6 Dynamical Processes on Networks

1. *Dynamical Processes on Complex Networks*,
   A. Barrat, M. Barthélemy and A. Vespignani,
   Cambridge University Press (2008)
2. *Connectivity of Growing Random Networks*,
   P. L. Krapivsky *et al.*, PRL 85 (2000).
3. *Statistical physics of social dynamics*,
   Castellano *et al.*, Rev. Mod. Phys. 81 (2009).

6.1 General Framework – Background

- Opinions, disease states, etc. modelled as variables on nodes, (e.g. binary variable: [uninfected, infected], [uninformed, informed], [against, for]...)
  - dynamics determined by social interactions.
- Microscopic processes lead to *cooperative phenomena*,
  - interaction and alignment of variables,
    - e.g. synchronisation of coupled oscillators.
- Interested in transition between *disordered* and *ordered* phases, where nodes share similar states;
  - can use tools of *statistical mechanics* and *population dynamics.*
- Results can often be translated to different contexts.

6.2 General Framework – Basics

- Associate variables x_i with states on node i, can be:
  - continuous variable, or set of variables $\mathbf{x_i} = (x_{1i}, x_{2i}, ...)$;
  - enumerated states $x_i \in \{1, 2, 3, ..., s\}$ (same meaning);
  - system configuration: $\mathbf{x} = (x_1, x_2, ..., x_N)$,
  - can define *macro-states*, e.g.: mean value $\langle \mathbf{x} \rangle$.
- Wish to understand evolution of system:
  $\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_N(t))$, or $\langle \mathbf{x} \rangle(t)$, or...
- Dynamical systems theory gives tools to do this, given:
  - intial values $\mathbf{x}_0$,
  - dynamical equations: $x'_i = f_i(\mathbf{x})$, for future state x'.
    - e.g.: $\dot{\mathbf{x}} = f(\mathbf{x}) + \sigma L\mathbf{x}$; reaction diffusion system.

6.3 General Framework – Issues

1. System size N often too large to solve directly,
   - analytically unwieldy, numerically impractical:
   - complete understanding requires phase-space map for all s^N initial configurations!
2. Network structure not known precisely, or different for each realisation:
   - cities, friendship networks, computer networks, biological or ecological networks...
3. Initial conditions not known precisely,
   - outcomes can be very sensitive to IC,
   - often know only certain features, e.g. distributions of numbers S in each state.

6.4 General Framework – Problem

- How can we deal with such systems?
- Micro-simulation of small-scale models, Monte-Carlo, ensemble averages...
  - gain some statistical insight,
  - study factors affecting outcomes by studying initial conditions.
- to gain more understanding need to simplify in meaningful way,
  - make analytically feasible.

6.5 Analytical Methods

- Know that dynamics tends to reduce variability and increase order:
  - individuals end up sharing same technology, language, opinion, velocity...
  - i.e. correlation of neighbouring states.
  - familiar from statistical physics...
- Use concepts and tools from statistical mechanics,
  - need to extend to deal with complex topologies.
- Ask questions such as:
  - what are likely equilibrium states, if they exist?
  - what are mechanisms driving ordered/polarised state?
  - what are thresholds for transitions?

6.6 Analytical Methods - Master Equation

- Dynamical process described by state transitions:
  - $\mathbf{x}^a \to \mathbf{x}^b$; where $\mathbf{x}^a, \mathbf{x}^b$ are different configurations.
- Often don't know precise dynamics due to unknown factors or noise.
  - instead focus on probability $P(\mathbf{x}, t)$, for particular config.
- Master equation (ME):

$$\partial_t P(\mathbf{x}, t) = \sum_{\mathbf{x}'} \left[P(\mathbf{x}', t) W(\mathbf{x}' \to \mathbf{x}) - P(\mathbf{x}, t) W(\mathbf{x} \to \mathbf{x}') \right],$$

  $W(\mathbf{x}^a \to \mathbf{x}^b)$ are transition rates.

6.7 ME e.g.: Growing Networks

Master Equation for number of nodes N_k with degree k:

- $\partial_t N_k = r_{k-1\to k} N_{k-1} - r_{k\to k+1} N_k + \delta_{k,m}$

For the simple Barabási-Albert model:

- each new node i has a edges to nodes j with probability $\propto k_j$, degree of j
- ME: $\partial_t N_k = \frac{1}{N}[(k-1)N_{k-1} - kN_k] + \delta_{k,1}$
  - note: $t = N$, so $N_k = tn_k$, obtaining recurrence relation:
- $n_k = n_{k-1}\frac{(k-1)}{(k+1)}$; and can solve to give:
- $n_k = \dfrac{4}{k(k+1)(k+2)}$. I.e. $P(k) \propto k^{-3}$.

