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FROM A WEIGHTLESS BENT WIRE COAT HANGER TO SHELL STRUCTURES VIA THE BELTRAMI STRESS TENSOR

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ABSTRACT

A weightless wire coat hanger bent out of its plane is one of the simplest possible structures. All it does is transfer a force and moment around a closed space curve. Assembling a family of coat hangers enables us to build up trusses and frames and if we allow an infinite number of coat hangers which overlap we can assemble plates, shells and fully three dimensional structures. We can apply loads to these structures via a loading frame and wires, also made from coat hangers.

A wire carrying an electric current produces a magnetic field in the space surrounding the wire. For the bent coat hanger we can imagine that there is a vector field surrounding the wire as a result of the force and moment in the wire. We can use this vector field to obtain expressions for the forces and moments in a shell in equilibrium with applied loads.

Keywords: *Shell structures, Beltrami stress function, Airy stress function, coat hanger, Cauchy stress, Cosserat moment*

1. INTRODUCTION

It is well known [17] that the equations of static equilibrium of an unloaded 2 dimensional continuum are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$
$$\tau_{yx} = \tau_{xy}$$

and that these equations are automatically satisfied by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$$
$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$$
$$\tau_{xy} = \tau_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

in which ϕ is the Airy stress function [2, 3, 17].

Thus stress corresponds to 'curvature' of the ϕ surface if ϕ is plotted in the third dimension, perpendicular to the $x - y$ plane. A concentrated axial force in a structural member represented by a straight or curved line, such as a truss element or cable, corresponds to a fold or concentrated curvature in the ϕ surface.

What is perhaps less well known is that the bending moment in a member corresponds to a discontinuity in the ϕ surface, which is effectively two infinite (in terms of the tangent of the slope) folds separated by a vertical surface [19]. This is consistent with the fact that a bending moment in a straight or curved line is produced by infinite tensile and compressive forces separated by an infinitesimal lever arm.

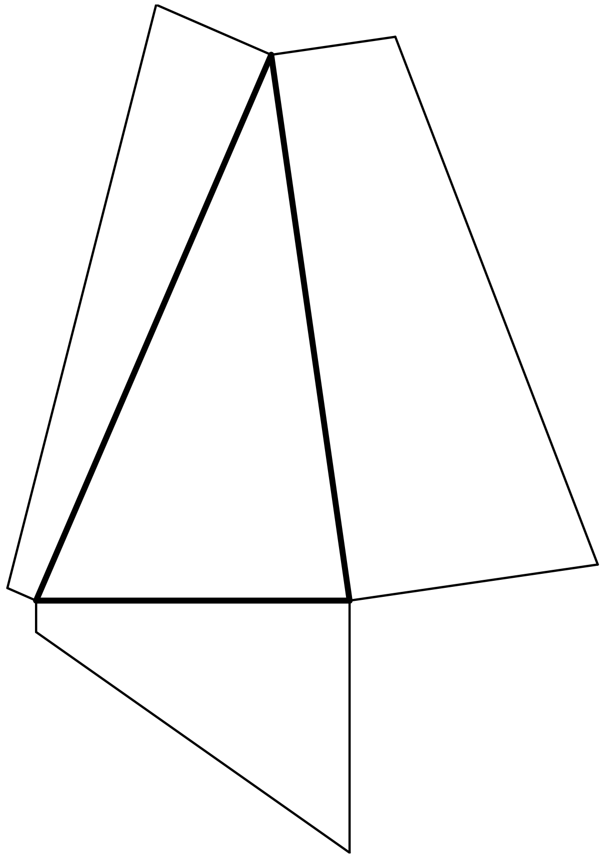

Figure 1: *Bending moment in a flat triangular wire*

This is illustrated in figures 1 and 2. Figure 1 shows a triangular wire lying in a plane together with the bending moment in the wire due to some forming process. In 2 dimensions the structure is 3 times statically indeterminate. The bending moment and forces in the wire are represented by the Airy stress function ϕ which is the third vertical coordinate in figure 2 showing a triangular 'island' surrounded by an infinite flat horizontal 'sea'. The magnitude of the force in the wire is equal to the tangent of the slope of the plane representing the 'land' and its direction is parallel to a contour line on the plane. The variable height of the 'cliffs' is equal to the bending moment. The line of action of the force is where an extrapolation of the 'land' plane enters the 'sea'.

Here we have shown a triangular wire, but it can be of any shape with a straight or curved sides, provided that it lies in a plane.

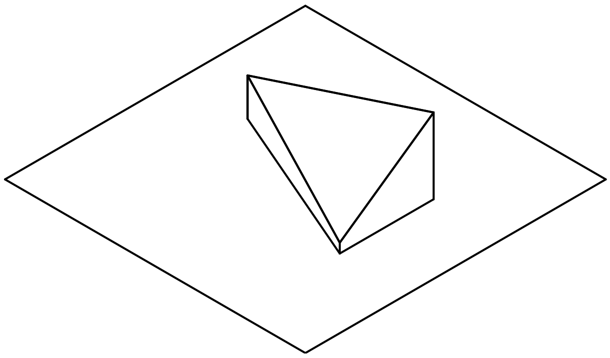

Figure 2: *Airy stress function is a triangular 'island' with sloping plane 'land' and vertical 'cliffs' surrounded by an infinite horizontal 'sea'. Here we have shown a triangular wire, but it can be of any shape with a straight or curved sides, provided that it lies in a plane*

By assembling Airy stress function islands side by side we could model the truss and loading frame shown in figure 3 in which the arrows represent the loads on the truss from the loading frame. If there is no bending moment in a member there is no discontinuity in ϕ, the cliffs are of the same height on adjoining islands.

