide. In Kerr black holes there's the ergoregion in between, where escape to infinity is still possible but there's some kind of problem with time for static objects. What happens there?

Consider light ($ds^2 = 0$) moving only in the ϕ direction "along a line of latitude" ($dr = d\theta = 0$). The Kerr metric becomes

$$0 = g_{\phi\phi} d\phi^2 + 2 g_{\phi t} d\phi dt + g_{tt} dt^2$$

which is a quadratic in $d\phi/dt$, with solutions

* *The figure has poetic licence: r is (of course) <u>not</u> a radius measured from the centre!*

$$\frac{d\phi}{dt} \equiv \omega_{\pm} = -\frac{g_{\phi t}}{g_{\phi\phi}}\left\{1 \pm \left[1 - \frac{g_{\phi\phi}}{g_{\phi t}^{2}} g_{tt}\right]^{1/2}\right\}$$

always +ve

Since nothing travels faster than light, particle worldlines are bounded by

$$\omega_{-} \leq \frac{d\phi}{dt} \leq \omega_{+}$$

What signs can the metric coeffs have? See p. 105:

$g_{\phi\phi}$ is always +ve, $g_{\phi t}$ is always -ve

g_{tt} is -ve outside the static limit and +ve inside the static limit

So ω_+ is always positive. For large $r > r_E$, ω_- is negative and $d\phi/dt$ (bounded by ω_+ and ω_-) can have either sign. But, for $r < r_E$, both ω_+ and ω_- are positive and *$d\phi/dt$ must be positive*.

$\Rightarrow$ In the ergoregion, matter *must orbit in the same ϕ direction as the spin of the black hole*. (This resolves the problem of time for static objects: objects cannot be static in the ergoregion, hence the term "static limit"). However, it is outside the horizon and is still free to move inward or outward in r (or escape to infinity).

This is an example of *frame dragging* - in the ergoregion, the black hole's spin drags inertial frames around it so fast that not even light can orbit the "wrong" way. (The fact that particles moving with the spin can stay outside the horizon is due to gravito-magnetism: the repulsive contribution to gravity between co-moving masses that we briefly encountered in L1.)

Orbits around Kerr black holes

For simplicity we'll only consider orbits in the equatorial plane ($\theta = \pi/2$, $d\theta = 0$) from now on. In Sch. spacetime we did this without loss of generality but, because Kerr black holes have axial not spherical symmetry, it's very much a special case here. Here are the $\theta = \pi/2$ versions of the results from the last lecture:

$$g_{tt} = -\left(1 - \frac{r_s}{r}\right) \qquad g_{\phi\phi} = r^2 + a^2 + \frac{a^2 r_s}{r}$$

$$g_{rr} = \frac{r^2}{\Delta} \qquad g_{\phi t} = -\frac{r_s a}{r} \qquad \text{[metric coeffs]}$$

$$\Delta = r^2 + a^2 - r_s r$$

$$r_H = GM + \left[(GM)^2 - a^2\right]^{1/2} \qquad \text{[event horizon, where } \Delta = 0\text{]}$$

$$r_E = 2GM = r_s \qquad \text{[static limit, where } g_{tt} = 0\text{]}$$

It's also possible to prove this Very Useful Identity*:

$$g_{\phi t}^2 - g_{\phi\phi} g_{tt} \equiv \Delta \sin^2\theta \qquad \text{[in general]}$$

$$= \Delta \qquad \text{[equatorial, } \theta = \pi/2\text{]}$$

- Equations of motion

The only coord that the metric coeffs depend on is r, but the metric is not diagonal because $g_{\phi t} \neq 0$. This means we can use simplifications #1 and #3, but not #2, on p. 62-63:

$$\Rightarrow \qquad \sum_{\beta=0}^{3} g_{\alpha\beta} \frac{dx^\beta}{d\tau} \qquad \text{is constant if } x^\alpha = t \text{ or } \phi$$

* *It doesn't seem to have a name, so I'll just call it the VUI.*

$x^\alpha = t$: $$g_{tt}\frac{dt}{d\tau} + g_{\phi t}\frac{d\phi}{d\tau} = -e$$

$x^\alpha = \phi$: $$g_{\phi t}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau} = l$$

give the constants names

For large r, $$e = \frac{dt}{d\tau} = \gamma$$ $\Rightarrow$ specific relativistic energy e

$$l = r^2\frac{d\phi}{d\tau}$$ $\Rightarrow$ specific angular momentum* l

Solve the simultaneous equations, with the aid of the VUI:

$$\Rightarrow \quad \left.\begin{aligned} \frac{dt}{d\tau} &= \frac{g_{\phi\phi}e + g_{\phi t}l}{\Delta} \\ \frac{d\phi}{d\tau} &= \frac{-g_{\phi t}e - g_{tt}l}{\Delta} \end{aligned}\right\}$$ [t and ϕ equations of motion]

