Coupled propagation of light and matter waves: solitons and transverse instabilities

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1 Introduction

In 1962 Askar’yán introduced the notion of self-focusing of radiation in a nonlinear medium$^1$. The physical situation he considered was that of the gradient force due to spatial variations in the average value of the radiation intensity pushing electrons or atoms into or out of the center of the beam. As he noted in the final paragraph of his paper “... actions of intense radiation in the medium can be so strong that a gradient is produced in the properties of the medium in the beam and outside of it, which results in the waveguide propagation of the beam...”. The waveguiding envisioned in [1] was due to local changes in the density of the medium. Although spatial solitons have been studied in the intervening years in a great variety of media, it is uncertain whether the mechanism of Askar’yán has in fact been observed. Most experimental studies of self-focusing have exploited electronic or orientational nonlinearities that do not rely upon a density change. In media with thermal nonlinearity the density does change due to the presence of the beam, although it is the intensity not the gradient of the intensity that is responsible for the nonlinearity. As thermal media are generally self-defocusing, only dark optical solitons have been observed in this case$^2$. Photorefractive media which have been the object of much work in recent years come perhaps closest to supporting the type of waveguiding envisaged by Askar’yán. There the induced nonlinearity is caused by a modification of the trapped charge density that is driven by the gradient of the optical intensity$^3$.

In the years since 1962 tremendous advances have been made in techniques for the optical manipulation of neutral atoms $^4,5$. The compression and rarefaction described in [1] have been observed using a thermal beam of neutral atoms copropagating with a focussed laser beam$^6$. Furthermore it is now a routine matter to optically cool atoms to sub-recoil temperatures where the atomic momentum is even less than the momentum due to a single photon. The wave and diffractive aspects of atomic propagation have been observed unambiguously $^7$, and sources of spatially coherent atomic waves or atom lasers are being actively developed $^8–10$. 
These developments in atomic and optical physics make it possible today to study the nonlinear propagation of radiation and matter in a unified fashion. Although experimental observations remain a topic for the (not too distant) future, several authors have already studied the nonlinear propagation of light and matter theoretically. Most of these recent studies of the interaction of cold atoms with light have been done in the approximation where the spatial dependence of the optical field was assumed unaffected by the atomic dynamics. This approximation was used successfully to describe trapping and cooling of atoms by optical fields. The basic model describing this process in the case of a near resonant optical field is a set of coupled Schrödinger equations for the atomic wave functions of the ground and excited states, where the interaction with the optical field is described in the dipole approximation [11,12]. If the atomic medium is sufficiently dense then multiple scattering of photons by atoms becomes an important process and dipole-dipole corrections have to be taken into account [13–15]. The latter corrections result in nonlinear potential terms in the Schrödinger equation, which have been shown to lead to the formation of atomic solitons in a traveling wave optical field [14], or to the formation of gap atomic solitons in a standing wave field [13]. Another source of atomic nonlinearity is collisions, which in the zero temperature approximation also lead to the appearance of a cubic nonlinearity in the Schrödinger equation, transforming it into the so-called Gross-Pitaevskii (GP) equation. The GP equation has been widely used to describe the recent wave of experimental results on Bose-Einstein condensation (BEC) in alkali vapors[16].

Although a rigorous theoretical approach to describing the joint dynamics of optical and atomic fields in the quantum regime has been developed by several groups of authors [12,13,15,17–20], there have been only a few investigations of the coupled dynamics. In [21] both the longitudinal and transverse dynamics of the atomic beam were accounted for, but transverse variations of the optical field were neglected. In this approximation the effects of mutual trapping of the atomic and optical fields do not appear. A first calculation of mutual trapping of an optical beam and a hot atomic gas was given in [22]. In another work [23] diffraction of both optical and atomic fields was taken into account and filamentation of coupled fields was studied. However, the model used in [23] was not derived from first principles and therefore a more rigorous treatment is desirable. Below we use a quantum model that describes the mutual spatio-temporal dynamics of an optical field and an ultracold Bose gas of two-level atoms [20]. To study solutions of these equations we use the mean-field approximation, i.e. we replace all operators by their expectation values. We develop a self-consistent description of the optical and atomic nonlinearities that goes beyond the adiabatic procedure used previously to eliminate the wave function of the excited state. As a result of this procedure we arrive at coupled mean-field equations of the nonlinear Schrödinger (NLS) type describing interaction of the paraxial optical and atomic beams.
We study stability of the continuous wave solutions and coupled solitonic solutions of the resulting system.

