7 Phase Transitions and Opinion Dynamics

1. *Dynamical Processes on Complex Networks* Chapter 5,
   A. Barrat, M. Barthélemy and A. Vespignani,
   Cambridge University Press (2008)

2. *Statistical physics of social dynamics*,
   C. Castellano, S. Fortunato and V. Loreto,
   Rev. Mod. Phys. 81 (2009).

7.1 Phase Transitions

7.1.1 Statistical Mechanics

- Statistical mechanics studies macroscopic peoperties emerging from microscopic interaction rules.

- Particularly looks at phase-transitions:
  - change in macroscopic behaviour under variation of parameter.
    - e.g. water to ice phase transition; bulk magnetisation.

- Ising model is simplified model of phase transitions:
  - usually on lattices,
  - can be translated to complex networks.

7.1.2 The Ising Model

- Model for interaction of magnetic dipole *"spins"*:
  - each *site* ("atom") i has a spin $\sigma_i \in [-1, +1]$:
    - called *"spin down"* and *"spin up"* states.
  - spins interact with nearest neighbours and can align,
  - bulk *magnetisation* M is *"order parameter"*:
    - $M = \sum_j \sigma_j / N$ quantifies degree of order in system.

- Ising model is defined by Hamiltonian:

$$H = -\sum_i \sum_{j \neq i} J_{ij} \sigma_i \sigma_j \tag{1}$$

  J_{ij} is energy reduction if spins aligned: $J_{ij} > 0$ if i and j are neighbours, $J_{ij} = 0$ otherwise.
  - can see from (1) that minimum H is where all spins are aligned up or down.

- Introduce thermal noise in system: temperature $T > 0$, encouraging disorder.

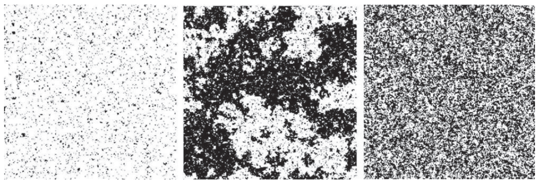


Figure 1: Below critical T_c states align; at T_c length-scales diverge; above T_c find disorder.

- At eqm. prob. finding syst. in state σ given by:

$$P(\sigma) = \frac{1}{Z} \exp\left[-\frac{H(\sigma)}{T}\right], \tag{2}$$

  partition function: $Z = \sum_\sigma \exp(-H(\sigma)/T)$

- Algorithm (ensuring above prob.):
  1. Set initially random σ_i values,
  2. randomly choose i and calculate δH if $\sigma_i \to -\sigma_i$,
  3. accept spin-flip **if** $\delta H < 0$; **else if** $\exp\left[-\frac{\delta H}{T}\right] \leq \nu$, random number $0 \leq \nu \leq 1$
  4. repeat above two steps until eqm. reached.

- See existence of *phase transition* (in 2D[1]):

7.1.3 Mean Field Treatment

- Assume all spins under equal influence of all others:

$$-\sigma_i \sum_j J_{ij}\sigma_j \to -J\sigma_i \sum_j \langle \sigma_j \rangle = -J\langle k \rangle M \sigma_i$$

  since $M = \langle \sigma_j \rangle$.

- $M = \dfrac{1}{Z} \sum\limits_{\sigma_i = \pm 1} \sigma_i \exp\left(-\dfrac{J\langle k \rangle}{T} M \sigma_i\right) = \tanh\left(\dfrac{J\langle k \rangle}{T} M\right)$
  - $M = 0$ always a solution,
  - nonzero solutions exist for $T < J\langle k \rangle = T_c$,
  - can also show: $M \approx (T_c - T)^{1/2}$ near T_c.

7.1.4 Generalising Ising

- σ_i can represent (binary) opinions in a social system.
- J_{ij} can be replaced with adjacency matrix:
- can study Ising on complex topologies.

[1]phase transitions are not seen for 1D lattices: proof exists.

Watts-Strogatz

- Can interpolate between:
  - 1D lattice (no phase transition),
  - random network (with critical $T_{c_{\rm MF}}$),
  - dD lattice (with critical $T_{c_{dD}}$);
- mean-field type phase transition found for any finite p_r.

7.1.5 General Degree Distributions

Given a degree distribution $P(k)$:

- define *average over class of nodes of degree* k:

$$\langle \sigma \rangle_k = \frac{1}{N_k} \sum_{i, k_i = k} \langle \sigma_i \rangle,$$

- define *average over all neighbours of with degree* k:

$$u = \sum_k \frac{k P(k)}{\langle k \rangle} \langle \sigma \rangle_k,$$

- can be shown that: $T_c = J \dfrac{\langle k^2 \rangle}{\langle k \rangle}$.