For $x^\alpha = r$ use simplification #3 from p. 63:

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + 2g_{\phi t}\left(\frac{d\phi}{d\tau}\right)\left(\frac{dt}{d\tau}\right)$$

Subst for $dt/d\tau$ and $d\phi/d\tau$ and manipulate using the VUI $\Rightarrow$

$$\left[g_{rr}\left(\frac{dr}{d\tau}\right)^2 + 1\right]\Delta = e^2 g_{\phi\phi} + l^2 g_{tt} + 2elg_{\phi t}$$ [r eqn of motion]

We now have three equations of motion that can (in principle) be integrated (numerically?) for given e and l to give the particle's worldline $r(\tau)$, $\phi(\tau)$ and $t(\tau)$. But they are somewhat unlovely. We'll study two cases.

* *Reminder: we use J or a for the spin angular momentum of the central mass, and L or l for the orbital angular momentum of a particle moving around it.*

- Case 1: Free fall from rest at large r

From the large-r results on the previous page, and for the same reasons as the Sch. case on p. 76, $l = 0$ and $e = \gamma = 1$. From the equations of motion:

$$\frac{d\phi}{d\tau} = \frac{-g_{\phi t}}{\Delta} = \frac{r_s a}{r\Delta}$$

$$\frac{dt}{d\tau} = \frac{g_{\phi\phi}}{\Delta} = \frac{r^2 + a^2 + a^2 r_s / r}{\Delta}$$

$$\frac{dr}{d\tau} = \text{ ... algebra ... } = -\left\{\frac{r_s}{r^3}(a^2 + r^2)\right\}^{1/2}$$

The particle has a *non-zero angular velocity* (in the direction of the black hole's spin) despite having *zero angular momentum** ...

At the horizon ($r = r_H$), $dt/d\tau$ and $d\phi/d\tau \to \infty$ while $dr/d\tau$ is finite $\Rightarrow$ *both* coords t and ϕ behave badly at the horizon; it's a coordinate singularity. But

$$\frac{d\phi}{dt} = \frac{d\phi / d\tau}{dt / d\tau} = \frac{r_s a}{r\left(r^2 + a^2 + a^2 r_s / r\right)}$$ finite at the horizon

Shape of orbit:

$$\frac{d\phi}{dr} = \frac{d\phi / d\tau}{dr / d\tau} = -\frac{a}{\Delta}\left(\frac{r_s r}{r^2 + a^2}\right)^{1/2}$$ $\to \infty$ at the horizon ($\Delta = 0$)

$\Rightarrow$ the observer at ∞ sees the particle revolve around the black hole at a finite rate as it approaches the horizon, but it wraps around an infinite number of times.

ϕ is basically a failed coordinate at the horizon!

* *Indeed, a particle's angular velocity and angular momentum can be in opposite directions!*

- Case 2: Negative-energy motion ($e < 0$)

Meaning: to bring a particle with $e < 0$ to $r \to \infty$ (where $e = 1$ at rest) requires more energy than the particle's own rest-mass. It is energetically "cheaper" to abandon the particle and build a new one at infinity! For Sch. black holes, we have already seen that negative e is only possible inside the horizon (p. 87).

Rearrange the r equation of motion (p. 111) as a quadratic in e:

$$g_{\phi\phi}e^2 + 2lg_{\phi t}e + l^2 g_{tt} - Z\Delta = 0$$

where
$$Z = g_{rr}\left(\frac{dr}{d\tau}\right)^2 + 1$$

is +ve outside the horizon, and solve for the allowed energies e given l. After some algebra and the VUI:

$$e = -\frac{g_{\phi t}}{g_{\phi\phi}} l \pm \frac{\Delta^{1/2}}{g_{\phi\phi}}\left(l^2 + g_{\phi\phi}Z\right)^{1/2}$$

The minus sign gives $e \leq -1$ at $r \to \infty$, but $e \geq 1$ at infinity so this solution is absurd. Consider only the plus sign:

$$e = -\frac{g_{\phi t}}{g_{\phi\phi}} l + \frac{\Delta^{1/2}}{g_{\phi\phi}}\left(l^2 + g_{\phi\phi}Z\right)^{1/2}$$

$g_{\phi\phi} > 0$ and $g_{\phi t} < 0$ always, so both circled parts of the equation must be positive. The only way e can be negative is if

$$g_{\phi t} l > \Delta^{1/2}\left(l^2 + g_{\phi\phi}Z\right)^{1/2}$$

which requires $l < 0$: the particle's angular momentum is directed against the black hole's spin.