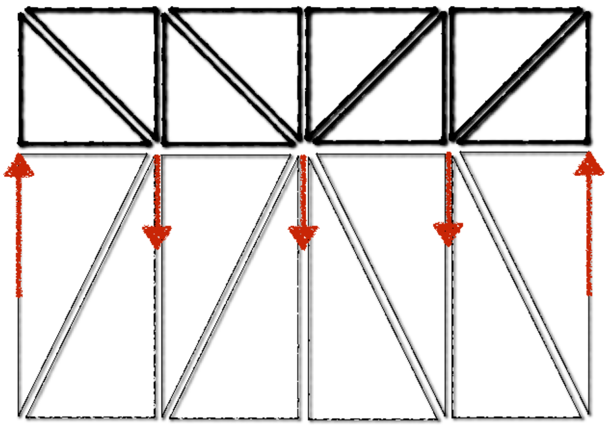

Figure 3: *Truss and loading frame made from coat hangers. Truss shown with thick lines, loading frame shown with thin lines and loads applied to truss are indicated by the arrows. Forces and moments are added for members running alongside each other. The moments cancel for a pin-jointed truss*

2. EXTENSION TO THREE DIMENSIONS

We shall refer to closed wire frames as 'coat hangers'. So far we have only considered a coat hanger or system of coat hangers all lying in the same 2 dimensional plane. We will now consider a coat hanger bent out of its plane to form a 3 dimensional space curve as shown in figure 4, which also shows two green and yellow plastic hoops, one going around the wire and the other not.

If the coat hanger were to be cut with a pair of pliers, the two sides of the cut would spring apart and we would need to apply a force and a moment to bring the two sides back together and to line up. The force may well be much larger than the own weight of the hanger, and so for purity we shall imagine that the hanger is floating weightless.

Because the coat hanger is assumed to be weightless, the force and moment about a given fixed point are both constant, regardless of where the cut is made. Of course the moment about the cut itself does vary, according to the force and lever arm from the fixed point.

There is no stress in the space around the coat hanger, but within the wire itself there is a state of stress resulting in the force and moment. The plastic hoop going around the wire encircles this force and moment, whereas the other plastic hoop encircles no force or moment. We are not interested in the actual stress in the wire or how thick the wire is, it could be very fat provided that the plastic hoop goes fully around it. This suggests that equilibrium is best discussed by considering a line integral around a curve, in this case the curve being one or other of the plastic hoops, not the coat hanger.

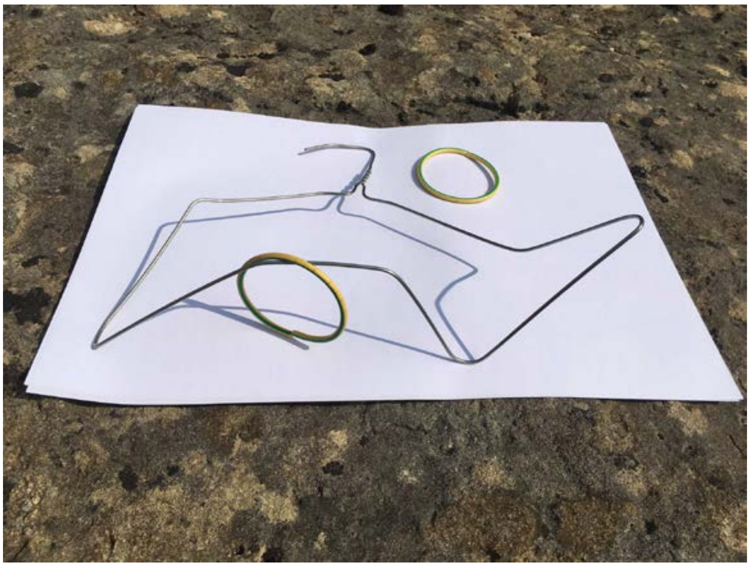

Figure 4: *Wire coat hanger bent into 3 dimensional space curve and two green and yellow plastic hoops, one going around the wire and the other not*

Figure 5: *Hard-broom (walis-tingting) stall in the Philippines. Photo: Thamizhpparithi Maari, Department of Journalism and Mass Communication, Periyar University, India*

Figure 5 shows some brooms. If we were to apply a bending moment to the collection of strands making up the broom head, it would be carried partly by moments in the individual strands and partly by a tension in some strands and compression in others, as in the classical Euler-Bernoulli theory of bending of beams. The moment in the individual strands would be termed Cosserat moment in continuum mechanics, after the brothers François and Eugène Cosserat. If one were to encircle a broom with a green and yellow plastic hoop one would only be interested in the total force and moment being transferred through the hoop.

It should be emphasized that nowhere in this paper do we make any assumption regarding material properties. The material may be linear or non-linear elastic, there may be plastic and other forms of deformation. We are only concerned with static equilibrium in some deformed geometry, which may be very different from some 'initial' geometry, perhaps the straight wire from which a coat hanger was originally made.

Before discussing the coat hanger in more detail, we need some preliminaries in order to fully understand the concepts of equilibrium of force and moment.

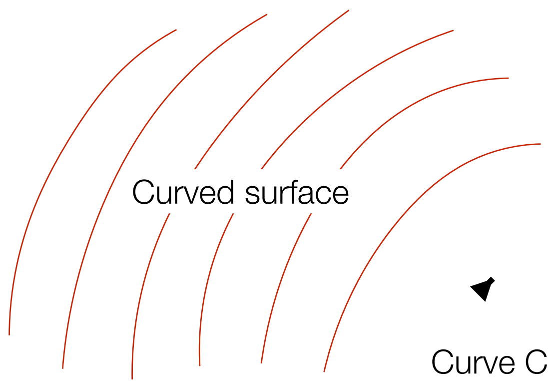

Figure 6: *Portion of curved surface bounded by a closed space curve C, all contained within an un- loaded material, or combination of materials, subject to stress, but which may contain stress-free regions*

3. EQUILIBRIUM IN 3 DIMENSIONS

Let us imagine a weightless 3 dimensional material, or combination of materials, in which forces and moments are distributed, and which may contain stress free regions like the air around our coat hanger, ignoring air pressure.

