The Serpentine Route to Patterns on Networks

Nick McCullen[@]
Thomas Wagenknecht
with Matthias Wolfrum at WIAS

School of Mathematics
University of Leeds

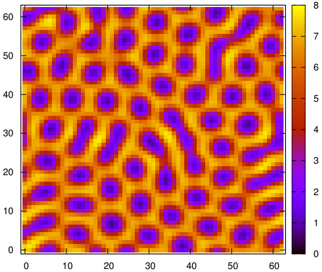
[@]Research Unit for Energy and the Design of Environments (EDEn)
Department of Architecture and Civil Engineering
University of Bath, UK

Coupled Networks, Patterns and Complexity
WIAS, Berlin
21st November 2012

Thomas Wagenknecht (1974–2012)

Patterns on Regular Networks

Turing Patterns



Reaction-diffusion system

$$\dot{u}_i = f(u_i, v_i) - D \sum_{j=1}^{N} L_{(i,j)} u_j$$

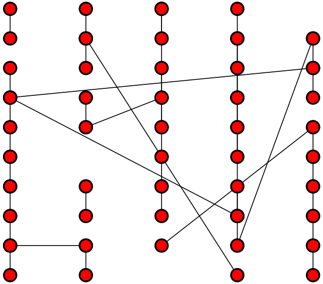
$$\dot{v}_i = g(u_i, v_i) - \sigma D \sum_{j=1}^{N} L_{(i,j)} v_j$$

- u activator, v inhibitor, $L = (L_{(i,j)})$ network Laplacian.

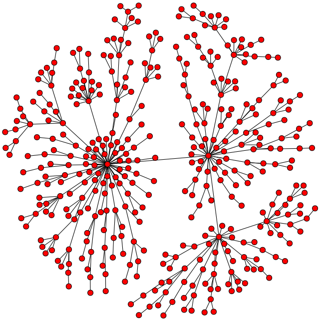
- Turing instability: stable equilibrium of $u_t = f(u, v)$, $v_t = g(u, v)$ destabilized on increase of σ
- $\Rightarrow$ emergence of alternating activator-rich and activator-low domains (periodic Turing pattern)

Complex Networks and Patterns

Watts–Strogatz (small world)



Barabasí–Albert (scale-free)



- What does it mean to have pattern on such networks?
- How can we understand the origin and spread of patterns?

Mimura-Murray model

Reaction-diffusion system:

$$\begin{aligned} \dot{u} &= f(u,v) - DLu \\ \dot{v} &= g(u,v) - \sigma DLv \end{aligned}$$

with

$$f(u,v) = \frac{au + bu^2 - u^3}{c} - uv, \quad g(u,v) = uv - v - dv^2$$

at the parameter values $a = 35$, $b = 16$, $c = 9$, $d = 2/5$.

Mimura-Murray model

Reaction-diffusion system:

$$\begin{aligned}\dot{u} &= f(u,v) - DLu \\ \dot{v} &= g(u,v) - \sigma DLv\end{aligned}$$

with

$$f(u,v) = \frac{au + bu^2 - u^3}{c} - uv, \quad g(u,v) = uv - v - dv^2$$

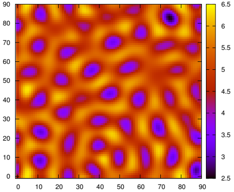
at the parameter values $a = 35$, $b = 16$, $c = 9$, $d = 2/5$.

- the model has an equilibrium at $(\bar{u}, \bar{v}) = (5, 10)$, which undergoes a *supercritical* Turing bifurcation at $\sigma = \sigma_T \approx 15.5$.

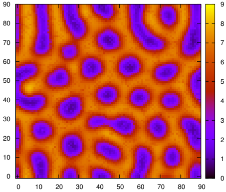
How did the leopard lose his spots?

