Complex Systems and Networks

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General References

1. *Networks: An Introduction*,
   M. E. J. Newman, Oxford University Press (2010).

2. *The structure and function of complex networks*,
   M. E. J. Newman, SIAM review (2003).

3. *Complex networks: Structure and dynamics*,
   Boccaletti *et al.*, Physics Reports 424 (2006).

4. *Statistical physics of social dynamics*,
   Castellano *et al.*, Rev. Mod. Phys. 81 (2009).

1 Introduction to Complex Systems

For a comprehensive survey of literature on many aspects of complex systems see: *Complex systems: A survey*, M. E. J. Newman, Am. J. Phys. 79 (2011).

1.1 Basic Definition of a Complex System

A Complex System is defined as a system composed of many components, where the behaviour of system depends strongly on the interaction between the components. The system is therefore defined not just in terms of the components and their properties but the relationships between the components. The components have properties and associated dynamical variables, and the time-evolution of these properties is described by *rules* or equations, determining the behaviour of the system. Coupled dynamical systems are therefore complex systems in this context, and indeed nonlinear dynamics forms one of the core components of complex systems theory.

The behaviour of a complex system at larger scales is found to emerge naturally through these interactions; that is to say it isn't somehow engineered from the top-down by assigning individual roles to the components in some overall scheme (even though complex *adaptive* systems - where the components' behaviour can change to optimise some criteria - can evolve so that the components come to play unique roles in the overall function of the system). These higher level phenomena or functions which appear are referred to as "emergent properties" of the system.

1.2 Examples of Complex Systems

Complex systems are found throughout the natural sciences, and their study pre-dates what is currently called "complex systems" research. Traditional fields which are now regarded as complex systems include condensed matter physics: looking at coherent domains and effective material properties brought about by interactions at the microscopic level. An example is the Ising model for magnetisation of materials, where the spins of domains on a lattice align with their neighbours to give rise to domains of coherence and bulk magnetisation.

In biology ecosystems can be regarded as complex systems, and the climate system also has multiple interacting constituent parts, giving rise to system-level behaviour such as tropical storms and El-Niño. Cities are also good examples of complex systems, where we can look at the relationship between different components towards the working of the city. This can include the flow of traffic, goods and services and money. In this way another human example of a complex system is the economy itself, with millions of traders interacting to give rise to emergent effects such as bubbles and crashes.

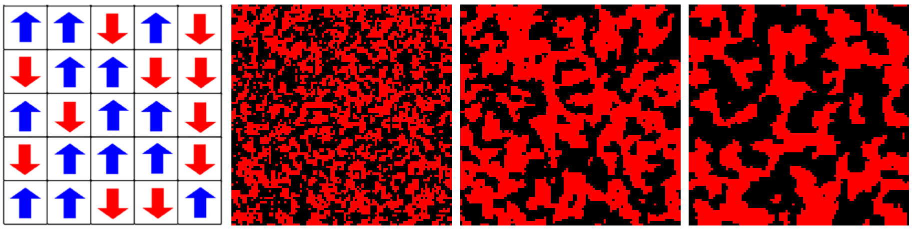


Figure 1: Ising model of magnetisation.

1.3 Emergent Properties

Emergent properties are hard to define precisely, as they come in many forms. One thing we can say is that they are features, effects or collective behaviours at scales larger than the component level interactions that they emerge from. Some obvious examples are self-organisation such as pattern formation in space and time. A good example is segregation in oscillated granular materials; another example of what are now regarded as complex systems. Fireflies interacting with a few local neighbours to produce patterns of synchronised flashing over very large scales is also a classic example of an emergent phenomenon.

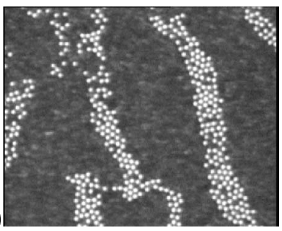
(a)  (b) 

Figure 2: Two examples of emergent patterns: (a), stripe formation in a binary granular mixture; (b) synchronised flashing of fireflies.

1.4 Theory and Modelling of Complex Systems

There are two main approaches to modelling complex systems, although there is often considerable overlap.

1. Detailed Simulation approach, e.g.:
   - agent-based simulation,
   - Monte Carlo simulation,
   - . . .

   These are large-scale computer models, including many details of the real system.

2. "Simplified" Models, e.g.:
   - Dynamical systems theory,
   - networks,
   - cellular automata,
   - . . .