7.2 Opinion Dynamics Models

7.2.1 The Voter Model

- Model of competition of species or opinions.
- Agents have binary variable: $s_i = \pm 1$,
- select node i and neighbour j and set $s_i = s_j$:
  - imitation rule (no noise).

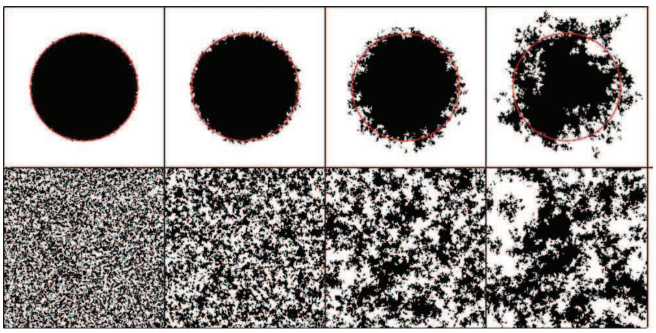


FIG. 2. (Color online) Evolution of a two-dimensional voter model starting from a droplet (top) or a fully disordered configuration (bottom). From Dornic *et al.*, 2001.

7.2.2 Modifications of the Voter Model

- Potts model:
  - multiple variables.
- "Zealots" [ref: Mobilia]:
  - Individuals who don't change their opinions, biasing local neighbourhood.
- Majority Rule:
  - select groups and take majority opinion.

7.2.3 Voter on Networks

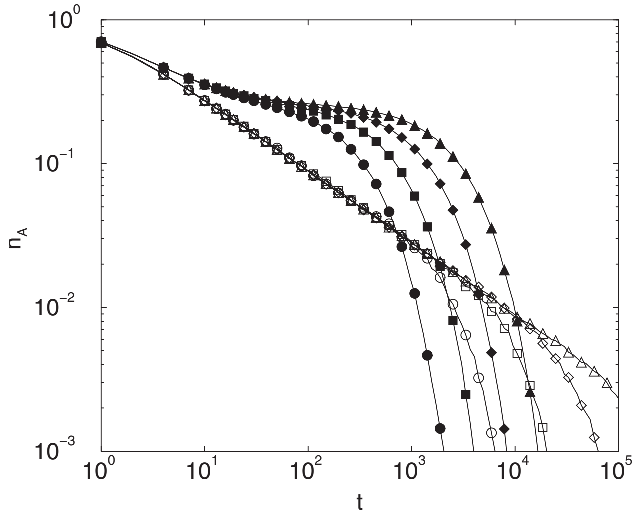


FIG. 3. Log-log plot of the fraction n_A of active bonds between nodes with different opinions. Empty symbols are for the one-dimensional case ($p=0$). Filled symbols are for rewiring probability $p=0.05$. Data are for $N=200$ (circles), $N=400$ (squares), $N=800$ (diamonds), $N=1600$ (triangles up), and $N=3200$ (triangles left). From Castellano *et al.*, 2003.

7.2.4 The Axelrod Model

- Model of cultural similarity and attraction[2]:
- The state of node i is a vector of F components:

$$(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{iF}).$$

- Each σ_{if} can take any of the q integer values $(1, \ldots, q)$, initially random.
- The time-discrete dynamics is an iteration:
  1. Select at random a pair of connected nodes: (i, j),
  2. calculate the *overlap*: $l(i,j) = \sum_{f=1}^{F} \delta(\sigma_{if}, \sigma_{jf})$,
  3. If $0 < l(i,j) < F$ the bond is *active* and nodes i and j interact with probability $l(i,j)/F$:
     - choose g randomly such that $\sigma_{ig} \neq \sigma_{jg}$ and set $\sigma_{ig} = \sigma_{jg}$.
- In any finite network the dynamics settles into an *absorbing state* with no active bonds.

7.2.5 The Axelrod Model on Watts Strogatz Networks

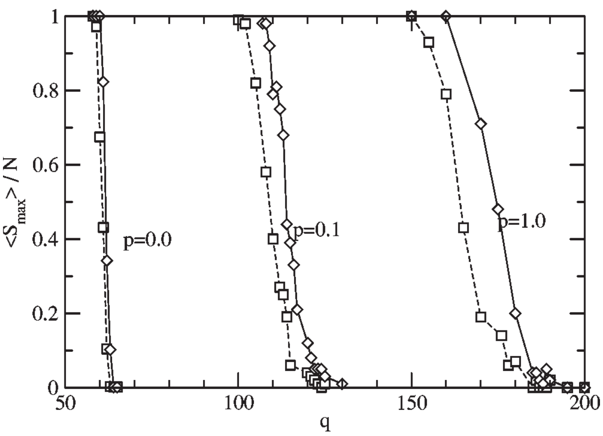


FIG. 3. The average order parameter $\langle S_{max} \rangle / N$ as a function of q for three different values of the small-world parameter p. System sizes are $N = 500^2$ (squares) and $N = 1000^2$ (diamonds); number of features $F = 10$. Each plotted value is an average over 100 runs with independent rewiring ($p > 0$) and independent initial conditions.