The chapter is organized as follows. In Sect. 2 we derive the coupled propagation equations using a a two-level atomic model. We point out the analogy with parametric three wave mixing in the low-density limit. Adiabatic elimination of the excited state then leads to coupled equations for the optical field and the atomic ground state that we use as a basis for studies of modulational instability and solitary solutions. In Sect. 3 we study the linear instability of plane wave solutions that leads to convective instabilities of the coupled fields. In Sect. 4 we study nonlinear solitonic solutions of the coupled equations. In Sect. 5 we consider the observability of modulational instability and opto-atomic solitons in relation to present experimental limits on atom laser sources.

2 Governing Equations

Mean values of the components of the matter field wave function $\tilde{\varphi} = \tilde{\varphi}_g e^{-i E_g t/\hbar} + \tilde{\varphi}_e e^{-i E_e t/\hbar}$ of a Bose gas of two level atoms evolving in the classical optical field $(\tilde{A} e^{-i \omega_1 t} + \tilde{A}^* e^{i \omega_1 t})/2$ obey a system of coupled Schrödinger equations [12,13,20]

\[
\begin{align*}
\hbar \partial_t \tilde{\varphi}_g & = -\frac{\hbar^2}{2m} \nabla^2 \tilde{\varphi}_g - \frac{1}{2} \mu \tilde{A}^* \tilde{\varphi}_e e^{i \Delta t} + \frac{4\pi \hbar^2 a_{gg}}{m} |\tilde{\varphi}_g|^2 \tilde{\varphi}_g, \\
\hbar \partial_t \tilde{\varphi}_e & = -\frac{\hbar^2}{2m} \nabla^2 \tilde{\varphi}_e - \frac{1}{2} \mu \tilde{A}_e \tilde{\varphi}_g e^{-i \Delta t} - i\hbar \gamma/2 \tilde{\varphi}_e,
\end{align*}
\]

where indices $g$ and $e$ denote ground and excited states, $\mu$ is the matrix element of the atomic dipole moment, $m$ is the mass of an atom, $\Delta = \omega_1 - \omega_2$, $\hbar \omega_2 = E_e - E_g$, $\gamma$ is the spontaneous decay rate of the excited atoms, $a_{gg}$ is the scattering length corresponding to interactions of the ground-state atoms, and $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$. The wavefucns are normalized so that $\int d^3 x |\tilde{\varphi}_g|^2 + |\tilde{\varphi}_e|^2 = N$, with $N$ the number of atoms. A term corresponding to population of the ground state due to spontaneous decay of the excited state has been neglected in Eq. (1a), as it will be negligible in the limit of large detuning considered below. The latter limit also allows to eliminate $\tilde{\varphi}_e$ adiabatically. As we will show below, our adiabatic method allows comparison of the atom-optical nonlinearities due to dipole, dipole-dipole and collisional interactions and due to the Kerr effect. Taking the collisional term in the form of a cubic nonlinearity of the atomic wave function we have made an implicit assumption that atoms in the ground state are at zero temperature.

The equation for the slowly varying amplitude of the optical field is

\[
\frac{2\omega_1 n^2}{c^2} \partial_t A = -\nabla^2 A - \frac{\omega_1^2 n^2}{c^2} A - \frac{\omega_1^2}{c^2 \varepsilon_0} P,
\]

(2)
where $c$ is the velocity of light in vacuum and $n$ is the index of refraction of the medium. The polarization due to dipole interaction with the atoms is

$$P = \mu \Phi^* \Phi e^{i \Delta t}. \quad (3)$$

Equations (1-3) can also be used to describe optical manipulation of cold atoms above the critical temperature for condensation[11], but the effect of atomic collisions in this regime should be disregarded: $a_{gg} = 0$. In this regime the wave function $\Phi$ is that of a single atom and the atomic polarization should be multiplied by the density of atoms.

In dense media the optical field $A$ should include corrections due to the induced polarization. A self-consistent quantum treatment of these local field corrections can be found in [19,20]. The results of those works can be recovered by formal substitution of the classical expression [24]

$$A \rightarrow A + \frac{P}{3\varepsilon_0}$$

into the equations for the atomic wave functions.