In that case, $e < 0$ at the horizon ($\Delta = 0$). For small -ve l, the inequality holds only close to the horizon: $\Delta^{1/2}$ increases with r and the RHS soon overtakes the LHS. As l becomes more -ve, r must be bigger before the RHS beats the LHS, and the range where $e < 0$ widens. The extreme case is when l is large and -ve, in which case we can neglect the Z term on the RHS and substitute for $g_{\phi t}$:

$$-\frac{r_s a}{r} l > \Delta^{1/2} \, |\, l \,| \qquad \text{[modulus to keep RHS +ve]}$$

$$\Rightarrow \qquad \left(\frac{r_s a}{r}\right)^2 = \Delta = r^2 + a^2 - r_s r \qquad \text{[square]}$$

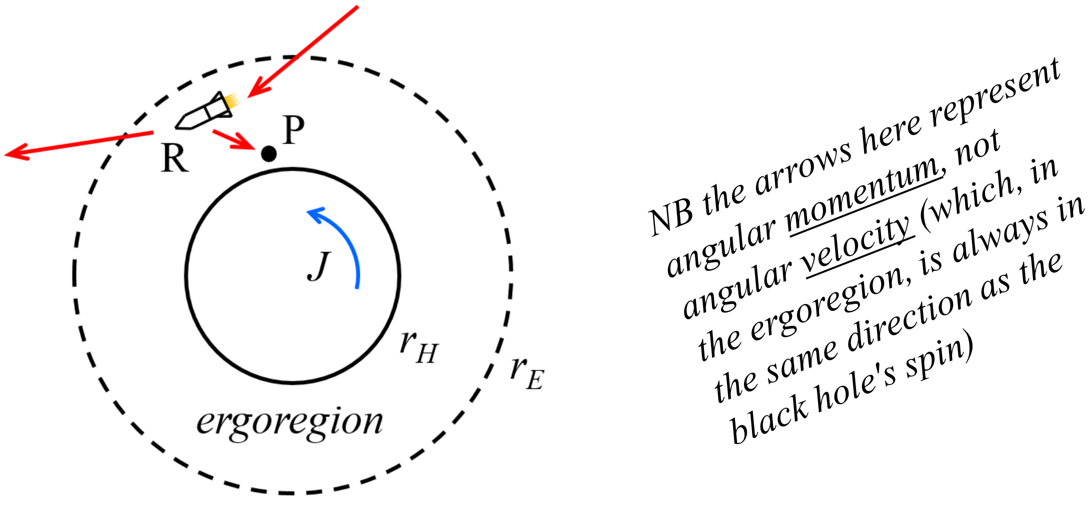
$$\Rightarrow \qquad (r - r_s)\left(r + \frac{a^2}{r} + \frac{r_s a^2}{r^2}\right) = 0 \qquad \text{[factorise]}$$

$$\Rightarrow \qquad r = r_s = r_E \qquad \text{[2nd factor has no +ve roots]}$$

Summary: For Kerr black holes, negative energy e is possible outside the horizon, but only in the ergoregion and only for negative l.

- The Penrose process (using negative energy)

Send rocket R from a far-away base to the ergoregion, where R dispatches payload P into an $e < 0$ trajectory. R returns to base, while P falls through the horizon. R has lost a -ve energy payload so it returns with *more energy* than it started with. The payload's -ve energy and -ve ang. mom. are added to the black hole's, *reducing its mass* and spin. This *Penrose process* therefore mines the rotational energy of a "live" Kerr black hole.



At the base, R's extra kinetic energy can be used to do work, then a new P attached and the process repeated. This is a very efficient energy generation scheme for a technologically-advanced civilisation*. (It's also good for waste disposal, if P is filled with junk.) The ultimate limit is when all of the black hole's rotational energy has been extracted, leaving a "dead" Schwarzschild black hole.