- critical q_c separating ordered and the disordered state,
- q_c increases with disorder from p.

[2] Klemm *et al.*, PRE (2003)

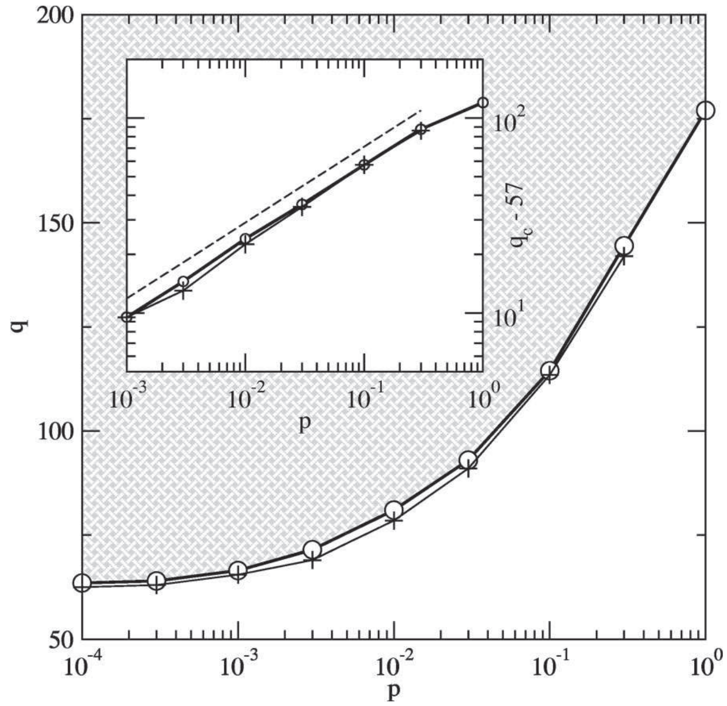


FIG. 4. Phase diagram for the Axelrod model in a small-world network. The curve separates parameter values (p,q) which produce a disordered state (shaded area) from those with ordered outcome (white area). For a given p the plotted value q_c is the one for which the value of the order parameter is closest to the, somewhat arbitrary but small, value 0.1 for system size $N=500^2$ and $F=10$. Inset: After subtraction of a bias $q_c(p=0)=57$, $q_c(p)$ follows a power law $\propto p^{0.39}$ (dashed line).

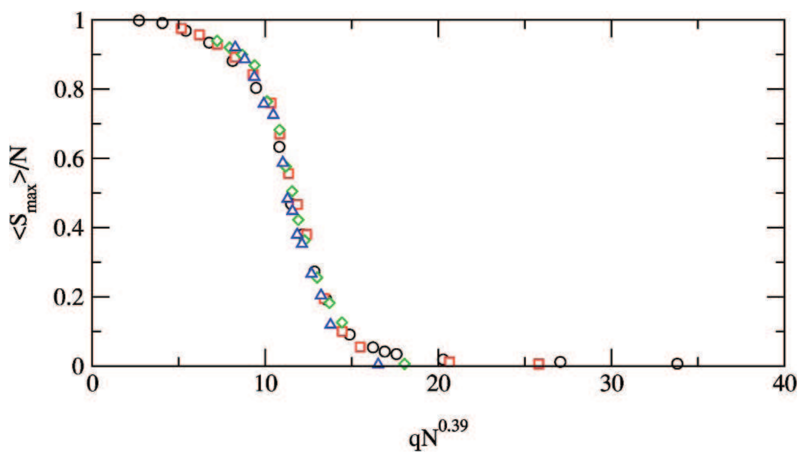


Figure 2: The average order parameter $\langle S_{max}\rangle/N$ in scale-free networks for $F = 10$. $N = 1000$ (circles), $N = 2000$ (squares), $N = 5000$ (diamonds), and $N = 10000$ (triangles).