Small-world networks

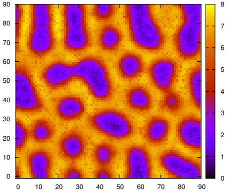
$p_r = 0$



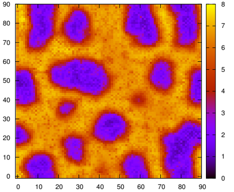
$p_r = 0.02$



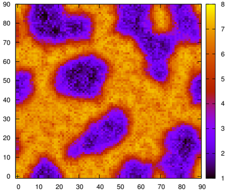
$p_r = 0.04$



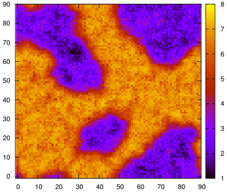
$p_r = 0.06$



$p_r = 0.08$



$p_r = 0.1$

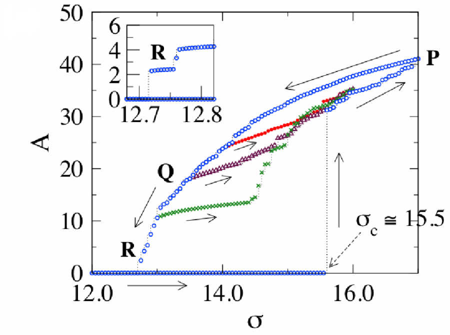


Lattice re-wired with probability p_r

Mimura-Murray on scale-free networks

Nakao and Mikhailov:

- Turing instability in large scale-free networks
- interesting differences to the continuous case:



- stable patterns exist before the homogeneous equilibrium becomes unstable (subcritical bifurcation)
- coexistence and multi-stability of a huge variety of patterns

Nakao, H. and Mikhailov, A.S., "Turing patterns in network-organized activator-inhibitor systems", *Nature Physics* (2010).

Single Differentiated Node (SDN) states

Fix all nodes at $(\bar{u}, \bar{v})$ except node k

$$\begin{aligned} \dot{u}_k &= f(u_k, v_k) + \beta(\bar{u} - u_k) \\ \dot{v}_k &= g(u_k, v_k) + \sigma\beta(\bar{v} - v_k) \end{aligned}$$

$\beta = d_k D$ (d_k = degree of node k).

- $(\bar{u}, \bar{v})$ stable for $\sigma < \sigma_T$ and unstable for $\sigma > \sigma_T$.

Wolfrum, M. "The Turing bifurcation in network systems: Collective patterns and single differentiated nodes", *Physica D: Nonlinear Phenomena*, (2012).

Single Differentiated Node (SDN) states

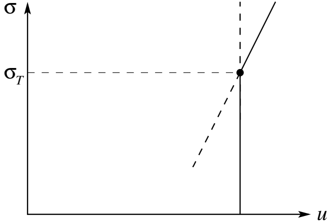
Fix all nodes at $(\bar{u}, \bar{v})$ except node k

$$\begin{aligned} \dot{u}_k &= f(u_k, v_k) + \beta(\bar{u} - u_k) \\ \dot{v}_k &= g(u_k, v_k) + \sigma\beta(\bar{v} - v_k) \end{aligned}$$

$\beta = d_k D$ (d_k = degree of node k).

- $(\bar{u}, \bar{v})$ stable for $\sigma < \sigma_T$ and unstable for $\sigma > \sigma_T$.

Transcritical bifurcation



Wolfrum, M. "The Turing bifurcation in network systems: Collective patterns and single differentiated nodes",

Physica D: Nonlinear Phenomena, (2012).

Single Differentiated Node (SDN) states

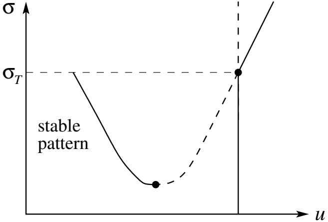
Fix all nodes at $(\bar{u}, \bar{v})$ except node k

$$\begin{aligned} \dot{u}_k &= f(u_k, v_k) + \beta(\bar{u} - u_k) \\ \dot{v}_k &= g(u_k, v_k) + \sigma\beta(\bar{v} - v_k) \end{aligned}$$

$\beta = d_k D$ (d_k = degree of node k).

- $(\bar{u}, \bar{v})$ stable for $\sigma < \sigma_T$ and unstable for $\sigma > \sigma_T$.

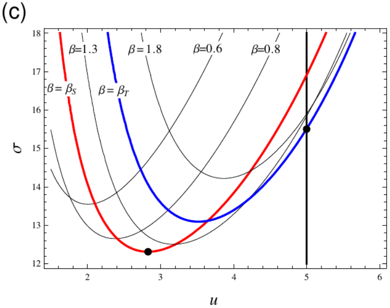
Transcritical bifurcation



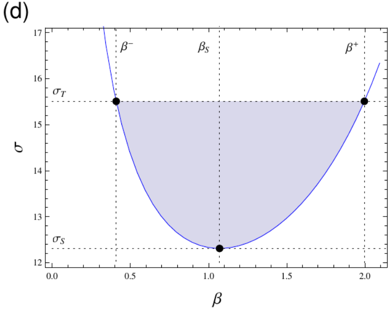
Wolfrum, M. "The Turing bifurcation in network systems: Collective patterns and single differentiated nodes", *Physica D: Nonlinear Phenomena*, (2012).