Figure 6 shows a portion of curved surface inside our material bounded by a closed space curve C. One can imagine many different portions of surface all bounded by the same curve. The curve C is analogous to the plastic hoop going around the wire in figure 4, rather than the coat hanger itself since we are interested in the force and moment passing through the curve.

The total surface area, treated as a vector, of any of these portions of surface bounded by C must be the same. Similarly, for equilibrium, the total force and moment about a fixed point crossing all surfaces bounded by the curve C must be the same. If we imagine a ruled surface defined by straight lines passing through the curve and a fixed point **R**, then using the properties of the vector product we have the total area, treated as a vector,

$$\begin{aligned}\mathbf{A}_{\mathrm{C}} &= \frac{1}{2}\oint_{\mathrm{C}} (\mathbf{r}-\mathbf{R}) \times d\mathbf{r} \\ &= \frac{1}{2}\oint_{\mathrm{C}} \mathbf{r} \times d\mathbf{r} - \frac{1}{2}\mathbf{R} \times \oint_{\mathrm{C}} d\mathbf{r}\end{aligned} \tag{1}$$

in which $\mathbf{r}$ is the position vector of a typical point on the curve C and $d\mathbf{r}$ is the vector joining two adjacent points on C.

However

$$\oint_{\mathrm{C}} d\mathbf{r} = 0$$

so that $\mathbf{R}$ can be any constant vector without influencing the result for $\mathbf{A}_{\mathrm{C}}$. Note that reversing the direction of travel around the curve changes the sign of the area and the total area of a closed surface, made up of two different surfaces passing through curve C, is zero.

We can use the permutation pseudo-tensor $\boldsymbol{\epsilon}$, whose components Green and Zerna [7] describe as ϵ-systems, to perform the vector product

$$\begin{aligned}\mathbf{c} &= \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot \mathbf{a} \\ &= (\mathbf{ab}) \cdot\cdot\, \boldsymbol{\epsilon} = \boldsymbol{\epsilon} \cdot\cdot\, (\mathbf{ab}) \\ &= a^i b^j \mathbf{g}_i \times \mathbf{g}_j \\ &= a^i b^j \epsilon_{ijk} \mathbf{g}^k \\ &= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} \\ &\quad + (a_x b_y - a_y b_x)\mathbf{k}.\end{aligned}$$

where $\mathbf{g}_i$ and $\mathbf{g}^k$ are the covariant and contravariant base vectors and we have used the Einstein summation convention [7]. **i**, **j** and **k** are unit vectors in the x, y and z directions. Hence we can write equation (1) as

$$\mathbf{A}_\mathrm{C} = \frac{1}{2}\oint_\mathrm{C} \big(d\mathbf{r}(\mathbf{r}-\mathbf{R})\big) \cdot\cdot\, \epsilon. \qquad (2)$$

The advantage of using the permutation pseudo-tensor is that it enables us to perform operations such as the vector triple product,

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \boldsymbol{\epsilon}\cdot\cdot\big(\mathbf{a}\boldsymbol{\varepsilon}\cdot\cdot(\mathbf{bc})\big)\\ &= \varepsilon^{ijk}a_j\varepsilon_{kmn}b^m c^n\mathbf{g}_i\\ &= \big(\delta^i_m\delta^j_n - \delta^j_m\delta^i_n\big)a_j b^m c^n\mathbf{g}_i\\ &= a_j\big(b^i c^j - b^j c^i\big)\mathbf{g}_i\\ &= (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\end{aligned}$$

in which δ^i_m is the Kronecker delta. We can also write the curl of the vector field **u** as $\boldsymbol{\varepsilon}\cdot\cdot\nabla\mathbf{u}$ in which $\nabla\mathbf{u}$ is the gradient of **u** and is the second order tensor defined by $d\mathbf{u} = d\mathbf{r}\cdot\nabla\mathbf{u}$.

Hence we can write equation (1) as

$$\mathbf{A}_\mathrm{C} = -\frac{1}{2}\oint_\mathrm{C} \Big(\big(d\mathbf{r}(\mathbf{r}-\mathbf{R})\big)\cdot\cdot\,\boldsymbol{\epsilon}\Big). \qquad (2)$$

Since the force due to stress must be the same for all areas bounded by C, we can introduce the second order tensor **S** and write

$$\mathbf{f}_\mathrm{C} = \oint_\mathrm{C} (d\mathbf{r}\cdot\mathbf{S}) \qquad (3)$$

as the force 'passing through' the curve C. This is perhaps not immediately obvious until we realize that reversing the direction of travel around the curve changes the sign and so the total force on a closed surface is automatically zero, remember that our material is weightless.