- Electrically-charged black holes (for completeness)

The other characteristic a black hole can have besides mass and spin is electric charge Q. However, charged black holes are of theoretical interest only. Here are the names of their metrics:

	$Q = 0$	$Q \neq 0$
$J = 0$	Schwarzschild	Reissner-Nordström
$J \neq 0$	Kerr	Kerr-Newman

** A hypothetical society with "advanced technology" is one that obeys the laws of physics, but can do indefinitely more within them than we can.*

10. GR and quantum mechanics

The thermodynamics of black holes

The thermodynamic states of black holes are very simple. Whereas the structure of an ordinary star encompasses huge numbers of moving particles, the only characteristics of a black hole are its mass M, spin J and electric charge Q. All other information about what formed it, or fell in afterwards, is lost.

Wheeler (again!) expressed this as: "A black hole has *no hair*."

If a black hole has no other degrees of freedom in its structure, it has no statistical-mechanical microstates. So, *what happens to the entropy* of matter that falls through the horizon? Do black holes violate $\Delta S \geq 0$, the second law of thermodynamics?

- Irreversibility in black holes

Entropy is about irreversible processes, so what's irreversible about a black hole? Although things can only pass inwards through the event horizon, a black hole's mass doesn't always increase: the Penrose process is a counter-example. However, S Hawking showed that the *area* A of the event horizon (or the combined area if several black holes interact) can never decrease by classical physical processes. J Bekenstein then proposed that black holes have an entropy proportional to the area: $S \propto A$. The second law (for black holes + everything else) survives if

$$S = \frac{k_B A}{4G\hbar} \qquad \text{[in general]}$$

$$= \frac{4\pi k_B G M^2}{\hbar} \qquad \text{[Sch. black hole*, } A = 4\pi r_s^2\text{]}$$

** Because (p.51) the event horizon of a Sch. black hole has the geometry of a sphere of radius r_s and hence a surface area of $4\pi r_s^2$*

- Temperature of black holes

A thermodynamic system whose entropy S depends on its internal energy U has a *temperature* T :

$$dU = TdS \qquad \Rightarrow \qquad T = 1/_{dS\,/\,dU} \qquad\qquad [\text{1st law, } \text{đ}W = 0]$$

But in relativity mass is a form of internal energy: $U = M$. Since S (previous page) depends on M, black holes have a temperature:

$$T = \frac{\hbar}{8\pi k_B GM} \qquad\qquad [\text{differentiate } dS/dM]$$

A black hole is a black body (it absorbs all incident radiation) so it must emit *black-body radiation* according to Planck's law for temp T! Hawking used this to oppose Bekenstein's entropy idea - *obviously* nothing comes out of a horizon, so $T = 0$. But then he discovered a quantum mechanism for black holes to radiate.

- Hawking radiation

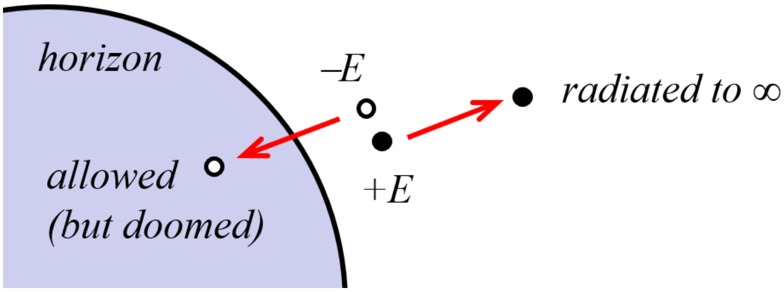
According to quantum field theory, vacuum fluctuations continually produce *virtual* particle-antiparticle pairs. One has +ve energy $+E$ and the other -ve energy $-E$. Since -ve energy is forbidden outside the horizon, the particles exist only briefly before recombining in a time given by the uncertainty principle:

$$\Delta E \Delta t \sim \hbar \qquad \Rightarrow \qquad \tau \sim \hbar / E$$

But if they are so close to the horizon that the $-E$ particle falls in within this time, its energy is now allowed* and the particles become *real*. The $-E$ particle reduces the black hole's mass, and the $+E$ particle can escape to infinity as Hawking radiation.

Hawking calculated the temperature of a black hole from this idea, and *got the same answer* as derived from Bekenstein's entropy. Thus he changed his mind about Bekenstein's proposal.