Bifurcations of SDNs

Multiple bifurcation curves

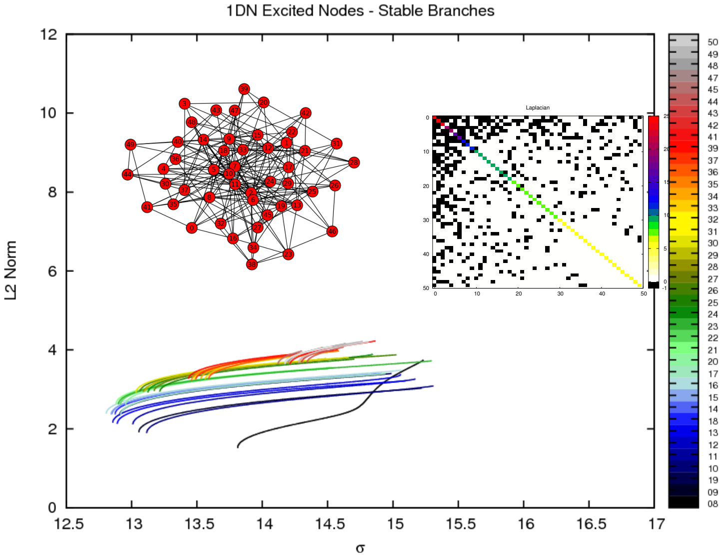


Existence region for SDNs



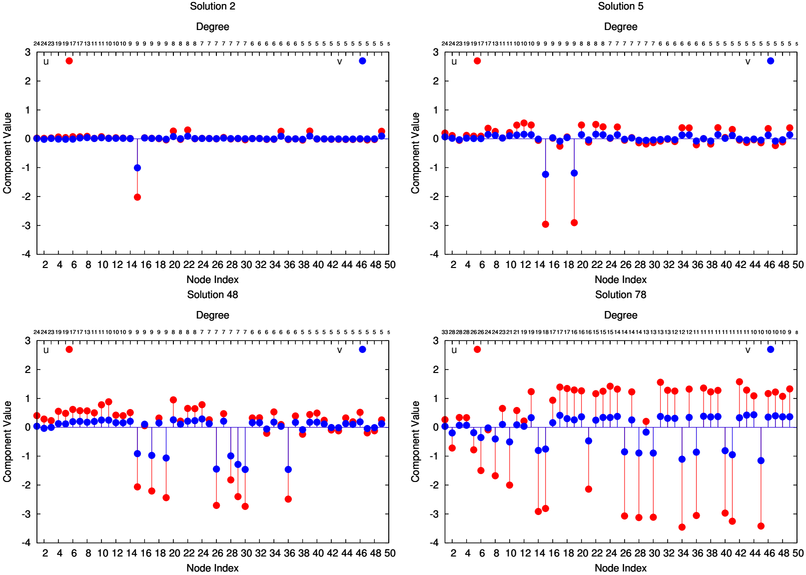
Wolfrum, M. "The Turing bifurcation in network systems: Collective patterns and single differentiated nodes", *Physica D: Nonlinear Phenomena*, (2012).

Stable SDNs on a scale-free network



⇒ good agreement with analytical region of existence

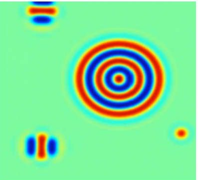
The transition to large scale activity



Localised patterns

Richter

Hunt et al.