We can rewrite (3) using the gradient of **S**,

$$\begin{aligned}\mathbf{f}_\mathrm{C} &= \oint_\mathrm{C}\big(d\big((\mathbf{r}-\mathbf{R})\cdot\mathbf{S}\big) - (\mathbf{r}-\mathbf{R})\cdot d\mathbf{S}\big)\\ &= -\oint_\mathrm{C}\big((\mathbf{r}-\mathbf{R})\cdot d\mathbf{S}\big)\\ &= -\oint_\mathrm{C}\Big(\big(d\mathbf{r}(\mathbf{r}-\mathbf{R})\big)\cdot\cdot\,\nabla\mathbf{S}\Big).\end{aligned} \qquad (4)$$

If $\boldsymbol{\sigma}$ is the stress tensor in the material, we have from (4) and (2)

$$\begin{aligned}-\frac{1}{2}\big(d\mathbf{r}(\mathbf{r}-\mathbf{R})\big)\cdot\cdot\,\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma} &= -\big(d\mathbf{r}(\mathbf{r}-\mathbf{R})\big)\cdot\cdot\,\nabla\mathbf{S}\\ \frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma} &= \nabla\mathbf{S}\end{aligned}$$

so that

$$\boldsymbol{\sigma} = \boldsymbol{\epsilon}\cdot\cdot\,\nabla\mathbf{S}. \qquad (5)$$

Therefore

$$\nabla\cdot\mathbf{S} = 0 \qquad (6)$$

confirming that equilibrium of force is satisfied. Note that we are in Euclidean space so that the fourth order tensor $\nabla\nabla\mathbf{S}$ is symmetric in its first two parts.

Using a similar argument to that for area and force, the moment about the point **R** crossing all surfaces through the curve C must be the same and equal to

$$\begin{aligned}\mathbf{m}_{\mathbf{CR}} &= \oint_\mathrm{C}\Big(d\mathbf{r}\cdot\big(\mathbf{B}+\mathbf{S}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\big)\Big)\\ &= \oint_\mathrm{C}\Big(d\mathbf{r}\cdot\big(\mathbf{B}+\mathbf{S}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{X})\big)\Big)\\ &\quad+\mathbf{f}_\mathrm{C}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{X}-\mathbf{R})\end{aligned} \qquad (7)$$

in which we have introduced the second order tensor **B**, which we will later see contains the Beltrami stress functions as its components. The term leading to $\mathbf{f}_\mathrm{C}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{X}-\mathbf{R})$ is necessary because it is the moment due to the force $\mathbf{f}_\mathrm{C}$ acting through the point **X** about **R**. If we were to change **R** by $d\mathbf{R}$ then $\mathbf{m}_{\mathbf{CR}}$ would change by $\mathbf{f}_\mathrm{C}\cdot\boldsymbol{\epsilon}\cdot(-d\mathbf{R}) = d\mathbf{R}\times\mathbf{f}_\mathrm{C}$ as we would expect.

Thus

$$\begin{aligned}\mathbf{m}_{\mathbf{CR}} &= \oint_\mathrm{C}\Big(d\big((\mathbf{r}-\mathbf{R})\cdot\mathbf{B}\big)\Big)\\ &\quad-\oint_\mathrm{C}\big((\mathbf{r}-\mathbf{R})\cdot d\mathbf{B}\big)\\ &\quad-\oint_\mathrm{C}\Big(\big(d\mathbf{r}(\mathbf{r}-\mathbf{R})\cdot\cdot\,(\mathbf{S}\cdot\boldsymbol{\epsilon})\big)\Big)\\ &= -\oint_\mathrm{C}\Big(\big(d\mathbf{r}(\mathbf{r}-\mathbf{R})\big)\cdot\cdot\,(\nabla\mathbf{B}+\mathbf{S}\cdot\boldsymbol{\epsilon})\Big)\end{aligned} \qquad (8)$$

and using the same argument that lead to the stress tensor, we have the Cosserat moment tensor,

$$\boldsymbol{\mu} = \boldsymbol{\epsilon} \cdot\cdot (\nabla \mathbf{B} + \mathbf{S} \cdot \boldsymbol{\epsilon}), \tag{9}$$

which gives the moment per unit area crossing a surface.

Hence, after a little manipulation, we have

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\epsilon} \cdot\cdot \boldsymbol{\sigma} = 0, \tag{10}$$

confirming that equilibrium of moments is automatically satisfied. Note that if $\nabla \cdot \boldsymbol{\mu} = 0$ then the stress tensor $\boldsymbol{\sigma}$ is symmetric, as is assumed in non-Cosserat continuum mechanics.

Equations (6) and (10) can be found as equations (47a) and (47b) in Das Cosserat Kontinuum by Hermann Schaefer [14] translated by David Delphenich [1].

If the curve C is exceedingly small and close to the point **R** and **S** is finite, then equation (7) reduces to

$$\mathbf{m}_{\mathrm{C}} = \oint_{\mathrm{C}} (d\mathbf{r} \cdot \mathbf{B}). \tag{11}$$

4. THE SIGNIFICANCE OF TENSORS S AND B

The stress $\boldsymbol{\sigma}$ is defined in terms of **S** in equation (5) and the Cosserat moment tensor $\boldsymbol{\mu}$ is defined in terms of **S** and **B** by equation (9). Since equation (5) contains the gradient of **S**, then **S** can be finite while $\boldsymbol{\sigma}$ is infinite, as happens in the theory of shells when we imagine that a shell is a surface of zero thickness and we have a step change in **S** across the surface. The same applies to $\boldsymbol{\mu}$ and **B** and the bending moment in a shell corresponds to a step change in **B**. Schaefer [13, 14, 1] imagines a continuum surrounded by a ‘crust’ in which a step change in **S** and **B** resist the stresses contained within.

Clearly $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are zero in empty space, but this does not necessarily apply to **S** and **B**. The plastic hoop surrounding the wire of the coat hanger is in empty space (forgetting about the air), yet when we integrate around the curve of the hoop we obtain the force and moment in the wire.