In the following we will assume monochromatic and paraxial optical and atomic fields and therefore make the substitutions

$$A(x, y, z, t) \rightarrow A(x, y, z)e^{ik_{1t}z},$$

$$\Phi_g(x, y, z, t) \rightarrow \Phi_g(x, y, z)e^{i[k_{3g}z-\omega t]},$$

$$\Phi_e(x, y, z, t) \rightarrow \Phi_e(x, y, z)e^{i[k_{3g}z-(\omega + \Delta)t]}, \quad (4)$$

where $k_{1t} = \omega_{lt}n/c$ is the optical wavenumber, $k_{3g} = mv_{a}/\hbar$ is the atomic wavenumber, $m$ is the atomic mass, $v_a$ is the velocity of the atomic beam, and $\hbar \omega = m v_a^2/2$. The detuning is now given by $\Delta = \omega_{lt} - \omega_n - k_{1t}v_a$ which accounts for the Doppler shift due to the mean atomic motion. With these substitutions Eqs.(1-3) take the form

$$i\sigma \partial_t + \partial_{x}^2 + \partial_{y}^2) \Phi_g = -i[\Omega^{*} \Phi_g - \frac{2\hbar^2 a_{gg}}{m} \Phi_y^2 \Phi_g - \frac{1}{6\varepsilon_0} |\Phi_e|^2 \Phi_g], \quad (5a)$$

$$i\sigma \partial_t + \partial_{x}^2 + \partial_{y}^2) \Phi_e = -\sqrt{|\Omega|} \Phi_g - \frac{m u_L^2}{\hbar^2} \frac{\mu^2}{6\varepsilon_0} |\Phi_g|^2 \Phi_e - \delta \left(1 + \frac{m u_L^2}{\hbar^2} \frac{\gamma}{2\delta} \right) \Phi_e, \quad (5b)$$

$$i\partial_t + \partial_{x}^2 + \partial_{y}^2) \Omega = \frac{k_{1t}^2 w_L^4}{4\varepsilon_0 h^2 \sqrt{|\delta|}} \Phi^* \Phi_g \Phi_e, \quad (5c)$$

where

$$\delta = k_{1t}^2 w_L^2 \left(\frac{\Delta}{2\omega_{lt}} - \sigma\right), \quad (6)$$
Table 1. Parameters of several atomic species for which Bose-Einstein condensation has been observed. $a_0 = 5.3 \times 10^{-11}$ m is the Bohr radius.

<table>
<thead>
<tr>
<th>parameter</th>
<th>units</th>
<th>$^7$Li</th>
<th>$^{23}$Na</th>
<th>$^{87}$Rb</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \times 10^{26}$</td>
<td>[kg]</td>
<td>1.2</td>
<td>3.9</td>
<td>15</td>
</tr>
<tr>
<td>$\lambda_{\text{cooling}}$</td>
<td>[$\mu$m]</td>
<td>.671</td>
<td>.589</td>
<td>.780</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>[MHz]</td>
<td>36.6</td>
<td>61.0</td>
<td>37.7</td>
</tr>
<tr>
<td>$\omega_R/2\pi$</td>
<td>[kHz]</td>
<td>62.8</td>
<td>24.8</td>
<td>3.74</td>
</tr>
<tr>
<td>$I_{\text{sat}}$</td>
<td>[W/m$^2$]</td>
<td>25.3</td>
<td>61.9</td>
<td>16.5</td>
</tr>
<tr>
<td>$\mu \times 10^{29}$</td>
<td>[C m]</td>
<td>2.0</td>
<td>2.1</td>
<td>2.5</td>
</tr>
<tr>
<td>$a_{\text{gg}}$</td>
<td>[$a_0$]</td>
<td>-27.6</td>
<td>260.</td>
<td>110.</td>
</tr>
<tr>
<td>$4\pi\hbar^2 a_{\text{gg}}/m \times 10^{29}$</td>
<td>[J s m$^3$]</td>
<td>-.17</td>
<td>.50</td>
<td>.056</td>
</tr>
<tr>
<td>$\mu^2/6\varepsilon_0 \times 10^{49}$</td>
<td>[J s m$^3$]</td>
<td>74.</td>
<td>83.</td>
<td>119.</td>
</tr>
</tbody>
</table>

$\omega_R = \hbar k_L^2/(2m)$ is the single photon recoil frequency, $\sigma = k_A/k_L$, $\zeta = z/(k_L u_L^2)$, $(\xi, \eta) = \sqrt{2}(x, y)/u_L$, and $u_L$ is a characteristic transverse size of the beams.

$$\Omega = \frac{\mu A k_0 u_L^2}{2\hbar} \frac{\sigma}{\sqrt{\theta}}$$

is the Rabi frequency multiplied by the time required for an atom to travel one diffraction length of the optical beam and by the dimensionless constant $\sigma/\sqrt{\theta}$.