6.8 Master Equation – Simplifications

- but still s^N equations with s^N terms!
- need to find ways to simplify...
- Do not need whole configurations:
  $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{x}' = (x'_1, x'_2, ..., x'_N)$
- State x_i of node i depends only on its neighbours j so:

$$W(\mathbf{x}' \to \mathbf{x}) = \prod_i w(x'_i \to x_i | x_j),$$

  where w are trans. rates conditional only on neighbours.
- Network topology now plays a role in dynamics.

- In principle possible to compute expectation value of function $A(\mathbf{x})$:

$$\langle A(t) \rangle = \sum_{\mathbf{x}} A(\mathbf{x}) P(\mathbf{x}, t).$$

- In practice impossible to solve ME in most cases!
- Additionally most real-world applications are non-equilibrium
  - not possible to use equilibrium thermodynamics and ergodic hypothesis.

6.9 Approx. Solutions of Master Equation

1. Consider appropriate projection:
   - e.g.: average number in state $x_i = a$ at time t:

   $$\langle N_a(t) \rangle = \sum_{\mathbf{x}} \sum_i \delta_{x_i, a} P(\mathbf{x}, t),$$

   δ is Kronecker delta.
   - average quantity $\Rightarrow$ deterministic.
2. Neglect network structure, assume homogeneous system with no correlations; $\text{prob}(x_i = a) = p_a$, for each i:

$$P(\mathbf{x}) = \prod_i p_{x_i}.$$

- *Mean Field* (MF) approximation.

6.10 (In-)Validity of Mean Field Approx.

MF valid when:

- Variables (degrees of freedom) of system are iid:
  - generally not true, as interactions are *by definition* dependencies!
  - can get around using *pair approximation* schemes:
    * assume neighbours j only dependent on self i.
- Homogeneous mixing; i.e. all i have equal chance of interacting with all j,
  - again, network structure invalidates this,
  - can assume independence of neighbourhoods:
    * neighbours of i are *on average* same as system average.

However:

- produces analytically tractable results,
- in many cases useful ones.

6.11 Example of Mean Field Approx. Scheme

- Consider simple system:
  - two states: $x_i = A$ and $x_i = B$,
  - Dynamics: $A + B \to 2B$, rate β.
  - $w(A \to A|x_j = A) = w(B \to B|x_j = A)$
    $= w(B \to B|x_j = B) = 1$, $w(A \to B|x_j = B) = \beta$
- Using: $\langle N_A(t)\rangle = \sum_{\mathbf{x}} \sum_i \delta_{x_i,A} P(\mathbf{x}, t)$
  and $\langle N_B(t)\rangle = \sum_{\mathbf{x}} \sum_i \delta_{x_i,B} P(\mathbf{x}, t)$
- ME: $\partial_t \langle N_B(t)\rangle = \sum_{\mathbf{x}} \sum_i \delta_{x_i,B} \partial_t P(\mathbf{x}, t)$

$$\partial_t \langle N_B(t)\rangle = \sum_i \sum_{\mathbf{x}'} \sum_{\mathbf{x}} \left[\delta_{x_i,B} \prod_k w(x'_k \to x_k|x'_j) P(\mathbf{x}', t) - \delta_{x_i,B} \prod_k w(x_k \to x'_k|x_j) P(\mathbf{x}, t) \right],$$

and using the normalisation conditions:
$\sum_{\mathbf{x}'} \prod_k w(x_k \to x'_k|x_j) = 1$, $\sum_{\mathbf{x}} \delta_{x_i,B} \prod_k w(x'_k \to x_k|x'_j) = w(x'_i \to x_i = B|x'_j)$

$$\partial_t \langle N_B(t)\rangle = \sum_i \sum_{\mathbf{x}'} \left[w(x'_i \to x_i = B|x'_j) P(\mathbf{x}', t) \right] - \langle N_B(t)\rangle$$

Now using MF: $p_A = N_A/N$, $p_B = N_B/N$ and $P(\mathbf{x}') = \prod_i p_{x'_i}$:

$$\sum_{\mathbf{x}'} w(x'_i \to x_i = B|x'_j) P(\mathbf{x}', t)$$
$$= \sum_{x'_j} \left[w(x'_i = A \to x_i = B|x'_j) p_A \prod_{j \in \mathcal{V}(i)} p_{x'_j} + w(x'_i = B \to x_i = B|x'_j) p_B \prod_{j \in \mathcal{V}(i)} p_{x'_j} \right],$$

LHS $w = \beta$ with prob $1 - (1 - p_B)^k$, and RHS $w = 1$ always, so:

$$\partial_t \langle N_B(t)\rangle = \beta N_A (1 - (1 - N_B/N)^k).$$