5. THE BELTRAMI STRESS TENSOR

If we were make the usual assumptions leading to the Cauchy stress tensor [4, 5, 6, 8, 9, 10, 11, 12, 16, 18], then $\boldsymbol{\mu} = 0$, which does not imply that $\nabla \mathbf{B} + \mathbf{S} \cdot \boldsymbol{\epsilon}$ is zero in (9), only that it is symmetric in its first two parts. Then

$$\mathbf{S} = (\boldsymbol{\epsilon} \cdot\cdot \nabla \mathbf{B})^{\mathrm{T}} - \frac{1}{2} \boldsymbol{\epsilon} \cdot\cdot\cdot \nabla \mathbf{B} \mathbf{I} \tag{12}$$

in which **I** is the unit tensor, and in Cartesian coordinates $\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$. The superscript ‘T’ means ‘transpose’. Then from equation (5)

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{1}{2} \boldsymbol{\epsilon} \cdot\cdot \nabla \left(\boldsymbol{\epsilon} \cdot\cdot \nabla \left(\mathbf{B} + \mathbf{B}^{\mathrm{T}} \right) \right) \\ &= \frac{1}{2} \epsilon^{ipm} \epsilon^{jqn} \nabla_p \nabla_q (B_{mn} + B_{nm}) \mathbf{g}_i \mathbf{g}_j \end{aligned} \tag{13}$$

confirming that **B** contains the Beltrami stress functions as its components. $\boldsymbol{\sigma}$ is symmetric, and the antisymmetric part of **B** makes no contribution to $\boldsymbol{\sigma}$. ∇_p denotes the covariant derivative with respect to the curvilinear coordinate θ^p [7].

From this point on we shall assume the **B** is symmetric so that equations (12) and (13) become

$$\mathbf{S} = (\boldsymbol{\epsilon} \cdot\cdot \nabla \mathbf{B})^{\mathrm{T}} \tag{14}$$

and

$$\boldsymbol{\sigma} = \boldsymbol{\epsilon} \cdot\cdot \nabla (\boldsymbol{\epsilon} \cdot\cdot \nabla \mathbf{B}). \tag{15}$$

Of course the Airy, Maxwell, Morera and Prandtl stress functions are all special cases of the components of the Beltrami stress tensor. Airy [2, 3] used an integral in 2 dimensions to find the total force on an element and then the divergence theorem to find the differential equations of equilibrium that are satisfied by the stress function, which now bears his name. Airy did not use the term ‘divergence theorem’ but instead described it as an application of the calculus of variations. Both of Airy’s papers were read in 1862 but published in 1863, the Royal Society paper is an expanded version of the British Association for the Advancement of Science paper.

We will continue to make the assumption $\boldsymbol{\mu} = 0$ *in the remainder of this paper, but it is important to realize that this does not mean that the moment in the coat hanger wire is zero, or that the moments in the shells that we will consider later are zero.*

6. A VECTOR FIELD

Let us write

$$\mathbf{B} = \boldsymbol{\gamma} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right) \tag{16}$$

and

$$\boldsymbol{\omega} = \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{T}}\right) \tag{17}$$

where $\mathbf{u}$ is a vector field. If we were to imagine that $\mathbf{u}$ is a fluid velocity then $\boldsymbol{\gamma}$ would be the symmetric strain rate tensor and $\boldsymbol{\omega}$ would be the antisymmetric vorticity tensor. The corresponding vorticity vector is

$$\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot\, \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot\, \nabla\mathbf{u} \tag{18}$$

and

$$\boldsymbol{\omega} = \boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega} = \boldsymbol{\Omega}\cdot\boldsymbol{\varepsilon}. \tag{19}$$

It is sometimes more convenient to use the antisymmetric tensor form for vorticity, and sometimes the vector form. The same applies to moment, but all the moments in this paper are in the vector form.

For $\boldsymbol{\mu}$ to be zero we have from equation (14)

$$\begin{aligned}\mathbf{S} &= \frac{1}{2}\left(\boldsymbol{\epsilon} \cdot\cdot\, \nabla\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right)\right)^{\mathrm{T}} \\ &= \frac{1}{2}\left(\boldsymbol{\epsilon} \cdot\cdot\, \nabla\left((\nabla\mathbf{u})^{\mathrm{T}}\right)\right)^{\mathrm{T}} \\ &= \frac{1}{2}\epsilon^{ijk}\nabla_j\nabla_m u_k \mathbf{g}_i = \nabla\boldsymbol{\Omega}.\end{aligned} \tag{20}$$

From equation (5)

$$\boldsymbol{\sigma} = \boldsymbol{\epsilon} \cdot\cdot\, \nabla\nabla\boldsymbol{\Omega} = 0 \tag{21}$$

and therefore we can use equation (16) to define the Beltrami stress function in the unstressed region outside the wire of the coat hanger.

From equation (3) the force passing through a curve C is now

$$\mathbf{f}_{\mathrm{C}} = \oint_{\mathrm{C}} (d\mathbf{r}\cdot\nabla\boldsymbol{\Omega}) = \oint_{\mathrm{C}} d\boldsymbol{\Omega} \tag{22}$$

Using equation (7) and the fact that $\nabla\mathbf{r} = \mathbf{I}$, the unit tensor,

$$\begin{aligned}\mathbf{m}_{\mathrm{CR}} &= \oint_{\mathrm{C}} \left(d\mathbf{r}\cdot\left(\boldsymbol{\gamma} + \nabla\boldsymbol{\Omega}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\right)\right) \\ &= \oint_{\mathrm{C}} \left(d\mathbf{r}\cdot\left(\boldsymbol{\gamma} + \nabla\left(\boldsymbol{\Omega}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\right)\right) + \boldsymbol{\omega}\right) \\ &= \oint_{\mathrm{C}} \left(d\left(\mathbf{u} + (\mathbf{r}-\mathbf{R})\times\boldsymbol{\Omega}\right)\right)\end{aligned} \tag{23}$$

Now if we identify the curve C with one or other of the plastic hoops in figure 4, we can say that $\mathbf{f}_{\mathrm{C}}$ and $\mathbf{m}_{\mathrm{CR}}$ are both zero for the curve that does not contain the coat hanger wire.