Avitabile

Umbanhowar

Snaking Bifurcations

Swift-Hohenberg:
$u_t = ru - (1 + \partial_x^2)^2 u + b_2 u^2 - u^3,$

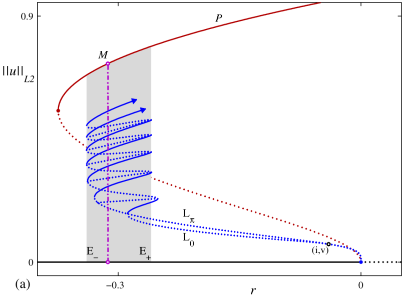


Figure: by Burke, Knobloch for $b_2 = 1.8$

Snaking Bifurcations

Swift-Hohenberg:
$u_t = ru - (1 + \partial_x^2)^2 u + b_2 u^2 - u^3,$

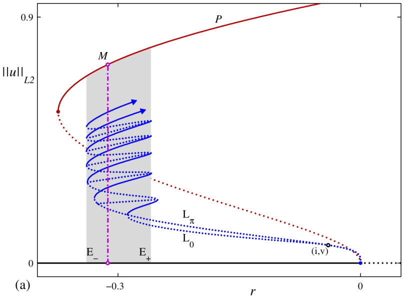


Figure: by Burke, Knobloch for $b_2 = 1.8$

Growth of Patterns

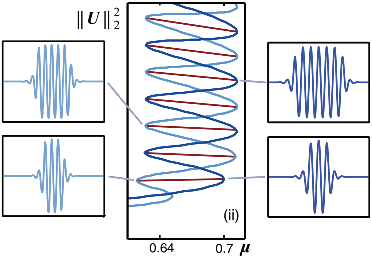
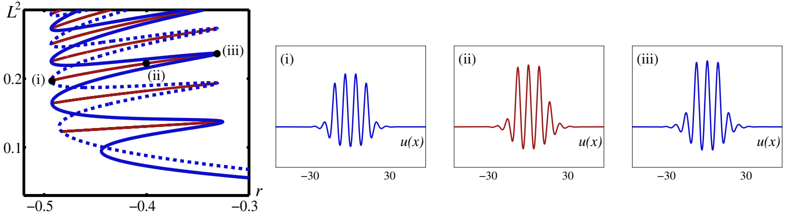
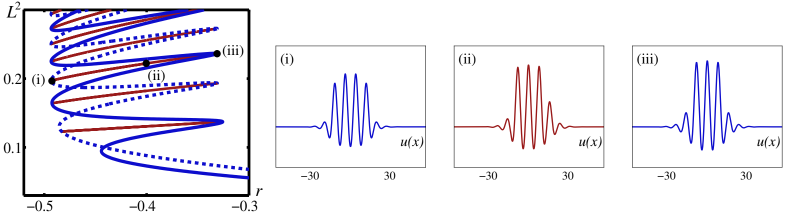


Figure: M. Beck et al., SIAM J. Math. Anal. (2009).

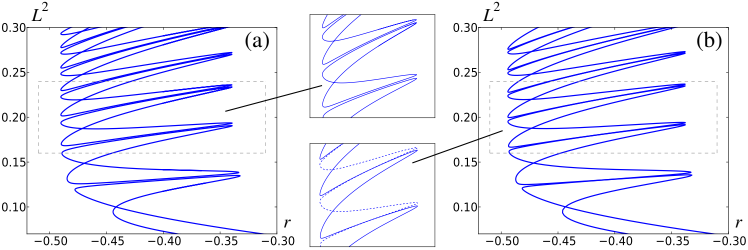
“Snakes and ladders”



“Snakes and ladders”



Symmetry Breaking



- Isolas and criss-cross snaking

Snaking Bifurcations

Swift-Hohenberg:
$u_t = ru - (1 + \partial_x^2)^2 u + b_2 u^2 - u^3,$

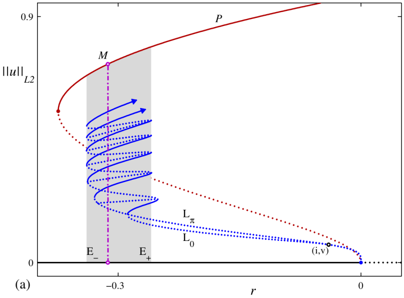


Figure: by Burke, Knobloch for $b_2 = 1.8$

Is this a snake?