   These are more familiar "traditional" approaches from the mathematical sciences and attempt to represent only the essential elements of the underlying mathematical laws and find "universal" features which can describe the properties of many different, but mathematically related, real systems.

1.5 Agent-Based Simulation

The first approach, which I'm not going to talk about too much, but which I'll just highlight briefly here, is ABS, often called "individual-based" simulation.

One of the more simple examples of ABS are attempts to model so called "flocking" or "swarming" behaviour seen in birds, fish and insects. Simple computational flocking rules can lead to behaviour similar to that seen in the natural world:

1. *Collision Avoidance*: short-range repulsion from local neighbours
2. *Velocity Matching*: steer towards average heading of neighbours
3. *Flock Centring*: steer towards center of mass of local neighbours

1.6 Conceptual Models of Complex Systems

In this course we'll be mainly looking at the type of models which aim to allow us to gain some analytical insight into the observed behaviours. They do this by abstracting the most important qualitative elements of systems into some solvable framework. To do this we must specify both the topology; i.e. who interacts with whom, in the form of a *network structure*, and the dynamics of the system, usually in the form of a set of rules or equations. These set out the behaviour of individual components and interaction between them. This is familiar as a coupled dynamical system, and indeed the dynamical systems theory forms a central component in the treatment of complex systems. Finally we use this framework to try to investigate the system and its behaviours to gain insight. The means of investigating complex dynamical systems can take the form of analysis of the systems, if possible, but more often the only available approach is numerical simulation, or statistical methods.

1.7 Basic Definition of Networks

To specify the structure of interactions, *networks* can be used to represent complex systems. A network is in mathematical terms a *graph*, with components of the system represented by the *vertices*: $v_1 \dots v_N$. These are also often called *"nodes"*, or *"sites"* in physics and *"agents"* in the social sciences. The relationships between the individual components are shown as *edges*, $e_{(i,j)}$, connecting the nodes in the network. These can also be referred to as *"links"*, *"bonds"* or *"ties"*.

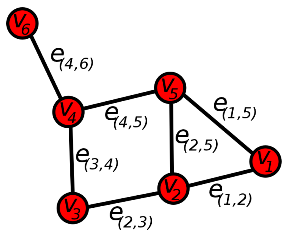


Figure 3: A simple six node graph with seven edges.

A network has many properties which can be measured, coming from graph-theory or social-network theories. These properties include things such as *node degree*; i.e. the number of links assigned to each node (vertices 2, 4 and 5 have degree three in the example). Also things such as the path-length connecting two vertices, or the centrality, which is a measure of the importance of a node or link in the network, and the relative damage to the network if it were to be removed. Another important feature is that of *clustering*, known in the social sciences as the "friend-of-a-friend" effect. In the graph shown the set of nodes [V_1, V_2, V_5] form a three-cluster.

1.8 Examples of Real World Networks

Simply plotting the networks can often clearly show some striking features of the structure.

(a) (b)

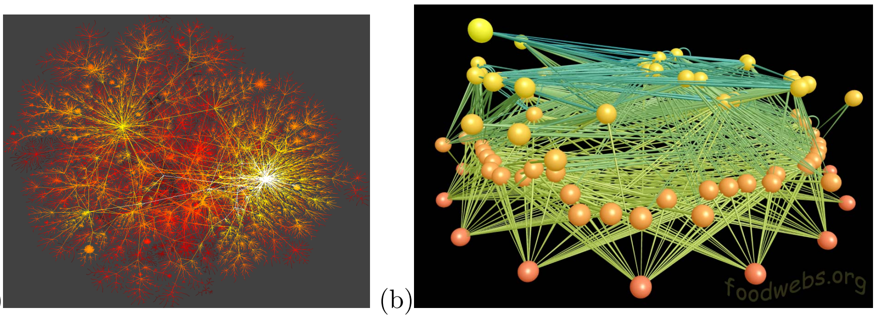


Figure 4: Real-world networks include the internet (a), and food-webs (b). Clear structural differences can be seen just in the visual representation.

