Transience & Recurrence of Markov Processes with Constrained Local Time

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Introduction & Background

- Motivation
- Local Time of a Markov Process
- Connection to Lévy Processes

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- Transient Case
- Recurrent Case
- Characterising Entropic Effects

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- References
Motivation

Domb-Joyce Model

- The “Domb-Joyce” model for long polymer chains involves simple random walks (for polymers without self-interaction), and self-avoiding walks (for polymers with excluded-volume self-interaction).
Motivation
Edwards Model

- In the “Edwards” model, Brownian motion is used in place of a random walk, and we can study the path behaviour of a 1-dimensional process in order to model our polymer.

- In the 1-dimensional setting, polymers can be modelled by restricting our process to avoid certain points. This type of process shall be the focus of this talk.
Motivation
Weak Avoidance

- We are interested in processes which are “weakly” restricted to avoid a certain point, so the process is allowed to visit the point, but is “discouraged” from doing so.

- Such processes can be used to model a copolymer made up of two different types of monomer, near a boundary between two fluids. The monomers may be attracted to or repelled by one of the fluids (e.g. hydrophobic/hydrophilic monomers), so the polymer tries to place as many monomers in their preferred fluid as possible.

- Examples: surfactants; emulsifiers; foaming/antifoaming agents.

(Lectures on Random Polymers, Caravenna, den Hollander, Pétrélis, 2012)
Motivation

Phase Transition

- Polymer physicists are especially interested in the occurrence of a “phase transition” between a “localised” phase, where the polymer remains close to the interface, and a “delocalised” phase, where it moves away from the interface. Typically, the goal is to understand when the transition occurs, as underlying model parameters vary.

- This motivates our study of the phase transition between transience and recurrence of various stochastic processes.
• In this talk, \((M_t)_{t \geq 0}\) denotes a strong Markov process, which takes values in \(\mathbb{R}^d\), and is homogeneous in time.

• Conditionally given the present state of the process, the future behaviour is independent of the past behaviour (the future only depends on the present).
Local Time
Discrete State Space

- The local time (at $y$) of a Markov process $(M_t)_{t \geq 0}$, with a discrete state space, is the process $(L^y_t)_{t \geq 0}$ defined by
  \[ L^y_t := \int_0^t \mathbb{1}_{\{M_s = y\}} ds \]

- The local time (here equivalent to occupation time) counts how long $M$ spends at $y$. 

![Diagram of Markov process and local time](image)
Local Time
Continuous State Space

• The local time (at $y$) of a Markov process $(M_t)_{t \geq 0}$ is the process $(L_t(y))_{t \geq 0}$ defined by

$$L_t(y) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|M_s-y|<\varepsilon\}} ds.$$ 

• Informally speaking, the local time is now a (rescaled) measure of how much time the Markov process $(M_t)_{t \geq 0}$ spends near the point $y$.

• Without loss of generality, we consider only $y = 0$, and denote $L_t := L_t(0)$. 

![Graph of $M_t$ and $L_t$ over time](image-url)
Connection to Lévy Processes
Excursions of Markov Processes

- We define an excursion interval of a Markov process away from 0 as an interval of time \((g, d)\) such that \(M_s \neq 0\) for all \(s \in (g, d)\).

- The value of the local time remains constant when \(M\) is in an excursion interval, and increases otherwise. In our cases of interest, there are generally infinitely many excursions of infinitesimally small size. Some large excursions are highlighted in red here.

\[ M_t \]

\[ L_t \]
Connection to Lévy Processes
Inverse Local Time

- The (right-continuous) inverse, $X_t := \inf\{s > 0 : L_s > t\}$, of the local time of a Markov process is a subordinator, i.e. a non-decreasing real-valued Lévy process.

- The jumps of $(X_t)_{t \geq 0}$ correspond to excursions of $(M_t)_{t \geq 0}$ away from zero.
Connection to Lévy Processes

Lévy Processes

- A Lévy process \((X_t)_{t \geq 0}\), is a \(\mathbb{R}^d\)-valued stochastic process which starts from 0 (almost surely), and has stationary, independent increments.