Eqs. (5) can be simplified slightly by considering typical parameter values for the atomic species used most widely in Bose-Einstein condensation experiments. Table 1 lists some intrinsic and derived values for Li, Na, and Rb atoms. For sufficiently large detunings we can neglect the dissipative correction to $\delta$ on the right hand side of Eq. (5b) that is due to spontaneous emission. To see this we note that for $\sigma \sim 1$ then $\delta \simeq \omega_R^2 m \Delta / \hbar$ and the magnitude of the imaginary correction in the last term of Eq. (5b) becomes $|\gamma/2\Delta|$. We can therefore neglect spontaneous decay for reasonably large detunings, say $\Delta \geq 10\gamma$.

With this simplification, and assuming a low density limit where the cubic terms in Eqs. (5a,5b) are small, we see that Eqs. (5) are equivalent to the equations of parametric three wave mixing with $\Phi_g$ and $\Omega$ the signal and idler waves, and $\Phi_s$ the sum frequency. It should be noted however that the detuning parameter $\delta$ for optical-atomic parametric mixing will typically be exceptionally large. For a fixed value of $\sigma$ we can choose $\omega_L$ to zero the term in parenthesis in (6). Let us assume that $\omega_L$ can be fixed with an accuracy corresponding to the recoil frequency, then the residual detuning will be $\delta_R = k_R^2 u_R^2/2$. Even for a narrow beam with a width of the order of several $\mu$m we find $\delta_R \sim 30$. For more realistic detunings with $\Delta \geq 10\gamma$ we find $\delta \gtrsim 10^3$, see Table 2.
Such large values of $\delta$ make the full three wave system unsuitable for numerical modelling. We will therefore proceed by adiabatically eliminating the excited atomic state. The extremely large value of $\delta$ justifies this approach. It should also be noted that in [19,20] the collisional term in (5a) was neglected in favor of the dipole-dipole interaction as physical estimates showed that even for a saturation parameter of only 0.01 the dipole-dipole nonlinearity was dominant. Although the dipole-dipole coefficient is much larger than the collisional coefficient, as can be seen in the last lines of Table 1, the dipole-dipole term is only dominant when the detuning is very small. For sufficiently large detuning the excited atomic state is depleted and the collisional nonlinearity always dominates, see Fig. 1. We therefore retain both the dipole-dipole and collisional terms. We consider the dipole-dipole terms and left-hand side in Eq. (5b) as a correction and after two steps of the adiabatic procedure we find

$$
\Phi_e = -s\Omega \Phi_0 \frac{1}{\sqrt{|\delta|}} \left[ 1 - \frac{1}{|\delta|} \left( \frac{s u^2_\perp m \mu^2}{6 \alpha_0 \hbar^2} |\Phi_e|^2 + |\Omega|^2 \right) + O \left( \frac{1}{|\delta|^2} \right) \right],
$$

(7)

where $s = \text{sgn}(\delta)$. Here we have also neglected the nonlinear diffraction term which is proportional to $(\sigma - 1)$. The coefficient on the right-hand side of Eq. (7) has the dimensions of density and can be used in scaling of the ground state wave function

$$
\psi = \Phi_e \frac{k_L u^2_\perp \mu}{2 \hbar} \sqrt{m/(\alpha_0 |\delta|)}
$$

which gives

$$
(i \sigma \partial_\xi + \partial^2_\xi + \partial^2_\eta)\psi = \left[ s \left( 1 - \frac{|\Omega|^2}{|\delta|} \right) |\Omega|^2 - 2 \beta_{\text{dd}} |\Omega|^2 |\psi|^2 + \beta_{\text{coll}} |\psi|^4 \right] \psi,
$$

(8a)

$$
(i \partial_\xi + \partial^2_\xi + \partial^2_\eta)\Omega = \left[ s \left( 1 - \frac{|\Omega|^2}{|\delta|} \right) |\psi|^2 - \beta_{\text{dd}} |\psi|^4 \right] \Omega
$$