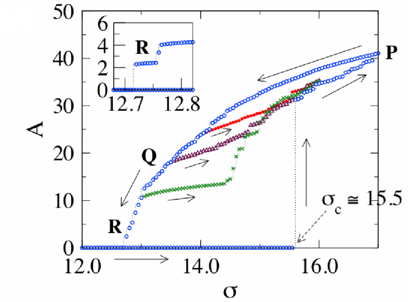
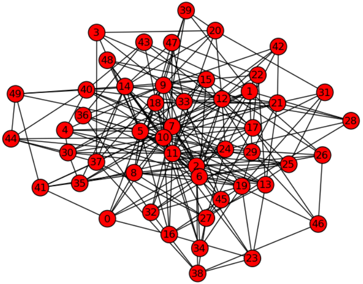
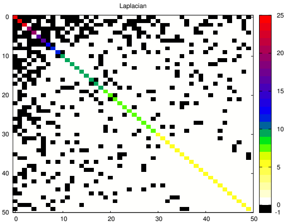


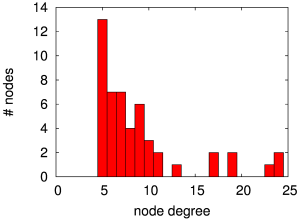
Figure: Nakao, H. and Mikhailov, A.S., (2010).

Catching the Snakes in a Net

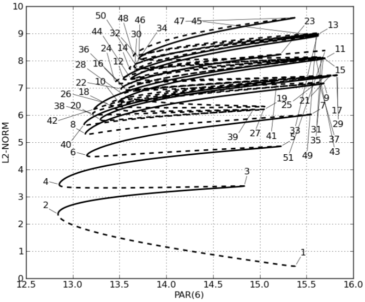
Scale-free network:



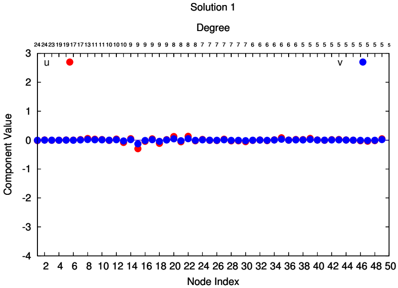




A Snake!

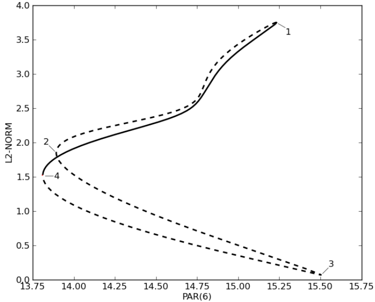


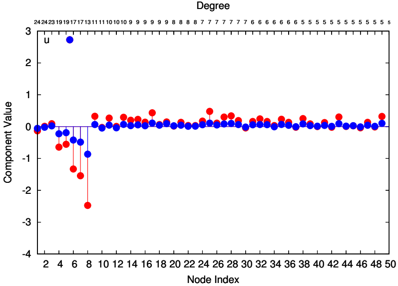
L2-NORM
PAR(6)

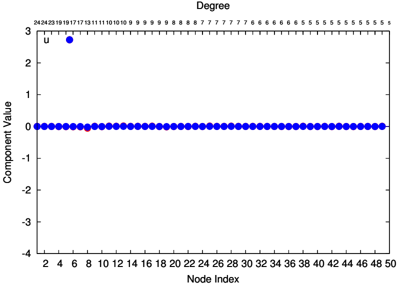


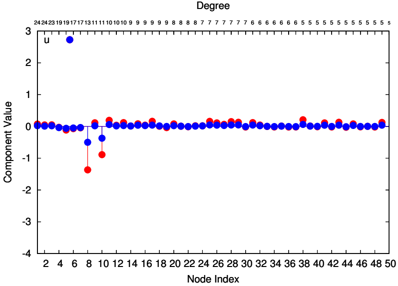
Solution 1
Degree
u
v
Component Value
Node Index

