7. Phase Transitions and Dynamical Models

1. *Dynamical Processes on Complex Networks*, A. Barrat, M. Barthélemy and A. Vespignani, Cambridge University Press (2008)
2. *Statistical physics of social dynamics*, C. Castellano, S. Fortunato and V. Loreto, Rev. Mod. Phys. 81 (2009).

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- Ising model is simplified model of phase transitions:
  - usually on lattices,
  - can be translated to complex networks.

The Ising Model

- Model for interaction of magnetic dipole *"spins"*:
  - each *site* ("atom") i has a spin $\sigma_i \in [-1, +1]$:
    - called *"spin down"* and *"spin up"* states.
  - spins interact with nearest neighbours and can align,
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- Ising model is defined by Hamiltonian:

$$H = -\sum_i \sum_{j \neq i} J_{ij} \sigma_i \sigma_j \quad (1)$$

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  - can see from (1) that minimum H is where all spins are aligned up or down.

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- Algorithm (ensuring above prob.):
  1. Set initially random σ_i values,
  2. randomly choose i and calculate δH if $\sigma_i \to -\sigma_i$,
  3. accept spin-flip **if** $\delta H < 0$; **else if** $\exp\left[-\frac{\delta H}{T}\right] \leq \nu$, random number $0 \leq \nu \leq 1$
  4. repeat above two steps until eqm. reached.

The Ising Model

- See existence of *phase transition* (in 2D[1]):

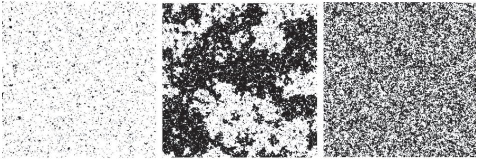

Figure: Below critical T_c states align; at T_c length-scales diverge; above T_c find disorder.

[1] phase transitions are not seen for 1D lattices: proof exists.

Mean Field Treatment

- Assume all spins under equal influence of all others:

$$-\sigma_i \sum_j J_{ij}\sigma_j \to -J\sigma_i \sum_j \langle \sigma_j \rangle = -J\langle k \rangle M\sigma_i$$

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  - $M = 0$ always a solution,
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  - can also show: $M \approx (T_c - T)^{1/2}$ near T_c.

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- mean-field type phase transition found for any finite p_r.

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Given a degree distribution $P(k)$:

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- can be shown that: $T_c = J \dfrac{\langle k^2 \rangle}{\langle k \rangle}$.

The Voter Model

- Model of competition of species or opinions.
- Agents have binary variable: $s_i = \pm 1$,
- select node i and neighbour j and set $s_i = s_j$:
    - imitation rule (no noise).

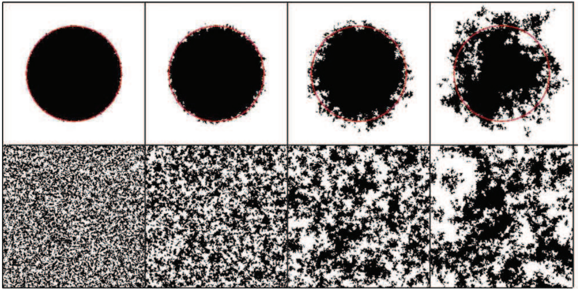

FIG. 2. (Color online) Evolution of a two-dimensional voter model starting from a droplet (top) or a fully disordered configuration (bottom). From Dornic *et al.*, 2001.

Modifications of the Voter Model

- Potts model:
  - multiple variables.
- "Zealots" [ref: Mobilia]:
  - Individuals who don't change their opinions, biasing local neighbourhood.
- Majority Rule:
  - select groups and take majority opinion.

Voter on Networks

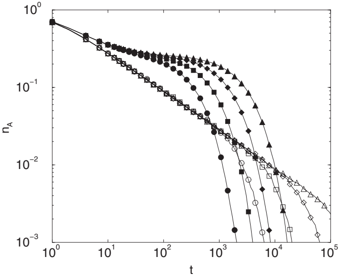

FIG. 3. Log-log plot of the fraction n_A of active bonds between nodes with different opinions. Empty symbols are for the one-dimensional case ($p=0$). Filled symbols are for rewiring probability $p=0.05$. Data are for $N=200$ (circles), $N=400$ (squares), $N=800$ (diamonds), $N=1600$ (triangles up), and $N=3200$ (triangles left). From Castellano *et al.*, 2003.

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     - choose g randomly such that $\sigma_{ig} \neq \sigma_{jg}$ and set $\sigma_{ig} = \sigma_{jg}$.
- In any finite network the dynamics settles into an *absorbing state* with no active bonds.