** -ve energy is allowed inside the horizon, see p. 87*



- Hawking's derivation

... is beyond us. But remarkably we can derive an approximate T from the uncertainty principle. A virtual pair fluctuates into existence just outside the horizon at $x = x_0$, where $r \equiv r_s + x$. Observe the pair in a reference frame free-falling from rest at that point. How long does it take the virtual $-E$ particle to reach the horizon at $x = 0$ and become real? From (ix) on p. 76:

$$\frac{dr}{d\tau} = -\left(E^2 - \left[1 - \frac{r_s}{r}\right]\right)^{1/2} \qquad \text{[vertical drop, with } e \to E\text{]}$$

Close to the horizon, $r \approx r_s$ and $x << r_s$, so:

$$1 - \frac{r_s}{r} = \frac{r - r_s}{r} = \frac{r_s + x - r_s}{r_s + x} = \frac{x}{r_s + x} \approx \frac{x}{r_s}$$

Start at rest at $x = x_0$: $\quad 0 = E^2 - \frac{x_0}{r_s} \quad \Rightarrow \quad E^2 = \frac{x_0}{r_s}$

$$\Rightarrow \qquad \frac{dr}{d\tau} = \frac{dx}{d\tau} = -\frac{(x_0 - x)^{1/2}}{r_s^{1/2}}$$

Integrate from the starting point $x = x_0$ to the horizon $x = 0$:

$$\int_0^\tau d\tau = -r_s^{1/2} \int_{x_0}^0 \frac{dx}{(x_0 - x)^{1/2}}$$

$\Rightarrow$ time to horizon $\quad \tau = r_s^{1/2} \left[\frac{(x_0 - x)^{1/2}}{1/2}\right]_{x_0}^0 = 2 r_s^{1/2} x_0^{1/2}$

The uncertainty principle $\Delta E \Delta t \sim \hbar$ allows the fluctuation to last this long if

$$E \sim \frac{\hbar}{\tau} = \frac{\hbar}{2 r_s^{1/2} x_0^{1/2}}$$

The $+E$ particle with this energy undergoes gravitational redshift as it travels to infinity, where it is observed to have energy E_∞

$$E_\infty = \left(1 - \frac{r_s}{r}\right)^{1/2} E \qquad \text{[from p. 58, } E = \hbar\omega\text{]}$$

$$\approx \frac{x_0^{1/2}}{r_s^{1/2}} E \qquad [E^2 \text{ from prev page}]$$

$$= \frac{x_0^{1/2}}{r_s^{1/2}} \times \frac{\hbar}{2 r_s^{1/2} x_0^{1/2}} = \frac{\hbar}{2 r_s} \qquad [\text{independent of } x_0 \text{ !}]$$

Characteristic temp corresponding to this energy $E_\infty = k_B T$:

$$T = \frac{\hbar}{2 k_B r_s} = \frac{\hbar}{4 k_B G M}$$

within 2π of Hawking's exact calculation!

- Black hole lifetime

If a black hole radiates, then (in a cold environment) it will lose mass and eventually evaporate completely. In problem E6 you'll use Hawking's temp & Stefan's law ($\rightarrow$ radiated intensity) & the area of the horizon ($\rightarrow$ radiated power) & $U = M$ ($\rightarrow$ rate of mass loss) to derive and solve a differential equation for dM/dt for a Sch. black hole in a Universe at absolute zero.

$\Rightarrow$ lifetime $\qquad t_0 = 2.1 \times 10^{67} \left(\dfrac{M}{M_S}\right)^3$ *years*