(8b)

where

$$
\beta_{\text{dd}} = \frac{2}{3 k_L^2 u^2_\perp},
$$

and

$$
\beta_{\text{coll}} = \frac{16 \pi \varepsilon_0 \hbar^2 a_{\text{gg}} |\delta|}{k_L^2 u^2_\perp m \mu^2}
$$

characterize the strengths of the dipole-dipole and collision interactions relative to the dipole interaction. Characteristic values of the nonlinear coefficients and field strengths are shown in Table 2.
Table 2. Nonlinear coefficients in Eqs. (8a,8b) for average atomic density \( \bar{N} = 10^{20} \text{ m}^{-3} \), intensity \( I = 0.1I_{\text{sat}} \), \( \Delta/\gamma = 100 \), \( \sigma = 1 \), and \( w_{L} = 2 \times 10^{-6} \text{ m} \).

| Atom | \( \beta_{dd} \) | \( \beta_{cd} \) | \( 1/|\delta| \) | \( |\Omega|^2 \) | \( |\psi|^4 \) |
|------|----------------|----------------|----------------|----------------|----------------|
| \( ^{7}\text{Li} \) | \( 1.9 \times 10^{-3} \) | -7.3 6.2 \times 10^{-4} | 2.0 | 1.0 |
| \( ^{23}\text{Na} \) | \( 1.5 \times 10^{-3} \) | 78 | 1.1 \times 10^{-7} | 11 | .88 |
| \( ^{87}\text{Rb} \) | \( 2.6 \times 10^{-3} \) | 25 | 4.8 \times 10^{-8} | 26 | 1.2 |

The nonlinear terms in Eqs. (8) correspond to distinct physical processes. In Eq. (8a) for the atomic dynamics \( s|\Omega|^2 \psi \) gives light induced focusing or defocusing due to the dipole interaction, the next term proportional to \( |\Omega|^4 \) is a higher order correction, \( -2\beta_{dd} |\Omega|^2 |\psi|^2 \) is a focusing nonlinearity (independent of the sign of \( \delta \)) due to dipole-dipole coupling, and \( \beta_{cd} |\psi|^2 \) is a focusing or defocusing collisional term depending on the sign of the scattering length \( a_{\text{agg}} \). In Eq. (8b) for the optical dynamics \( s|\psi|^2 \Omega \) gives a focusing or defocusing effect that is linear in the local atomic density, \( -(s/|\delta|)|\Omega|^2 |\psi|^2 \Omega \) is a focusing or defocusing Kerr nonlinearity that is cubic in the optical field, and \( -\beta_{dd} |\psi|^4 \Omega \) is a dipole-dipole correction that gives higher order focusing. Note that the method used in [19,20] for adiabatic elimination of the excited state failed to provide an explicit account for the Kerr nonlinearity of the far detuned two-level medium.

The nonlinear terms scale with different powers of \( |\delta| \) as is seen in Fig. 1. Considering the atomic equation first the collisional term is independent of \( \delta \) and so it will always dominate for sufficiently large detuning. The dipole term decays as \( |\delta|^{-1} \) and the dipole-dipole term falls off as \( |\delta|^{-2} \). For sufficiently high densities the dipole and dipole-dipole terms may be equal at some finite value of \( |\delta| \). The nonlinear terms in the optical equation (8b) scale as \( |\delta|^{-1} \), \( |\delta|^{-2} \), and \( |\delta|^{-3} \) for the density dependent term, dipole-dipole correction and Kerr nonlinearity respectively. We see that the Kerr nonlinearity is much weaker than all other terms for all values of \( |\delta| \) within the regime of applicability of the adiabatic elimination.

Equations (8) can be used to describe a number of different physical situations. Assuming \( |\psi| = \text{constant} \), Eq. (8b) reduces to the NLS equation used to describe propagation of paraxial beams in media with Kerr nonlinearity. Assuming \( |\Omega| = \text{constant} \), \( \beta_{dd} = \beta_{cd} = 0 \), Eq. (8a) reduces to the equation describing trapping and scattering of atoms by a far-detuned optical field. Assuming \( |\Omega| = \text{constant} \) and \( \beta_{dd} \neq 0 \), we arrive at the so-called nonlinear atom optics [14]. And finally for \( \Omega = 0 \) and \( \beta_{cd} \neq 0 \) we recover the paraxial version of the stationary GP equation describing Bose-Einstein condensates.