[2]Klemm *et al.*, PRE (2003)

The Axelrod Model on Watts Strogatz Networks

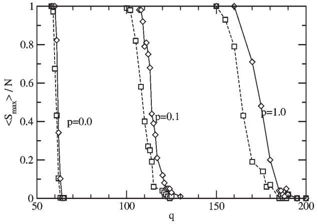

FIG. 3. The average order parameter $\langle S_{max} \rangle / N$ as a function of q for three different values of the small-world parameter p. System sizes are $N = 500^2$ (squares) and $N = 1000^2$ (diamonds); number of features $F = 10$. Each plotted value is an average over 100 runs with independent rewiring ($p > 0$) and independent initial conditions.

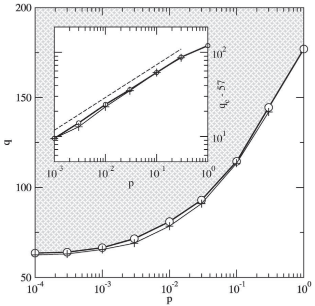

FIG. 4. Phase diagram for the Axelrod model in a small-world network. The curve separates parameter values (p, q) which produce a disordered state (shaded area) from those with ordered outcome (white area). For a given p the plotted value q_c is the one for which the value of the order parameter is closest to the, somewhat arbitrary but small, value 0.1 for system size $N = 500^2$ and $F = 10$. Inset: After subtraction of a bias $q_c(p=0) = 57$, $q_c(p)$ follows a power law $\propto p^{0.39}$ (dashed line).

- critical q_c separating ordered and the disordered state,
- q_c increases with disorder from p.

The Axelrod Model on Scale Free Networks

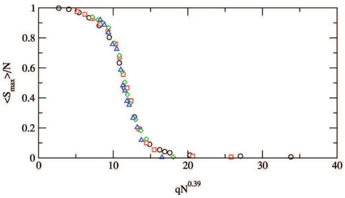

Figure: The average order parameter $\langle S_{max} \rangle / N$ in scale-free networks for $F = 10$. $N = 1000$ (circles), $N = 2000$ (squares), $N = 5000$ (diamonds), and $N = 10000$ (triangles).

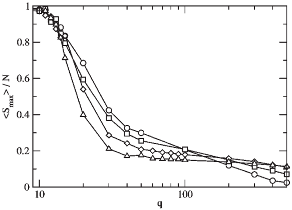

FIG. 8. The average order parameter $\langle S_{max} \rangle / N$ as a function of q for $F = 10$ in structured scale-free networks. The networks contained $N = 1000$ (circles), $N = 2000$ (squares), $N = 5000$ (diamonds), and $N = 10000$ (triangles) nodes with $F = 10$ features. Each data point is an average over 32 independent realizations.

The Kuramoto Model on Scale Free Networks[3]

- Model of synchronisation of many interacting individuals:
    - model each as a phase oscillator with attractive coupling:

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j \in \mathrm{nei}(i)}^{N} \sin(\theta_j - \theta_i) \qquad (3)$$

    - define order parameter:

$$r(t) = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \right| \qquad (4)$$

[3] Moreno and Pacheco, Europhys. Lett. (2004)

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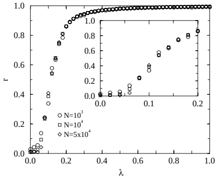


Fig. 1 – Coherence r as a function of λ for several system sizes. The onset of synchronization occurs at a critical value $\lambda_c = 0.05(1)$. Each value of r is the result of at least 10 network realizations and 1000 iterations for each N. The inset is a zoom around λ_c.

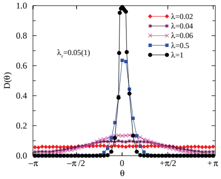
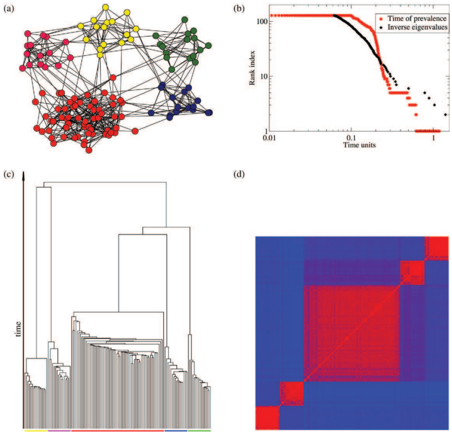


Fig. 2 – Normalized phase distributions $D(\theta)$ for different values of the control parameter λ. The curves depicted correspond to values of λ below, near and above λ_c as indicated. The network is made up of $N = 10^4$ nodes.

Community Detection using Synchronisation[4]



- Use *local order parameter*: $\rho_{ij}(t) = \langle \cos(\theta_i(t) - \theta_j(t)) \rangle$

[4] Arenas *et al.*, Physica D (2006)