# *Introduction to Dynamics on Networks*

1. *Dynamical Processes on Complex Networks*,
   A. Barrat, M. Barthélemy and A. Vespignani,
   Cambridge University Press (2008)
2. *Statistical physics of social dynamics*,
   Castellano *et al.*, Rev. Mod. Phys. 81 (2009).

## *General Framework – Background*

- Opinions, disease states, etc. modelled as variables on nodes, (e.g. binary variable: [uninfected, infected], [uninformed, informed], [against, for]...)
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- Interested in transition between *disordered* and *ordered* phases, where nodes share similar states;
  - can use tools of *statistical mechanics* and *population dynamics*.
- Results can often be translated to different contexts.

# *General Framework – Basics*

- Associate variables $x_i$ with states on node $i$, can be:
  - continuous variable, or set of variables $\mathbf{x_i} = (x_{1i}, x_{2i}, ...)$;
  - enumerated states $x_i \in \{1, 2, 3, ..., s\}$ (same meaning);
  - system configuration: $\mathbf{x} = (x_1, x_2, ..., x_N)$,
  - can define *macro-states*, e.g.: mean value $\langle \mathbf{x} \rangle$.

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- Dynamical systems theory gives tools to do this, given:
    - intial values $\mathbf{x}_0$,
    - dynamical equations: $x_i' = f_i(\mathbf{x})$, for future state $x'$.
        - e.g.: $\dot{\mathbf{x}} = f(\mathbf{x}) + \sigma L \mathbf{x}$; reaction diffusion system.

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2. Network structure not known precisely, or different for each realisation:
   - cities, friendship networks, computer networks, biological or ecological networks...
3. Initial conditions not known precisely,
   - outcomes can be very sensitive to IC,
   - often know only certain features, e.g. distributions of numbers $S$ in each state.

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  - study factors affecting outcomes by studying initial conditions.
- to gain more understanding need to simplify in meaningful way,
  - make analytically feasible.

# *Analytical Methods*

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    - individuals end up sharing same technology, language, opinion, velocity...
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- Ask questions such as:
  - what are likely equilibrium states, if they exist?
  - what are mechanisms driving ordered/polarised state?
  - what are thresholds for transitions?

# *Analytical Methods - Master Equation*

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    - instead focus on probability $P(\mathbf{x}, t)$, for particular config.
- Master equation (ME):

$$\partial_t P(\mathbf{x}, t) = \sum_{\mathbf{x}'} \left[ P(\mathbf{x}', t) W(\mathbf{x}' \to \mathbf{x}) - P(\mathbf{x}, t) W(\mathbf{x} \to \mathbf{x}') \right],$$

$W(\mathbf{x}^a \to \mathbf{x}^b)$ are transition rates.

# *ME e.g.: Growing Networks*

Master Equation for number of nodes $N_k$ with degree $k$:

- $\partial_t N_k = r_{k-1\to k} N_{k-1} - r_{k\to k+1} N_k + \delta_{k,m}$

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- ME: $\partial_t N_k = \frac{1}{N}[(1-k)N_{k-1} - kN_k] + \delta_{k,1}$
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- $n_k = n_{k-1}\frac{(k-1)}{(k+1)}$; and hence:
- $n_k = \dfrac{4}{k(k+1)(k+2)}$. I.e. $P(k) \propto k^{-3}$.

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  $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{x}' = (x_1', x_2', ..., x_N')$
- State $x_i$ of node $i$ depends only on its neighbours $j$ so:

$$W(\mathbf{x}' \to \mathbf{x}) = \prod_i w(x_i' \to x_i | x_j),$$

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  where $w$ are trans. rates conditional only on neighbours.
- Network topology now plays a role in dynamics.

- In principle possible to compute expectation value of function $A(\mathbf{x})$:

$$\langle A(t)\rangle = \sum_{\mathbf{x}} A(\mathbf{x}) P(\mathbf{x}, t).$$

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- In practice impossible to solve ME in most cases!
- Additionally most real-world applications are non-equilibrium
  - not possible to use equilibrium thermodynamics and ergodic hypothesis.

# *Approx. Solutions of Master Equation*

1. Consider appropriate projection:
   - e.g.: average number in state $x_i = a$ at time $t$:

$$\langle N_a(t) \rangle = \sum_{\mathbf{x}} \sum_i \delta_{x_i,a} P(\mathbf{x}, t),$$

     $\delta$ is Kronecker delta.
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   - average quantity $\Rightarrow$ deterministic.
2. Neglect network structure, assume homogeneous system with no correlations; prob$(x_i = a) = p_a$, for each $i$:

$$P(\mathbf{x}) = \prod_i p_{x_i}.$$

- *Mean Field* (MF) approximation.

# *(In-)Validity of Mean Field Approx.*

MF valid when:

- Variables (degrees of freedom) of system are iid:
    - generally not true, as interactions are *by definition* dependencies!
    - can get around using *pair approximation* schemes:
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- Homogeneous mixing; i.e. all $i$ have equal chance of interacting with all $j$,
  - again, network structure invalidates this,
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However:

- produces analytically tractable results,
- in many cases useful ones.

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    $= w(B \rightarrow B|x_j = B) = 1$, $w(A \rightarrow B|x_j = B) = \beta$

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- Using: $\langle N_A(t) \rangle = \sum_{\mathbf{x}} \sum_{i} \delta_{x_i,A} P(\mathbf{x}, t)$
  and $\langle N_B(t) \rangle = \sum_{\mathbf{x}} \sum_{i} \delta_{x_i,B} P(\mathbf{x}, t)$

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  and $\langle N_B(t) \rangle = \sum_{\mathbf{x}} \sum_i \delta_{x_i, B} P(\mathbf{x}, t)$
- ME: $\partial_t \langle N_B(t) \rangle = \sum_{\mathbf{x}} \sum_i \delta_{x_i, B} \delta_t P(\mathbf{x}, t)$

$$\partial_t \langle N_B(t) \rangle =$$

$$\sum_i \sum_{\mathbf{x}'} \sum_{\mathbf{x}} \left[ \delta_{x_i,B} \prod_k w(x'_k \to x_k | x'_j) P(\mathbf{x}', t) - \delta_{x_i,B} \prod_k w(x_k \to x'_k | x_j) P(\mathbf{x}, t) \right],$$

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and using the normalisation conditions:

$$\sum_{\mathbf{x}'} \prod_k w(x_k \to x'_k | x_j) = 1,$$

$$\sum_{\mathbf{x}} \delta_{x_i,B} \prod_k w(x'_k \to x_k | x'_j) = w(x'_i \to x_i = B | x'_j)$$

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LHS $w = \beta$ with prob $1 - (1 - p_B)^k$, and RHS $w = 1$ always, so:

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