- So for all \(0 < s < t\), the increments \(X_t - X_s\) and \(X'_t - X'_{t-s} - X'_0\) have the same distribution, where \(X'\) is an independent copy of \(X\) (stationarity).

- And for all \(0 < s < t < u < v\), the increments \(X_v - X_u\) and \(X_t - X_s\) are independent (independence of increments).
The distribution of a subordinator is best understood through its Laplace exponent $\varphi$,

$$\mathbb{E} \left[ e^{-\lambda X_t} \right] = e^{-t\Phi(\lambda)},$$

$$\varphi(\lambda) = \delta \lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx).$$

Here $\delta \geq 0$ is the linear drift, and $\Pi$ is the Lévy measure, which determines the size and rate of the jumps (discontinuities), and satisfies $\int_0^\infty (1 \wedge x)\Pi(dx) < \infty$. 
Brownian Motion with Constrained Local Time
A Necessary and Sufficient Condition for Transience

- $M$ is “transient” if $\sup\{t \geq 0 : M_t = 0\} < \infty$ almost surely, and “recurrent” otherwise.

**Theorem (Benjamini, Berestycki 2011)**

A sufficient condition for transience of 1-dimensional Brownian motion conditioned so that $L_t \leq f(t)$ for all $t$ is

$$I(f) := \int_{1}^{\infty} \frac{f(s)}{s^{3/2}} ds < \infty.$$

**Theorem (Kolb, Savov 2016)**

The above condition is necessary and sufficient for transience of the conditioned Brownian motion.
Doob’s h-Transform

- We wish to condition the process so that $L_u \leq f(u)$ for all $u$. Recalling that $X$ is the (right) inverse of the process $L$, this condition is equivalent to $X_s \geq f^{-1}(s)$ for all $s$.

- A key tool here is the Doob h-transform. For the measure $Q(\cdot) := \lim_{t \to \infty} \mathbb{P}(\cdot|\mathcal{O}_t)$ defined as the limit of $\mathbb{P}(\cdot)$ conditioned on $\mathcal{O}_t := \{X_s \geq f^{-1}(s), \text{ for all } 0 \leq s \leq t\}$,

$$Q(X_h \in dy) := \lim_{t \to \infty} \mathbb{P}(X_h \in dy|\mathcal{O}_t)$$
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$$Q(X_h \in dy) := \lim_{t \to \infty} \mathbb{P}(X_h \in dy | O_t)$$

$$= \lim_{t \to \infty} \mathbb{P}(X_h \in dy; O_h | O_t)$$
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$$= \lim_{t \to \infty} \frac{\mathbb{P}(O_t | X_h \in dy; O_h) \mathbb{P}(X_h \in dy; O_h)}{\mathbb{P}(O_t)}$$
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$$= \lim_{t \to \infty} \frac{\mathbb{P}(O_{t-h}^{h,y})}{\mathbb{P}(O_t)} \times \mathbb{P}(X_h \in dy; O_h)$$
Doob’s h-Transform

- We wish to condition the process so that \( L_u \leq f(u) \) for all \( u \). Recalling that \( X \) is the (right) inverse of the process \( L \), this condition is equivalent to \( X_s \geq f^{-1}(s) \) for all \( s \).

- A key tool here is the Doob h-transform. For the measure \( Q(\cdot) := \lim_{t \to \infty} \mathbb{P}(\cdot|O_t) \) defined as the limit of \( \mathbb{P}(\cdot) \) conditioned on \( O_t := \{X_s \geq f^{-1}(s), \text{ for all } 0 \leq s \leq t\} \),

\[
Q(X_h \in dy) := \lim_{t \to \infty} \mathbb{P}(X_h \in dy|O_t)
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\]

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\]

- So to understand the conditioned behaviour of the inverse local time process \( X \) under \( Q \), we can study the asymptotics of the probabilities \( \mathbb{P}(O_t^{h,y}) \) and \( \mathbb{P}(O_t) \) as \( t \to \infty \).
Main Results

Regularity Conditions

- We prove the upcoming results in two main cases of interest. Define the tail of the inverse local time’s Lévy measure as \( \Pi(x) := \int_x^\infty \Pi(dx) \).