However, to obtain non-zero force for the plastic hoop containing the coat hanger wire we need a constant discontinuity in $\boldsymbol{\Omega}$ equal to the force in the wire,

$$\mathbf{f}_{\mathrm{wire}} = \boldsymbol{\Omega}_{\mathrm{discontinuity}} \tag{24}$$

across a surface passing through the wire.

For the moment about $\mathbf{R}$ we need a constant discontinuity in $(\mathbf{u} + (\mathbf{r}-\mathbf{R})\times\boldsymbol{\Omega})$ equal to $\mathbf{m}_{\mathbf{R}}$ so that

$$\mathbf{u}_{\mathrm{discontinuity}} = \mathbf{m}_{\mathbf{R}} + \boldsymbol{\Omega}_{\mathrm{discontinuity}} \times (\mathbf{r}-\mathbf{R}) \tag{25}$$

where $\mathbf{r}$ is now a point on the surface. Thus $\mathbf{u}_{\mathrm{discontinuity}}$ corresponds to a rigid body motion since $\mathbf{m}_{\mathbf{R}}$ and $\boldsymbol{\Omega}_{\mathrm{discontinuity}}$ are both constant.

A plastic hoop encircling the coat hanger wire must pass through this surface an odd number of times, probably just once. However, a plastic hoop that does not pass around the coat hanger wire will pass through the surface an even number of times, or not at all.

Equation (23) gives the moment about the fixed point $\mathbf{R}$. However, if we imagine that the wire is described by a space curve representing its axis, then the moment in the wire about a point $\mathbf{P}$ on the axis is

$$\begin{aligned}\mathbf{m}_{\text{wire axis}} &= \mathbf{m}_{\mathbf{R}} + (\mathbf{R}-\mathbf{P})\times\mathbf{f}_{\mathrm{wire}} \\ &= \mathbf{u}_{\text{discontinuity at wire axis}}\,.\end{aligned} \tag{26}$$

In 3 dimensions the coat hanger is 6 times statically indeterminate, 3 components of force and 3 components of moment. These 6 quantities correspond to our choice of the constant vectors $\mathbf{m_R}$ and $\mathbf{\Omega}_{\text{discontinuity}}$ in equation (25).

7. PRESTRESSED PIN-JOINTED TRUSSES

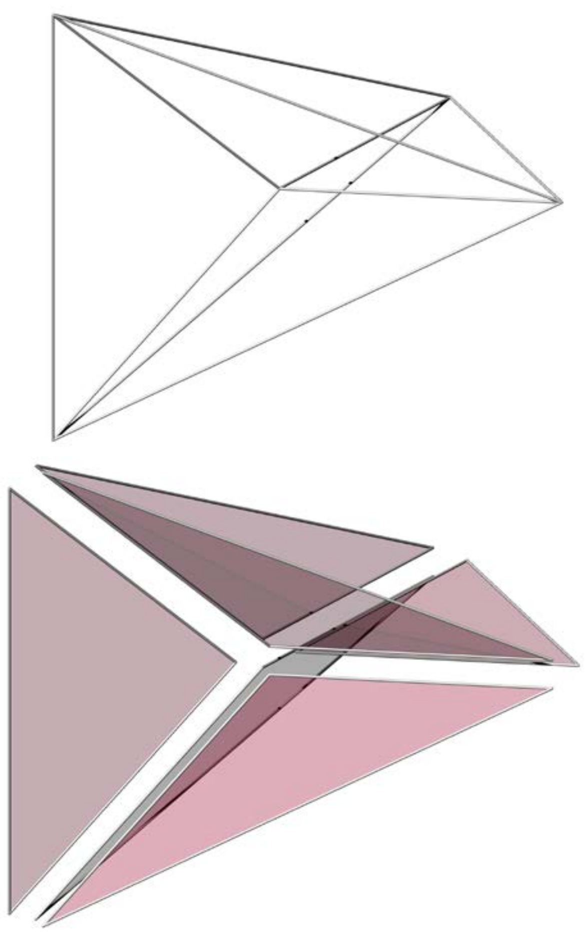

Figure 7: *Upper image shows pin-jointed truss with 5 nodes and 10 members. Lower image shows the truss 'exploded' into 6 'coat hangers'*

The simplest possible 3 dimensional prestressed pin jointed truss is shown in the upper image in figure 7. The truss is free to move in space and it has 5 nodes and 10 members. Given the 3 possible rigid body displacements and 3 possible rigid body rotations, there are 5 x 3 - 6 = 9 independent equilibrium equations. This confirms that choosing the tension or compression in any one member determines the tension or compression in all the others.

The lower image in figure 7 shows the truss 'exploded' into 6 'coat hangers', each with its corresponding surface of discontinuity. 6 truss members are made up of a single wire and the remaining 4 members have 3 wires combined. Along each edge formed by a single wire there is a discontinuity in angular velocity about an axis parallel to the wire and no discontinuity in velocity. If one of these angular velocities is specified, the remaining 5 can be determined by the requirement that the sum of the moments in the 3 wires making up the remaining 4 members is zero.

8. APPLICATION TO SHELLS

Let us now imagine that the discontinuities in $\mathbf{u}$ and its derivatives are no longer constant so that we have forces and moments in the surface of discontinuity itself as well as in the boundary coat hanger.