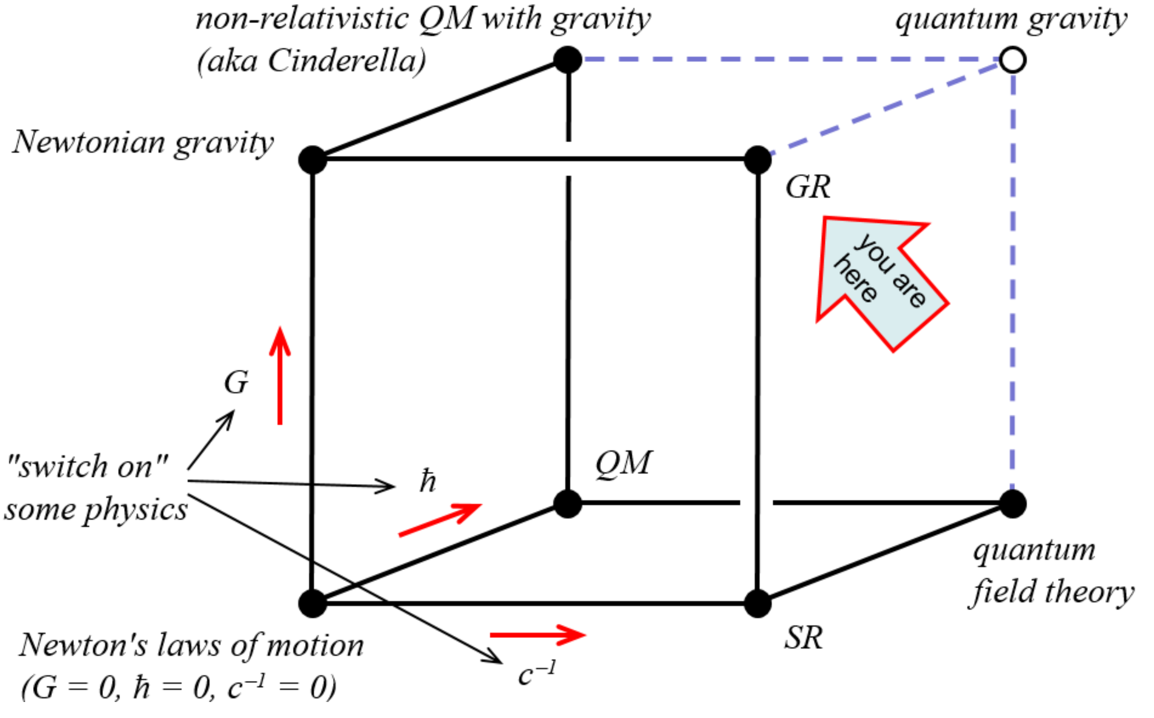
ie a very very long time. $\qquad$ mass of the Sun

Quantum gravity

> **Attention**: in this lecture (quantum gravity) we will revert to ordinary $c \neq 1$ units, with time in seconds.

GR is a classical theory that ignores quantum uncertainty. Bekenstein and Hawking needed QM (quantum mechanics) - their formulae include $\hbar$ - but they still used classical GR for the gravity parts of their derivations. We need to go beyond GR to a *quantum theory of gravity* to understand gravitational phenomena where quantum "fuzziness" on small scales is important - like the singularities of black holes.

The *cube of physics* illustrates how physical theories have been developed to encompass phenomena represented by non-zero values of G, $\hbar$ and c^{-1} (SR $\rightarrow$ Newton if c is *infinite*). Quantum gravity is the missing corner where all three are included:



- The Planck scale

The Planck units of mass, length and time* are *defined* (using dimensional analysis) to be the combinations of $\hbar$, c and G that have the right unit dimensions:

Planck mass $$M_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.177\times10^{-8}\ \text{kg}$$

Planck length $$L_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.616\times10^{-35}\ \text{m}$$

Planck time $$T_P = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.391\times10^{-44}\ \text{s}$$

These are extreme values: L_P and T_P are ridiculously small for any purpose, and M_P is both ridiculously big (for a fundamental particle) and ridiculously small (for a black hole).

- GR and QM conflict at the Planck scale

Consider the "effective size" of mass M under each theory.

QM: To localise a particle within Δx, the uncertainty principle gives it a momentum of $\sim\hbar/\Delta x$. But, if this exceeds $\sqrt{3}Mc \sim Mc$ [from $E^2 = p^2c^2 + M^2c^4$] there's enough spare energy to create a new particle, preventing us localising the original one. So the minimum measurable *quantum size* of "point" mass M is