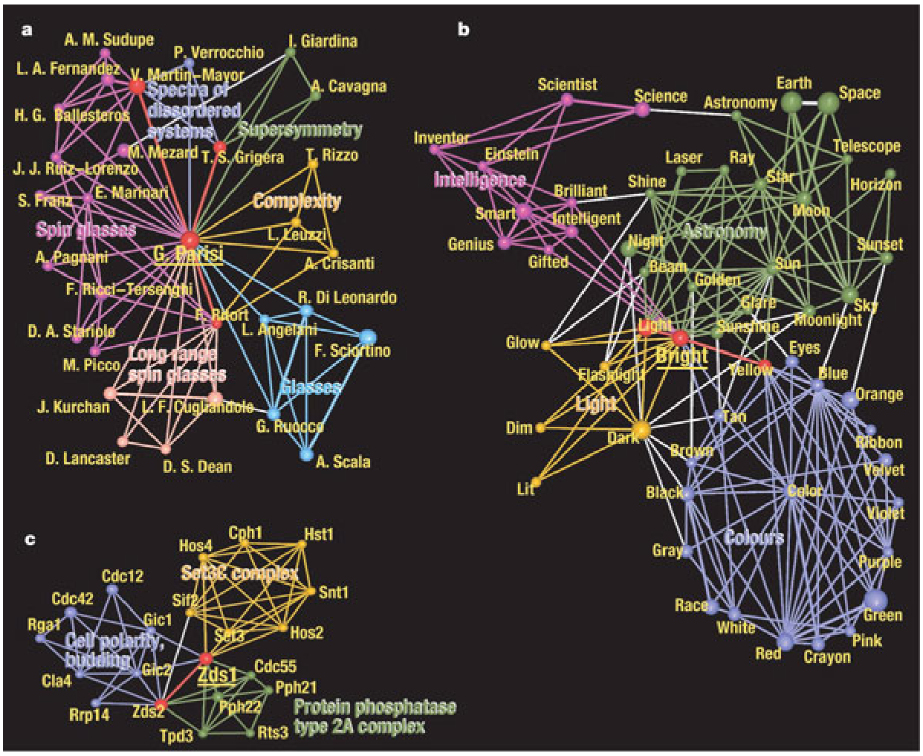


Figure 5: More networks. a, A co-authorship network. b, Word Association network. c, Protein–protein interactions in yeast. (Figure from: *Uncovering the overlapping community structure of complex networks in nature and society*, Palla *et al.*, Nature 435, 814-818 (9 June 2005))

For example, in the case of the internet, a feature that can be immediately seen is that there are some nodes which are central *"hubs"*, with a very high degree, whereas most have only a few edges. For such large systems it is often more useful to speak in terms of statistical properties of the network. Instead of listing the individual degrees of every node (the *degree sequence*), we look at the *degree distribution*. In the case of the internet, and many other networks, a wide degree distribution is found. In fact, in the case of the internet the degree distribution is found to be a power-law, with no characteristic scale as would be the case if the degree distribution were Poisson, as would be expected from a random assignment of edges. One thing we can do with large networks such as computer network systems or ecological food-webs is look at the robustness of the network. That is, we can investigate the effect of randomly or selectively removing nodes and look at how well information can propagate or the system can function in general. In this way we can try to understand the weaknesses of a system and where potential problems might arise.

Other networks show different features such as community structure. Communities are sub-graphs within the main graph in which the nodes are more strongly connected internally than with the rest of the network. The examples shown here are an indication of a property which is called *homophily*, i.e. nodes with a similar property or function tend to associate more with other nodes having that same characteristic.

1.9 Human Social Networks

Social networks are often studied, and social-network analysis has been around for a long time, although in a less mathematically rigorous way that recently. To study these in the field, individuals can be interviewed about their own personal social network. So called *egocentric* networks can be constructed by asking an individual (the *ego*) about their personal contacts (known as the *alters*) and the links between them. This obviously misses individuals who are not known to the interviewee, so further searching would need to be dome to expand this to full scale. However, it is useful for looking at community structures of society, which would become less visually clear if the network was scaled up.

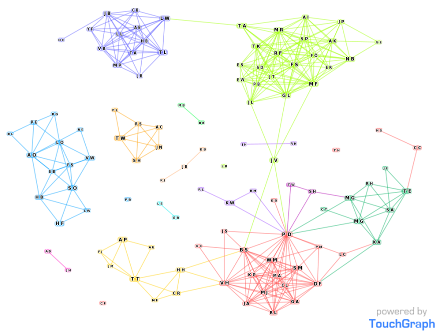


Figure 6: An *egocentric* network centred on one individual (not shown). Contacts and their inter-connections can be seen, as well as the communities to which the individual belongs. Each individual shown would also have a similar network adding to this, with many overlaps.

As well as the networks shown above, there are many more real-world and model examples, which are constructed to reproduce real-world and other theoretical features. These networks can then be used as the coupling in dynamical models of the time evolution of behaviour on the networks.