- Case (i): The tail \( \Pi \) is regularly varying at \( \infty \) with index \(-\alpha\), where \( \alpha \in (0, 1) \), and for some \( \beta > \frac{1+2\alpha}{\alpha(2+\alpha)} \), the function \( g := f^{-1} \) satisfies

\[
\lim_{t \to \infty} t\Pi\left( \frac{g(t)}{\log(t)^\beta} \right) = 0.
\]

- A function \( h \) is regularly varying at \( \infty \) with index \( \alpha \) if for all \( \lambda > 0 \),

\[
\lim_{t \to \infty} \frac{h(\lambda t)}{h(t)} = \lambda^\alpha.
\]
Regularity Conditions

• Case (ii): The tail satisfies $\beta(\Pi) > -1$ where $\beta(\Pi)$ is the lower Matuszewska index of $\Pi$, the tail $\Pi$ is a “CRV function”, and there exists $\varepsilon > 0$ such that $g := f^{-1}$ satisfies

$$\lim_{t \to \infty} t^{1+\varepsilon} \Pi (g(t)) = 0.$$  

• The lower Matuszewska index, $\beta(h)$, of a function $h$, is the infimum of $\beta \in \mathbb{R}$ such that

$$\exists C > 0 \text{ such that } \forall \Lambda > 1, \frac{h(\lambda x)}{h(x)} \geq C(1 + o(1)) \lambda^\beta \text{ as } x \to \infty, \text{ uniformly for } \lambda \in [1, \Lambda].$$

• A function $f$ is “CRV” at $\infty$ if $\lim_{\lambda \to 1} \lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = 1$.

• The above conditions provide much more generality than regular variation, but unfortunately the stronger condition $\lim_{t \to \infty} t^{1+\varepsilon} \Pi (g(t)) = 0$ means that we are not able to capture the recurrent case here.
Transient Case

- Recalling $g := f^{-1}$, we define the following events by
  \[ O_s := \{ X_u \geq g(u), \text{ for all } 0 \leq u \leq s \}, \]
  \[ O_{s}^{h,y} := \{ X_u \geq g_{y,h}(u), \text{ for all } 0 \leq u \leq s \}, \]
  where $g_{y,h}(u) := g(u + h) - y$.

- We define $\Phi(t) := \int_0^t P(O_s) ds$ and $\Phi_{y}^{h}(t) := \int_0^t P(O_{s}^{h,y}) ds$.

- In the case where $I(f) := \int_1^{\infty} f(x) \Pi(dx) < \infty$, we have $\Phi(\infty) < \infty$ and $\Phi_{y}^{h}(\infty) < \infty$.

- Applying Doob's h-transform, it follows that under $\mathbb{Q}$, the inverse local time $X$ has law
  \[ \mathbb{Q} \left( X_h \in dy \right) = \frac{\Phi_{y}^{h}(\infty)}{\Phi(\infty)} P \left( X_h \in dy; O_h \right). \]
Transient Case

**Theorem 1**

If \( I(f) := \int_1^\infty f(x)\Pi(dx) < \infty \), the conditioned process can be constructed as follows:

Until an independent random time \( \mathcal{C} \), run \( M \) conditioned so that \( L_t \leq f(t) \). From time \( \mathcal{C} \) onwards, the process behaves as \( Y_{t+\mathcal{C}} \), where \( Y \) is the Markov process \( M \) conditioned to avoid zero. The law of the last excursion time \( \mathcal{C} \) is given by

\[
P(\mathcal{C} \in ds) = \frac{P(\mathcal{O}_s)ds}{\int_0^\infty P(\mathcal{O}_v)dv} = \lim_{t \to \infty} P\left(\Delta_1^{g(t)} \in ds \mid \mathcal{O}_t\right),
\]

where \( \Delta_1^{g(t)} \) denotes the time of the first excursion of size at least \( g(t) \).