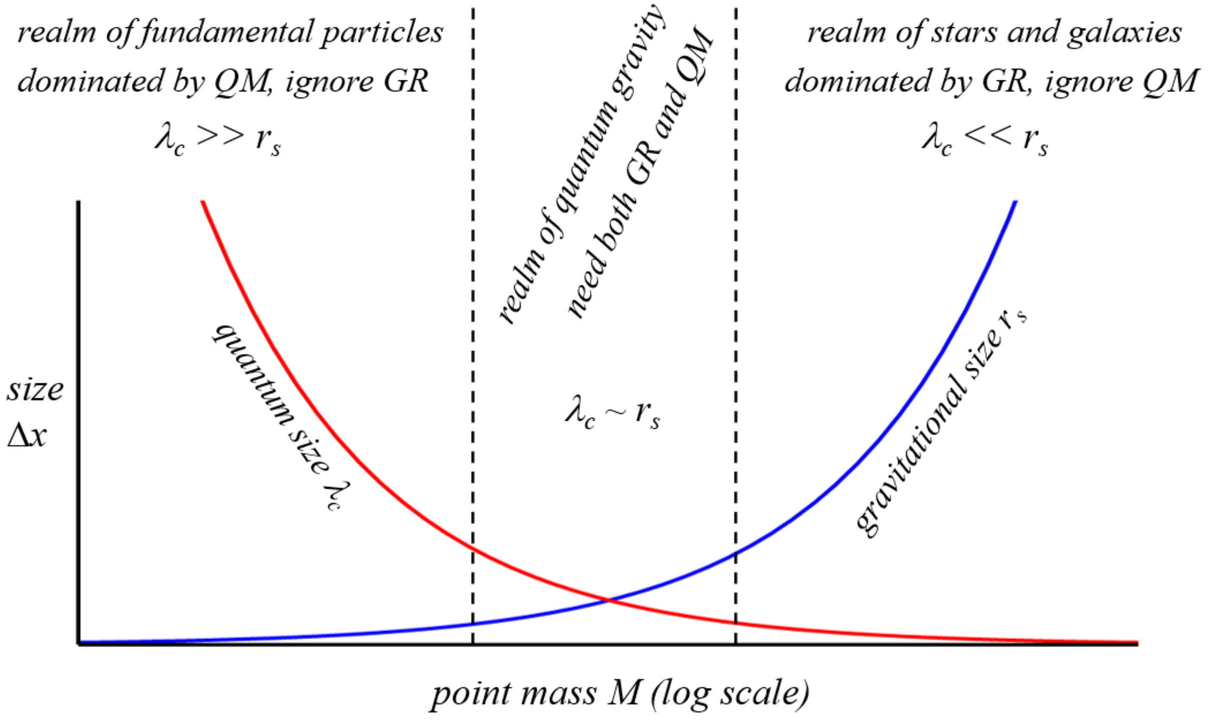
$$\Delta x_{QM} \sim \lambda_c = \frac{\hbar}{Mc}$$ [a.k.a. the *reduced Compton wavelength*]

GR: Meanwhile the minimum *gravitational size* of mass M is

$$\Delta x_{GR} \sim r_s = \frac{2GM}{c^2}$$ [the Sch. radius, in $c \neq 1$ units]

** Other derived Planck units can be obtained from these three, eg Planck energy M_Pc^2, Planck area L_P^2, etc.*

Note that quantum size $\propto 1/M$ whereas gravitational size $\propto M$:



The quantum and gravitational sizes become equal if

$$\lambda_c \sim r_s \qquad \Rightarrow \qquad M \sim \left(\frac{\hbar c}{G}\right)^{1/2} = M_P$$

ie the *Planck mass*. The effective size (λ_c or r_s) of a Planck-mass particle is the *Planck length*. Light travels the Planck length in the *Planck time*.

So the two theories disagree about the minimum sizes of point particles. We can (and usually do) ignore QM at the scale of stars and galaxies, or GR at the scale of fundamental particles, but this won't work at the Planck scale, where:

Gravitational and quantum phenomena are both important;

L_P is the smallest meaningful length in physics;

A point particle is heavy enough to be sufficiently localised (despite quantum uncertainty) to disappear inside its own event horizon and form a "micro black hole";

Attempts to measure down to L_P require so much energy that the measurement process *creates* micro black holes;

A black hole is small enough for quantum effects to become important - QM affects spacetime itself, not just particles;

Spacetime becomes like foam (Wheeler), or a bucket of dust (Wheeler), or a bubbling sea of virtual black holes (Hawking), or a weave of knots and tangles (Smolin), or whatever ...

- The need for a theory of quantum gravity

Quantum gravity is needed to make sense of physics at the Planck scale. Neither QM nor GR on its own is good enough, and they can't both be right.

Unfortunately there is as yet *no adequate theory of quantum gravity*. Candidate theories are largely speculation, untestable hypothesis, or even metaphysics.

Nor is there any prospect of guidance from observations: no foreseeable experiment could probe the small distances and times, or the heavy point particles, at the Planck scale. And, indeed, experimentally it's not a pressing problem.

So, for now, *classical GR remains our best theory of gravity*.

Nevertheless, quantum gravity is needed to explain:

- what happens at the singularity of a black hole;

- what happens before time T_P after the Big Bang;

- how black-hole evaporation ends, when $M \sim M_P$;

- the nature of the microstates that give black holes entropy;

- to reveal "unknown unknowns" in physics;

- to complete physics!

We can already look at some features a theory of quantum gravity may have:

- The black-hole information problem

(Sometimes called the black-hole information *paradox*.)