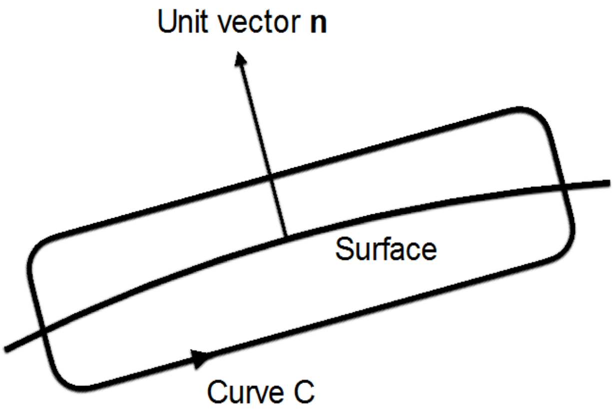

Figure 8: *Section through surface showing 3 dimensional curve C doubling back on itself*

It should be noted that we are only interested in the discontinuities in fluid velocity, vorticity and strain rate across the surface and not in the velocity field elsewhere, provided that the velocity and its gradient are continuous. The discontinuities in vorticity and strain rate are both determined by the discontinuity in the gradient $\nabla\mathbf{u}$. However, the discontinuity in $\mathbf{t}\cdot\nabla\mathbf{u}$, where $\mathbf{t}$ is any vector tangential to the surface, is determined by the discontinuity in $\mathbf{u}$. Thus we are only free to choose the discontinuities across the surface in the two vectors $\mathbf{u}$ and $\mathbf{n}\cdot\nabla\mathbf{u}$ where $\mathbf{n}$ is the unit normal to the surface.

So far there are no external loads, only internal forces and moments. However, we can add loads by adding yet more coat hangers to produce loading wires, attached to some sort of loading frame, again made of coat hangers. If we replace equation (16) by

$$\mathbf{B} = \boldsymbol{\gamma} + \mathbf{B}_{\text{load}} \tag{27}$$

in which we still have

$$\boldsymbol{\gamma} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right) \tag{28}$$

then we can use the discontinuity in $\mathbf{B}_{\text{load}}$ across the surface to represent the load. $\mathbf{B}_{\text{load}}$ is not uniquely determined by the load since we can add any strain rate type tensor to $\mathbf{B}_{\text{load}}$ without changing the load.

If the imaginary fluid velocity and $\mathbf{B}_{\text{load}}$ are zero above the surface shown in section in figure 8 and non-zero below, then it follows from equations (3), (11) and (14) that the force and moment crossing a cut $d\mathbf{r}$ in the surface are

$$d\mathbf{f} = d\mathbf{r} \cdot \left(\nabla\boldsymbol{\Omega} + (\boldsymbol{\epsilon} \cdot\cdot \nabla\mathbf{B}_{\text{load}})^{\mathrm{T}}\right) \tag{29}$$

and

$$d\mathbf{m} = d\mathbf{r} \cdot (\boldsymbol{\gamma} + \mathbf{B}_{\text{load}}). \tag{30}$$

Let us now introduce the second order tensors **F** and **M** such that

$$\mathbf{n} \cdot \mathbf{F} = 0, \tag{31}$$

$$\mathbf{n} \cdot \mathbf{M} = 0 \tag{32}$$

$$\mathrm{d}\mathbf{f} = (d\mathbf{r} \times \mathbf{n}) \cdot \mathbf{F} = d\mathbf{r} \cdot (\mathbf{n} \times \mathbf{F}) \tag{33}$$

and

$$\mathrm{d}\mathbf{m} = (d\mathbf{r} \times \mathbf{n}) \cdot \mathbf{M} = d\mathbf{r} \cdot (\mathbf{n} \times \mathbf{M}) \tag{34}$$

in which $\mathbf{n}$ is again the unit normal to the surface. Then it follows that

$$\mathbf{F} = -\mathbf{n} \times \left(\nabla\boldsymbol{\Omega} + (\boldsymbol{\epsilon} \cdot\cdot \nabla\mathbf{B}_{\text{load}})^{\mathrm{T}}\right) \tag{35}$$

and

$$\mathbf{M} = -\mathbf{n} \times (\boldsymbol{\gamma} + \mathbf{B}_{\text{load}}). \tag{36}$$

The tensors **F** and **M** represent the forces and moments in the shell. It should be noted that the forces will in general include normal shear forces as well as membrane stresses. The moments will include bending and twisting moments as well as in plane Cosserat moments which are important for structures such as the Mannheim Multihalle made from continuous timber laths (figure 9).

In some theories it is assumed that the surface tensor representing bending and twisting moments in a shell is symmetric. The corresponding assumption with our notation would be that the trace of the in surface part of **M** is zero.

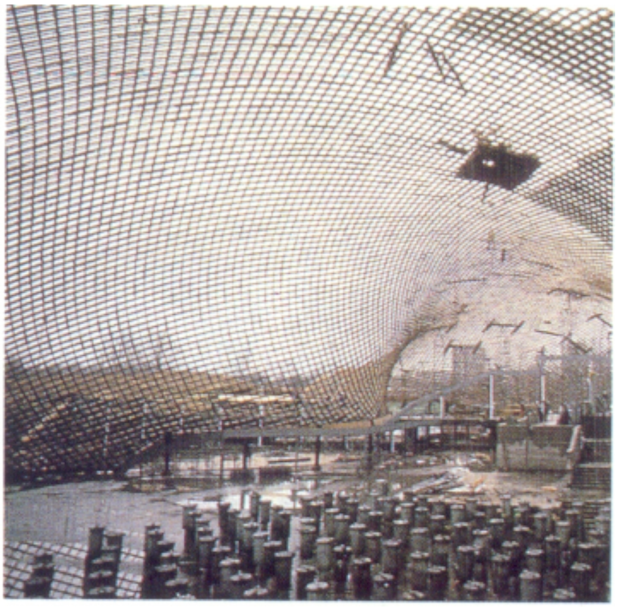

Figure 9: *Mannheim Multihalle made from continuous timber laths crossing at the nodes. Shell is being test loaded by wires supporting water filled dustbins*

We can choose the discontinuities in the imaginary velocity field $\mathbf{u}$ and in the vector $\mathbf{n} \cdot \nabla\mathbf{u}$ arbitrarily and still ensure equilibrium of a shell structure. For a real shell the discontinuities will be determined by the elastic and other properties of the structure.