- So from time \( \mathcal{C} \) onwards, the conditioned process never hits zero again, and hence our conditioned process is transient.
Recurrent Case

Theorem 2

When $I(f) = \int_{1}^{\infty} f(x)\Pi(dx) = \infty$, the law of the inverse local time $X$ under $\mathbb{Q}$ satisfies

$$\mathbb{Q}(X_h \in dy) = q_h(y)\mathbb{P}(X_h \in dy; \mathcal{O}_h),$$

where for $t_0 = f(Ay)$, $A > 3$, the function $q_h(y)$ is given by

$$q_h(y) = \frac{\Phi^h_y(t_0)}{\Phi(1)} \exp \left( \int_{t_0}^{\infty} \left( \Pi(g_{y,h}(s) + \rho^h_y(s)) - \Pi(g(s)) + \rho(s) \right) ds \right),$$

where $\rho^h_y(s)$ and $\rho(s)$ are smaller order integrable functions, uniformly as $y, h$ vary.

- In this case, the conditioned Markov process under the limit law $\mathbb{Q}$ is recurrent.
Understanding the Necessary and Sufficient Condition

- We have conditioned $M$ so that $L_u \leq f(u)$ for all $u$, or equivalently, $X_s \geq f^{-1}(s)$ for all $s$, and $M$ is transient if $I(f) := \int_1^\infty f(x)\Pi(dx) < \infty$, or recurrent if $I(f) = \infty$.

- Our necessary and sufficient condition relates to the following result on the rate of growth of subordinators:

**Theorem (Bertoin 1998)**

Under some regularity conditions, for a subordinator $X$ with Lévy measure $\Pi$,

$$I(f) = \int_1^\infty f(x)\Pi(dx) = \infty \iff \limsup_{t \to \infty} \frac{X_t}{f^{-1}(t)} = \infty, \text{ almost surely},$$

and if this fails, i.e. if $I(f) < \infty$, then $\lim_{t \to \infty} \frac{X_t}{f^{-1}(t)} = 0$, almost surely.

- So our process is transient if and only if we constrain the inverse local time $X$ to grow faster than a function $g = f^{-1}$ for which $X_t = o(g(t))$ as $t \to \infty$, almost surely.
Entropic Repulsion

- In the transient case, we bounded the rate of growth of the local time, while still allowing it to grow to infinity as $t \to \infty$. But in fact, we have seen that the local time ceases to grow at all after a certain time.

- Even in the recurrent case, the conditioned process generally doesn’t come close to using all of its allowance of local time.

- This phenomenon is related to entropic effects, which cause the process to stay far away from breaking the constraint to allow for more fluctuations.

- “Entropic repulsion” describes situations where the most likely way for a process to satisfy an imposed condition is for the process to satisfy an even stronger condition.
Entropic Repulsion

- In the recurrent case (so $I(f) = \infty$), we can study non-decreasing functions $w$ such that $\lim_{t \to \infty} w(t) = \infty$, and $\lim_{t \to \infty} \mathbb{Q}(X_t \geq w(t)g(t)) = 1$. The set of such functions is called the “entropic repulsion envelope” for the function $f = g^{-1}$, and is denoted by

$$R_f := \left\{ w : w \uparrow \infty, \lim_{t \to \infty} \mathbb{Q}(X_t \geq w(t)g(t)) = 1 \right\}.$$ 

**Theorem 3**

Under some further regularity conditions, with $I(f) = \infty$, $g = f^{-1}$,

$$w \in R_f \iff \lim_{h \to \infty} \int_h^\infty f(g(h)w(h)) \Pi(g(s)) ds = 0.$$ 

- From this necessary and sufficient condition, we can deduce that in all cases satisfying our regularity conditions, the entropic repulsion envelope is non-empty.
References