\(^{1}\) Figure 1 is plotted in terms of the physical detuning \( \Delta \). For the parameter regime shown in the figure and \( \sigma = 1 \), \( \Delta \) and \( \delta \) are proportional to each other.
Equations (8) conserve the normalized power of the optical beam and the number of atoms
\[
\int d\xi d\eta |\Omega|^2 = \frac{k_1^2 u_1^2 \sigma^2 \rho^2}{2 \alpha \hbar^2 c |\delta| v_a^2} P, \tag{9}
\]
\[
\int d\xi d\eta |\psi|^2 = \frac{k_1^2 u_1^2 m \rho^2}{2 \alpha \hbar^2 |\delta| v_a} F \tag{10}
\]
where \(P\) is the optical beam power in Watts and \(F\) is the atomic beam flux in atoms per second. The average atomic density is given by \(\bar{N} = F/(v_a A)\), where \(A\) is the cross sectional area of the beam.

3 Modulational instability

Before proceeding with analysis of the set (8) we recognize that the Kerr non-linearity can be safely neglected for physical parameters of interest (see Table 2). In the remainder of this chapter we will therefore use the equations
\[
(i \sigma \partial_\xi + \partial^2_\xi + \partial^2_\eta)\psi = [s|\Omega|^2 - 2 \beta_{\lambda\lambda} |\Omega|^2 |\psi|^2 + \beta_{\lambda\delta} |\psi|^2] \psi, \tag{11a}
\]
\[
(i \partial_\xi + \partial^2_\xi + \partial^2_\eta)\Omega = [s|\psi|^2 - \beta_{\lambda\delta} |\psi|^4] \Omega. \tag{11b}
\]

The simplest solutions of Eqs. (11) are plane waves which can be sought in the form \(\psi = \psi_0 e^{i \kappa_\xi \zeta}, \Omega = \Omega_0 e^{i \kappa_\delta \zeta}, \) where \(\psi_0, \kappa_\psi, \Omega_0\) and \(\kappa_\Omega\) are real constants. To study stability of these solutions we perturb them by the weak periodic modulations
\[
\psi = [\psi_0 + (U_\psi(\zeta) + i W_\psi(\zeta)) \cos k \mathbf{r}] e^{i \kappa_\xi \zeta},
\]
\[
\Omega = [\Omega_0 + (U_\Omega(\zeta) + i W_\Omega(\zeta)) \cos k \mathbf{r}] e^{i \kappa_\delta \zeta}, \tag{12}
\]
where \(k\) is the wavevector of a fourier component of the perturbation and \(k \mathbf{r} = k_\xi \xi + k_\eta \eta\). Substituting Eqs. (12) into Eqs. (11) and separating real and
imaginary parts of the linearised equations we arrive at the linear system

\[ \partial_t \varepsilon = \hat{L}(|k|^2)\varepsilon, \quad \varepsilon = (U\psi, W\psi, U_B, W_B)^T \]  

(13)

where \( \hat{L} \) is a four by four real matrix. The presence of any eigenvalue \( \lambda(|k|^2) \) of \( \hat{L}, \hat{L} \varepsilon = \lambda \varepsilon \), with a positive real part implies instability. We have found that the plane wave solutions are modulationally unstable for a wide range of experimentally relevant parameters. Details of this study will be presented elsewhere. Here we will restrict ourselves to showing results of numerical simulation of Eqs. (11) with realistic initial conditions in the form of spatially bounded atomic and optical beams. If the area of the beams is large enough then modulational instability develops in a fashion qualitatively similar to the case of spatially unbounded plane wave solutions.

For \( \beta_{col} < 0 \) the condensate is modulationally unstable even without an optical field. It breaks into a set of filaments, which collapse with increasing \( \zeta \). This collapse is due to the critical character of the cubic nonlinearity in propagation with two transverse dimensions [25]. However it is clear that our model can correctly predict only the initial stage of filamentation, but not collapse itself. This is due to the fact that the local atomic density inside a filament becomes so high, that three-body and higher order collisions become important leading to the necessity of taking into account quintic and higher nonlinearities of the ground state wave functions.

The effect of the optical field exhibits itself in the most striking way for \( \beta_{col} > 0 \), when in its absence the condensate is modulationally stable. In Figs. 2,3,4 we show numerically calculated dynamics of the coupled atomic and optical beams. The values of \( \beta_{at} \) and \( \beta_{col} \) given in the caption of Fig. 2.
were used in Figs. 2-6. We have found that when the optical intensity exceeds some critical value both beams develop filamentary structures, see Figs. 2,3. For unbounded plane wave solutions the threshold effect is absent, i.e. coupled waves are always unstable.