It is a feature of quantum-mechanical theories that physical information is conserved: so-called *unitarity*. However, GR's no-hair theorem says that all information carried by objects falling into a black hole (besides mass M, spin J and electric charge Q) is destroyed. There have been many proposals (and a famous bet by Stephen Hawking) addressing this disagreement, but no real solution yet. Quantum gravity should tell us.

- The cosmological constant (dark energy) problem

Simple QM calculations* predict a cosmological constant that is $\sim 10^{120}\times$ bigger than observed. This has been called the worst theoretical prediction in the history of physics! We'd like quantum gravity to fix it.

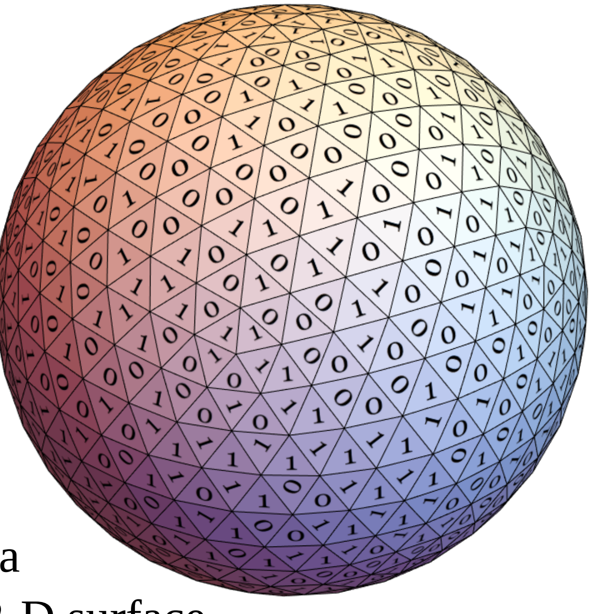
* *Basically, the gravity of the "$\frac{1}{2}\hbar\omega$" zero-point energy of the vacuum, with a high-frequency cutoff at $\omega \sim 1/T_P$ or else the answer is infinite ...*

- Origins of black-hole entropy

Bekenstein: $$S = \frac{c^3 k_B A}{4G\hbar}$$ [from p. 116, in $c \neq 1$ units]

Extensive variables in thermal physics scale with volume, but here S scales with *area*. And, given the no-hair theorem (p. 116), where are the microstates that give rise to S? Subst L_P:

$$S = k_B \frac{A}{(2L_P)^2}$$



One unit of entropy k_B for every ~Planck area $(2L_P)^2$ of the event horizon - something deep there! This suggests that a black hole stores information uniformly on its horizon, in tiny Planck-sized bits. This is known as the *holographic principle*, by analogy with the way a hologram stores a 3-D image on a 2-D surface.

But, the event horizon is not a material surface, it's just empty space - does this mean spacetime itself is granular?

- The graviton

The force-carrying quantum of gravity, like the photon is for electromagnetism. It has zero rest-mass like the photon, because both forces are long range. Tidal displacement has two-fold rotation symmetry, so the graviton is a *tensor boson* with spin 2. (The electric field vector has one-fold rotation symmetry, so the photon is a vector boson with spin 1.) Gravitons interact very weakly, so there is no prospect of detecting them experimentally.

- Current theories of quantum gravity

The most well-known is *string theory* / superstrings / M-theory (the same theory in different stages of development):

Particles are excitations of 1-D Planck-scale *strings* rather than the traditional 0-D points.

The theory *unifies all fundamental forces*, not just gravity.

It attempts to *eliminate free parameters* (like particle masses, charges, force strengths etc) from physics.

It introduces 6 or 7 *extra Planck-sized dimensions* to spacetime.

The maths has not been completed and is only approximate so far. Consequently there are ~10^{500} possible topologies for the extra compactified dimensions - a lot more free parameters!

Supersymmetry (hence *super*strings) predicts new "partner" particles, eg photinos, squarks, sleptons etc. The LHC hasn't found any of them yet - how long do we wait?

It is controversial, highly speculative, and has metaphysical baggage (extra dimensions, unfalsifiable multiverses, anthropic reasoning - see "Occam's razor"). But, simplified cases correctly yield the microstates needed for black-hole entropy, and it reproduces the holographic principle.

It is not testable for the foreseeable future, so is it really physics?

Other theories of quantum gravity are available. The most well-publicised is *loop quantum gravity*, in which spacetime itself is quantised on the Planck scale.