7.2.6 The Axelrod Model on Scale Free Networks

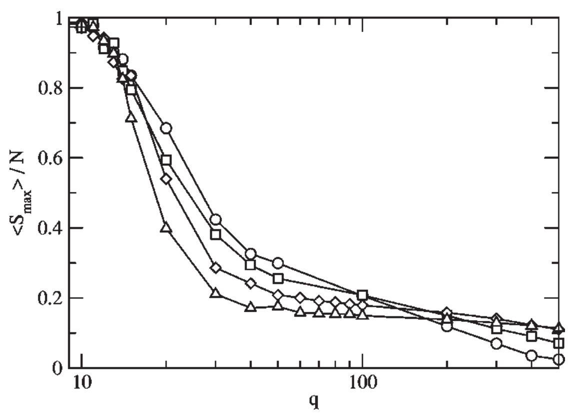


FIG. 8. The average order parameter $\langle S_{max}\rangle/N$ as a function of q for $F=10$ in structured scale-free networks. The networks contained $N=1000$ (circles), $N=2000$ (squares), $N=5000$ (diamonds), and $N=10000$ (triangles) nodes with $F=10$ features. Each data point is an average over 32 independent realizations.

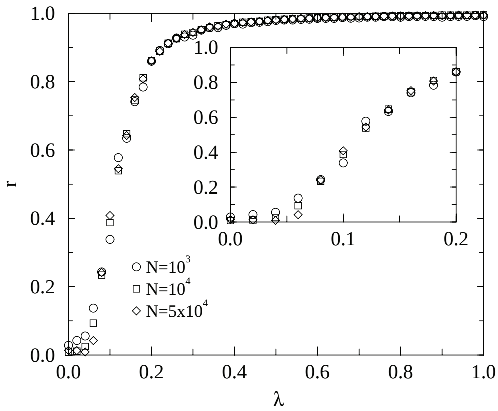


Fig. 1 – Coherence r as a function of λ for several system sizes. The onset of synchronization occurs at a critical value $\lambda_c = 0.05(1)$. Each value of r is the result of at least 10 network realizations and 1000 iterations for each N. The inset is a zoom around λ_c.

7.2.7 The Kuramoto Model on Scale Free Networks

[*Moreno and Pacheco, Europhys. Lett. (2004)*]

- Model of synchronisation of many interacting individuals:
  - model each as a phase oscillator with attractive coupling:

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j \in \mathrm{nei}(i)}^{N} \sin(\theta_j - \theta_i) \tag{3}$$

  - define order parameter:

$$r(t) = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \right| \tag{4}$$

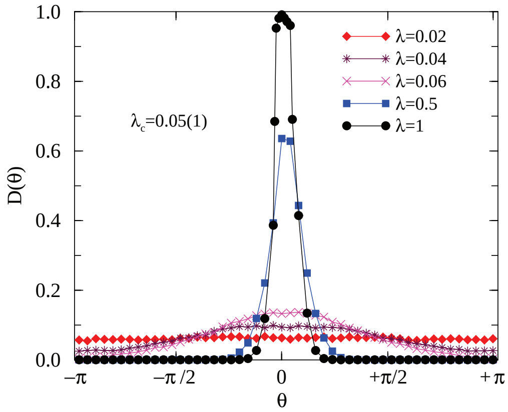
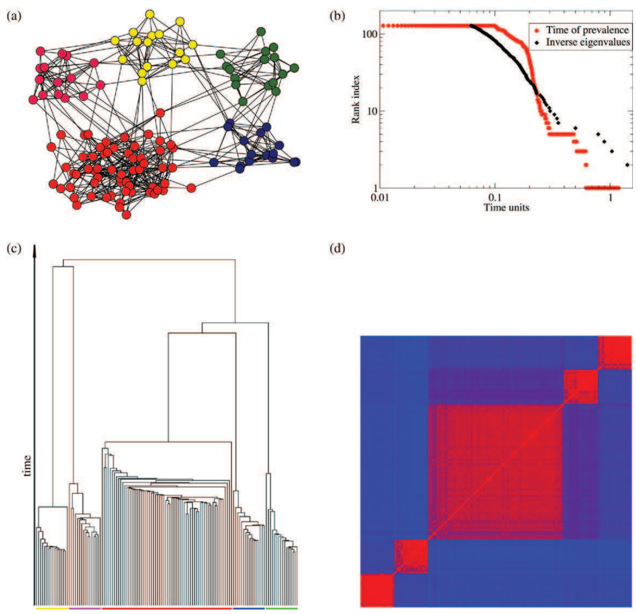


Fig. 2 – Normalized phase distributions $D(\theta)$ for different values of the control parameter λ. The curves depicted correspond to values of λ below, near and above λ_c as indicated. The network is made up of $N = 10^4$ nodes.

7.2.8 Community Detection using Synchronisation

[*Arenas* et al., *Physica D (2006)*]



- Use *local order parameter*: $\rho_{ij}(t) = \langle \cos(\theta_i(t) - \theta_j(t)) \rangle$