The character of the filamentary pattern depends dramatically on the sign of detuning, $s$. If the dipole interaction induces a defocusing nonlinearity, $s = 1$, then the maxima of the atomic field in the filamentary pattern coincide with minima of the optical field and vise versa, see Fig. 2. If, however, $s = -1$, i.e. the dipole nonlinearity is focusing, then maxima of the atomic and optical fields coincide, as well as their minima, see Fig. 3. This effect is similar to the modulational instability of coupled orthogonally polarised optical waves [26]. The major differences between the case under consideration and the purely optical one are, first, the focusing optical Kerr nonlinearity corresponds to the defocusing dipole nonlinearity and, second, in the optical case modulational instability is significantly affected by nonlinear self-action, while here nonlinear cross-coupling is the strongly dominating process. For $s = -1$ filaments collapse upon propagation, because dipole and dipole-dipole interactions lead to strong focusing effects which can not be outweighed by the defocusing due to collisions ($\beta_{\text{col}} > 0$) and the optical Kerr nonlinearity. Decreasing the transverse size of the atomic and optical beams leads to the collapse of the beams as whole without formation of multiple filaments as shown in Fig. 4.

The difference between filamentary patterns appearing for $s = 1$ and $s = -1$ can be interpreted on the basis of system symmetries, see [26] for more details. Modulational instability can be understood in terms of continuation of the neutral eigenvectors (i.e. those with zero eigenvalues) of $\hat{L}(|k|^2 = 0)$ into the region of nonzero $|k|$. The neutral eigenvectors can be generated by
Fig. 4. Mutual collapse of the coupled atomic (|ψ| is on the left) and optical (|Ω| is on the right) beams for s = -1. Initial conditions for numerical simulation: ψ(ζ = 0) = 4e^{-(ζ^2 + μ^2)/a}, Ω(ζ = 0) = 4e^{-(ζ^2 + ν^2)/a}. Size of the computational window is 15. Maximal values of |ψ| and |Ω| are, respectively, 10 and 18.

applying the infinitesimal phase transformations

\[
(\psi, \Omega) \rightarrow (\psi e^{i\alpha}, \Omega e^{i\alpha}), \quad (\psi, \Omega) \rightarrow (\psi e^{i\varphi}, \Omega e^{-i\varphi})
\] (14)

Infinitesimal variations of α and φ generate eigenvectors \( e_{\alpha} = (0, \psi_0, 0, \Omega_0)^T \) and \( e_{\phi} = (0, \psi_0, 0, -\Omega_0)^T \). It is clear that growth of perturbations along the direction in the phase space determined by the continuation of \( e_{\phi} \) generates a pattern with interleaving minima and maxima of the two fields and this scenario is realized for \( s = 1 \). Conversely, continuation of \( e_{\alpha} \) becomes unstable for \( s = -1 \) and a filamentary pattern with coincident peaks appears.

4 Solitons and vortices

The coupled equations (11) have a variety of different types of solitary solutions. This topic is still at the very beginning of its study and here we will merely enumerate some possible solutions and present numerical results verifying existence of two of them.

The simplest diffraction-free solitary solutions can be sought in the form

\[
\psi = \psi_0(\varphi)e^{ik_0\zeta + im_0\varphi}, \quad \Omega = \Omega_0(\varphi)e^{ik_0\zeta + im_0\varphi},
\] (15)

where \( \varphi = |\xi + in| \) is the distance from the origin and \( \theta = arg(\xi + in) \) is the polar angle. The amplitudes \( \psi_0(\varphi), \Omega_0(\varphi) \) now obey the stationary versions
of Eqs. (11) with $i\partial_t$ replaced by $-\kappa_\psi$ and $-\kappa_{\Omega}$, respectively. $m_\psi$ and $m_{\Omega}$ are integer numbers indicating the order of the phase singularities at the beam center. To fully specify solutions of the ordinary differential equations for $\psi_0(\varrho)$ and $\Omega_0(\varrho)$ one has to supplement them by boundary conditions. The boundary conditions at the origin are determined by the values of $m_\psi, m_{\Omega}$.

If $m$ is different from zero then the corresponding amplitude has to be zero at the origin to provide continuity of the solution. If $m$ is zero then the corresponding amplitude has a zero derivative at the origin. As $\varrho \to \infty$ the amplitudes $\psi_0$ and $\Omega_0$ should tend to either zero or to one of the plane wave solutions.