We have 6 components for each of the second order tensors **F** and **M**, subject to equations (31) and (32), and 3 equations of equilibrium of forces and 3 equations of equilibrium of moments. We therefore have $2 \times 6 + 2 \times 3 = 6$ ‘degrees of freedom’ which are the components of the discontinuities in the two vectors $\mathbf{u}$ and $\mathbf{n} \cdot \nabla\mathbf{u}$.

We can eliminate the discontinuities in $\mathbf{u}$ and $\mathbf{n}\cdot\nabla\mathbf{u}$ to obtain the equilibrium equations for a shell, although the derivation is long and complicated and will not be included here. It is better to use methods which derive the equilibrium equations by just considering the surface itself or alternatively treating the shell as a thin 3 dimensional continuum as, for example in Green and Zerna [7] and Steigmann [15].

9. FLAT PLATES

The derivation of the equilibrium equations for flat plates is much easier. If the plate lies in the $x-y$ plane, then

$$\mathbf{n} = \mathbf{k} \tag{37}$$

and let the imaginary fluid velocity,

$$\begin{aligned}&\mathbf{u} = 0 \text{ when } z > 0\\ &\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} \text{ when } z < 0.\end{aligned} \tag{38}$$

Then when $z < 0$,

$$\begin{aligned}\boldsymbol{\gamma} = {}& \frac{\partial u_x}{\partial x}\mathbf{ii} + \frac{\partial u_y}{\partial y}\mathbf{jj} + \frac{\partial u}{\partial z}\mathbf{kk}\\ &+ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)(\mathbf{ij} + \mathbf{ji})\\ &+ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)(\mathbf{jk} + \mathbf{kj})\\ &+ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)(\mathbf{ki} + \mathbf{ik})\end{aligned} \tag{39}$$

and

$$\begin{aligned}\boldsymbol{\Omega} = {}& \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i}\\ &+ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j}\\ &+ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}.\end{aligned} \tag{40}$$

If

$$\mathbf{B}_{\text{load}} = \frac{1}{2}q(x,y)(\mathbf{ij} + \mathbf{ji}) \tag{41}$$

then the plate is subject to an upwards load equal to

$$W = \frac{\partial^2 q}{\partial x \partial y} \tag{42}$$

per unit area. We therefore have plane stress in the plane of the plate plus bending out of the plane of the plate.

Differentiating

$$\begin{aligned}\nabla\mathbf{B}_{\text{load}} &= \frac{1}{2}\left(\frac{\partial q}{\partial x}\mathbf{i} + \frac{\partial q}{\partial y}\mathbf{j}\right)(\mathbf{ij} + \mathbf{ji})\\ \varepsilon \cdot\cdot \nabla\mathbf{B}_{\text{load}} &= \frac{1}{2}\left(\frac{\partial q}{\partial x}\mathbf{ki} - \frac{\partial q}{\partial y}\mathbf{kj}\right)\end{aligned}$$

so that from (35) and (36)

$$\begin{aligned}\mathbf{F} = {}& \mathbf{i}\frac{\partial \boldsymbol{\Omega}}{\partial y} - \mathbf{j}\frac{\partial \boldsymbol{\Omega}}{\partial x}(\mathbf{ij} + \mathbf{ji})\\ &- \frac{1}{2}\left(\frac{\partial q}{\partial y}\mathbf{i} + \frac{\partial q}{\partial x}\mathbf{j}\right)\mathbf{k}\end{aligned} \tag{43}$$

and

$$\begin{aligned}\mathbf{M} = {}& -\frac{\partial u_x}{\partial x}\mathbf{ji} + \frac{\partial u_y}{\partial y}\mathbf{ij}\\ &+ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)(-\mathbf{jj} + \mathbf{ii})\\ &+ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)\mathbf{ik}\\ &- \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\mathbf{jk}\\ &- \frac{1}{2}q(\mathbf{jj} - \mathbf{ii}).\end{aligned} \tag{44}$$

Now for simplicity let us consider the special case of plane stress when

$$\mathbf{B}_{\text{load}} = 0 \tag{45}$$

and

$$\begin{aligned}\phi &= \phi(x,y)\\ u_x &= -z\frac{\partial \phi}{\partial x}\\ u_y &= -z\frac{\partial \phi}{\partial y}\\ u_z &= \phi\end{aligned} \tag{46}$$

below the plate. Then

$$\mathbf{\Omega} = \frac{\partial \phi}{\partial y}\mathbf{i} - \frac{\partial \phi}{\partial x}\mathbf{j} \tag{47}$$

so that

$$\mathbf{F} = \frac{\partial^2 \phi}{\partial y^2}\mathbf{ii} - \frac{\partial^2 \phi}{\partial x \partial y}(\mathbf{ij} + \mathbf{ji}) + \frac{\partial^2 \phi}{\partial x^2}\mathbf{jj} \tag{48}$$

and, from the discontinuity across the plate when $z = 0$,

$$\mathbf{M} = 0 \tag{49}$$

confirming that ϕ is the Airy stress function.

10. CONCLUSIONS AND FURTHER WORK

We have demonstrated that the force and moment in an unloaded coat hanger bent out of its plane can be described by an imaginary fluid velocity field in the space outside the wire.

The theory leads to expressions for the equilibrium forces and moments in a shell and contains the Airy stress function as a special case.

The concept is quite difficult to understand and the application to further special cases will help clarify the matter.

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