Isola Solutions



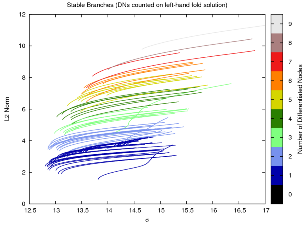




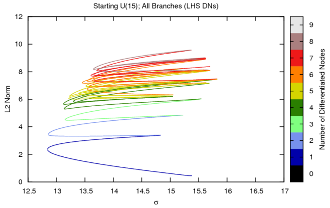


A zoo of bifurcation curves

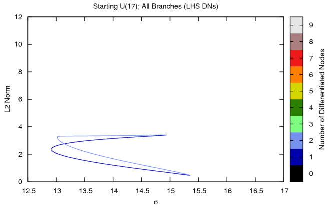
Stable Patterns



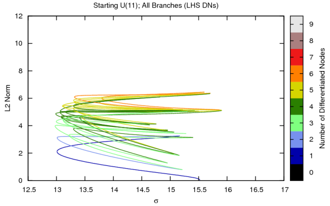
Snakes



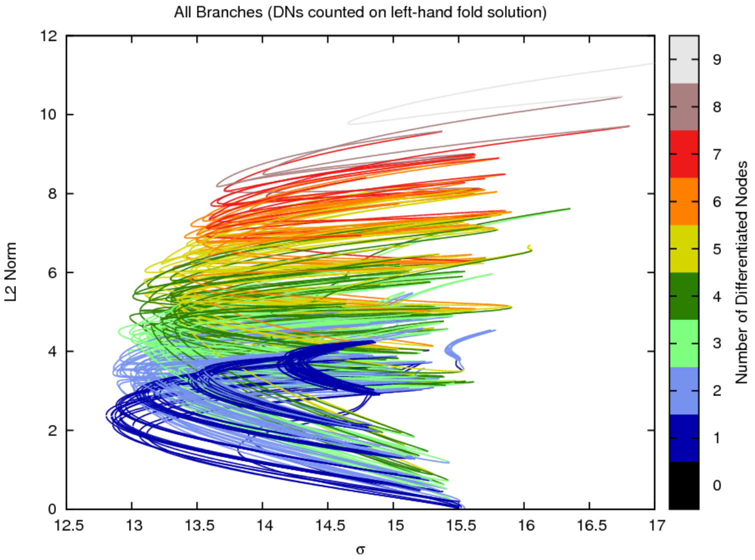
Isolas



Other Creatures

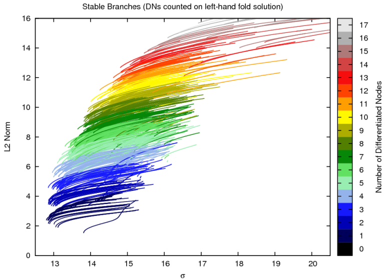


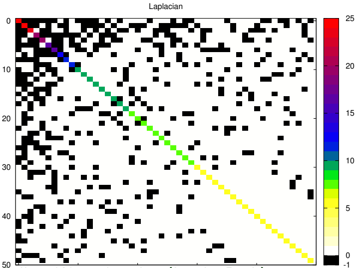
A work of art?


All Branches (DNs counted on left-hand fold solution)
L2 Norm
σ
Number of Differentiated Nodes
0
2
4
6
8
10
12
12.5
13
13.5
14
14.5
15
15.5
16
16.5
17
0
1
2
3
4
5
6
7
8
9

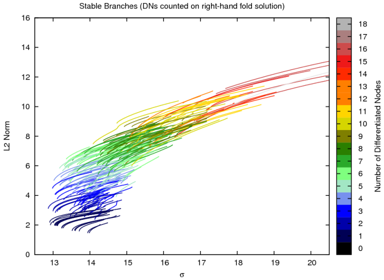
What else is going on?

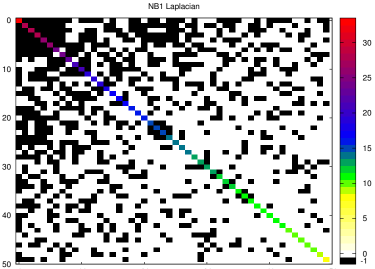
To fully developed patterns





On different networks





Conclusions

- Reaction-diffusion, activator-inhibitor models on networks
- rich set of coexisting solutions

Conclusions

- Reaction-diffusion, activator-inhibitor models on networks
- rich set of coexisting solutions
  - very complicated
  - making progress

Conclusions

- Reaction-diffusion, activator-inhibitor models on networks
- rich set of coexisting solutions
  - very complicated
  - making progress
- activity originating in SDN solutions
- development closely related to homoclinic snaking

Conclusions

- Reaction-diffusion, activator-inhibitor models on networks
- rich set of coexisting solutions
  - very complicated
  - making progress
- activity originating in SDN solutions
- development closely related to homoclinic snaking
- lots more to be investigated!

Conclusions

- Reaction-diffusion, activator-inhibitor models on networks
- rich set of coexisting solutions
  - very complicated
  - making progress
- activity originating in SDN solutions
- development closely related to homoclinic snaking
- lots more to be investigated!

Thanks to WIAS and Matthias Wolfrum

To be continued. . .