To gain intuitive insight into the problem of existence of coupled light-matter solitons it is useful to use the analogy between solitary waves and linear waveguides. For $\Omega = 0$ and $\beta_{\text{col}} < 0$ two-body collisions lead to the possibility of existence of purely atomic bright solitons and ring-like solutions with nested phase dislocation (vortex) [27]. Atomic vortices located on an infinite background [28] exist for $\Omega = 0$ and $\beta_{\text{col}} > 0$. The density distribution created by these solutions can be considered as an effective potential for the optical field. In the case of the vortex on an infinite background the potential is attractive for $s = 1$ and in the case of solitons on a zero background it is attractive for $s = -1$. Thus in these two cases the atomic solitons and vortices can guide bright optical beams. An example of such a configuration is shown.
in Fig. 5. Preliminary numerical simulations indicate that the structure in Fig. 5 is dynamically stable.

It is clear that bright-bright solitons existing for $\beta_{\text{col}} < 0$ and $s = -1$ are unstable because the focusing collisional nonlinearity is enhanced by the dipole and dipole-dipole interactions and the small defocusing Kerr term can not counterbalance it. However bright-bright solitons can also be found for $\beta_{\text{col}} > 0$ and $s = -1$, see Fig. 6. In this case the instability growth rate is significantly weakened because the collisional nonlinearity is defocusing.

5 Discussion

The results of the preceding sections demonstrate that a rich variety of spatial structures may appear in the coupled propagation of optical and atomic waves. Although the filamentation instability is deleterious to the transmission of well collimated beams with smooth wavefronts, spatial instabilities are potentially useful for splitting and combining optical and atomic waves. As a first step towards applications it is necessary to estimate physical parameters for which the structures shown in Figs. 2-6 could be observed. A detailed discussion will be given elsewhere. Our aim here is to indicate characteristic physical values for these phenomena.

The physical propagation lengths corresponding to the calculated filamentation patterns are estimated as follows. Assuming Rb atoms the nonlinear
coefficients used in Fig. 2 correspond to $u_L = 8 \ \mu$m and $\Delta/\gamma = 14$. The physical box size in Fig. 2 is then 140 $\mu$m on a side, and the physical propagation length is $z \simeq 0.26$ mm. The intensity at the center of the gaussian is $I_{\text{peak}}/I_{\text{sat}} \simeq 1.4 \times 10^{-2}$, and the peak density is $N_{\text{peak}} \simeq 1.4 \times 10^{20} \text{ m}^{-3}$. The transverse scale of the filamentation is of order 8 $\mu$m. These values indicate that observation of filamentation patterns should be possible with current atom laser sources.

Finally let us estimate the spatial size of the coupled atomic and optical solitons shown in Fig. 6. Using again parameters for Rb and $u_L = 19 \ \mu$m, $\Delta/\gamma = 14$, we get a physical box size of 85 $\mu$m on a side, $I_{\text{peak}}/I_{\text{sat}} \simeq 2.6 \times 10^{-3}$, and $N_{\text{peak}} \simeq 2.6 \times 10^{20} \text{ m}^{-3}$. Under these conditions the solitons have a transverse diameter of order 15 $\mu$m. The corresponding linear diffraction length is order 2 mm so unambiguous observation of solitary propagation would require an interaction length of at least half a cm. This would require a continuous atom laser source lasting about one second for a beam propagation velocity equal to the recoil velocity. The atom-laser source described in [9] provided pulses of up to 100 msec duration. We can conclude from the above estimates that observation of solitary propagation is not terribly far from the limits set by current atom laser experiments.

In conclusion the equations derived and studied here provide a basis for investigating the coupled spatial dynamics of coherent optical and atomic waves. Depending on the physical parameters in force, and in particular the amount of detuning between the optical field and the atomic transition, different nonlinear effects dominate. For small detunings and low atomic densities the dipole atom-field coupling dominates while for large detunings and high densities the two-body collisional nonlinearity is dominant. Local field effects give a dipole-dipole nonlinearity that appears as a correction to the above terms. Modulational instability is observed for wide beams. The nonlinear stage of the modulational instability itself, or localized initial conditions, lead to a rich variety of possible solitary structures. There is no doubt that much remains to be discovered while exploring the possibilities inherent in the coupled dynamics of optical and atomic fields.

References

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