1.10 Dynamical Systems

In continuous-time dynamical systems the network takes the form of a coupling between elements, for example:

$$\dot{x}_i = f(x_i) + \sum_{j \in K} \sigma_{i,j} g(x_j),$$

where the coupling of the ith node to its K network neighbours is given by the strengths, here shown as the $\sigma_{i,j}$ terms. The functions f and g aren't specified at this point.

An example of some of the behaviour of this type of system could be either complete or partial synchronisation of a chain of coupled nonlinear oscillators. As well as total synchronisation we can see the emergence of stationary patterns or spatio-temporal ones such as travelling waves, or the fireflies seen previously.

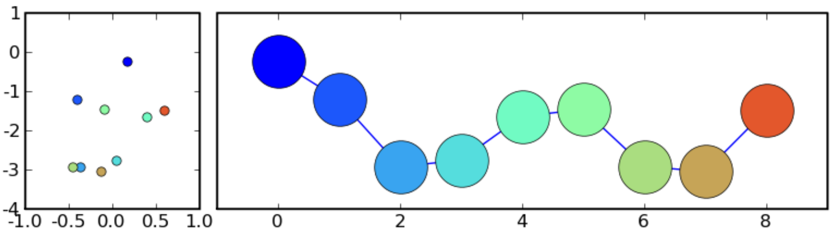


Figure 7: A period-two spatio-temporal pattern found in a one dimensional regular lattice of coupled nonlinear oscillators.

As well as simple lattice topologies, dynamical processes can also be studies on more complex network topologies, although these are less amenable to analytical attack.

1.11 Cellular Automata

As well as continuous-time, the dynamics can be given by discrete (time-discontinuous) rules. An example of this type of system are Cellular Automaton models. The scheme here is as follows:

- each "site" has a *rule* based on its current state and that of its neighbours,
- update at next time-step given by this rule.

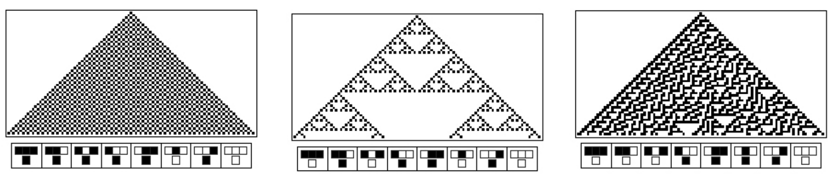


Figure 8: For the 1D lattices shown here, each row is a time-step, with increasing time shown from top to bottom. [S. Wolfram, *A New Kind of Science*, (2002).]

Different emergent patterns can be found for different rules and topologies.

1.12 Analytical Approaches

As a hint of where these modelling approaches can take us, let's consider the question of how macroscopic laws can be obtained from understanding the component interactions. It's possible to use methods borrowed from statistical mechanics. In traditional statistical mechanics, e.g. in thermodynamics we can derive from the microscopic quantities such as particle momenta and velocities the macroscopic variables such as pressure and temperature and the relationships between them. Mean field arguments can also be applied to networks, using the "homogeneous mixing" assumption to try to obtain macro-variables and understand dynamical behaviour. Homogeneous mixing can only be applied strictly for certain types of networks, but it can nonetheless give useful results.

An example where these statistical mechanics approaches have been usefully applied is to the susceptible infected, removed (or recovered) model of disease propagation via a contact network. A set of transition rates (λ, μ) can be used to describe transitions between the different states of individuals in a population:

$$S(\text{usceptible}) \xrightarrow{\lambda} I(\text{nfected}) \xrightarrow{\mu} R(\text{emoved}).$$

If we assume that homogeneous mixing occurs (the mean field assumption), which would be true in random or all-to-all networks then the following set of equations can be written:

$$\begin{aligned} \dot{n}_S &= -\lambda \bar{k} n_I(t) n_S(t) \\ \dot{n}_I &= -\mu n_I(t) + \lambda \bar{k} n_I(t) n_S(t) \\ \dot{n}_R &= \mu n_I(t) \end{aligned}$$

where n_S, n_I, n_R are the number density of S, I, R, respectively and $\bar{k}$ is the number of contacts per unit time, taken as constant due to the mean-field assumption. This gives a set of equations for macroscopic, system level variables, while losing information on the individual states.

Using this kind of approach we will see that it is possible to determine the critical ratio of rate constants for the successful transmission of the disease through the network.