The Emergence of Patterns on Complex Networks:

Turing Instability, Snakes and Other Animals

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Thomas Wagenknecht (1974–2012)

Patterns on Regular Networks

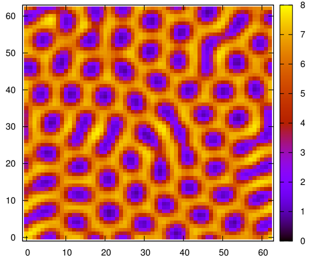
Complex Networks

Patterns of Spreading

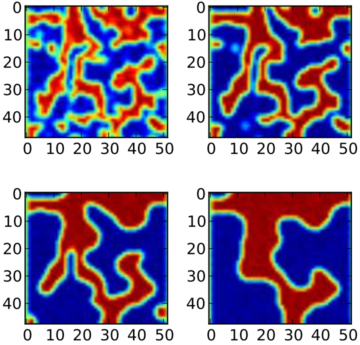
Turing Patterns on Networks

Patterns on Regular Networks

Static Patterns:

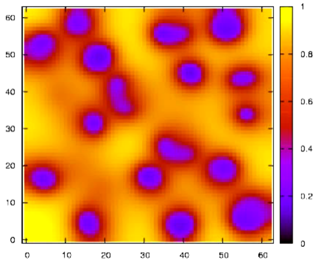


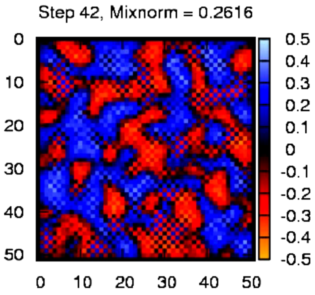
Coarsening Patterns:



Time-Delay and Dynamic Patterns

Gray-Scott model:

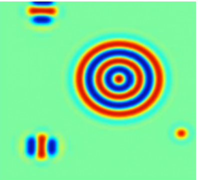




Localised patterns

Richter

Hunt et al.



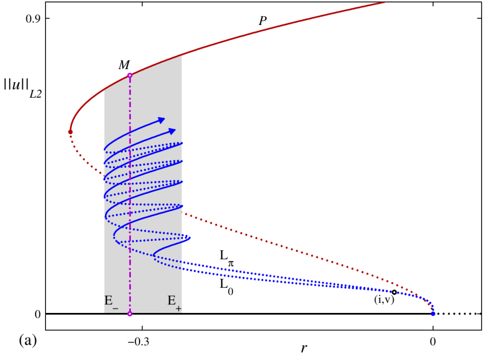
Avitabile

Umbanhowar

Snaking Bifurcations

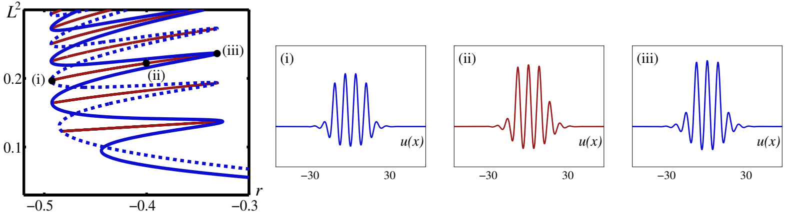
Quadratic-cubic Swift-Hohenberg equation (SH23):

$$u_t = ru - (1 + \partial_x^2)^2 u + b_2 u^2 - u^3, \qquad b_2 > 0$$

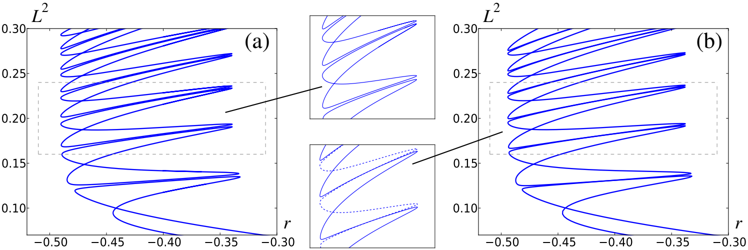


(numerical results by Burke, Knobloch for $b_2 = 1.8$)

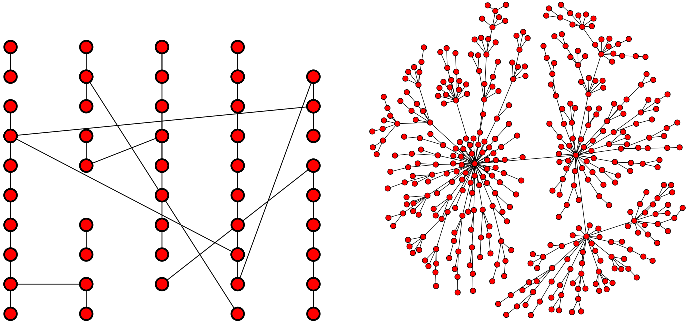
“Snakes and ladders”



Symmetry Breaking



- Isolas and criss-cross snaking

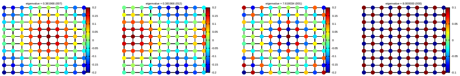


- What does it mean to have pattern on such networks?
- How can we understand the origin and spread of patterns?

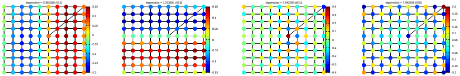
Eigenvectors

smallest two and largest two eigenvectors[1] with increasing rewiring probablility p

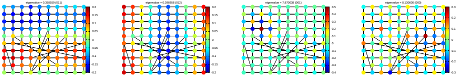
$p = 0$



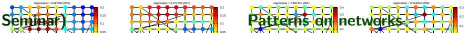
$p = 0.01$



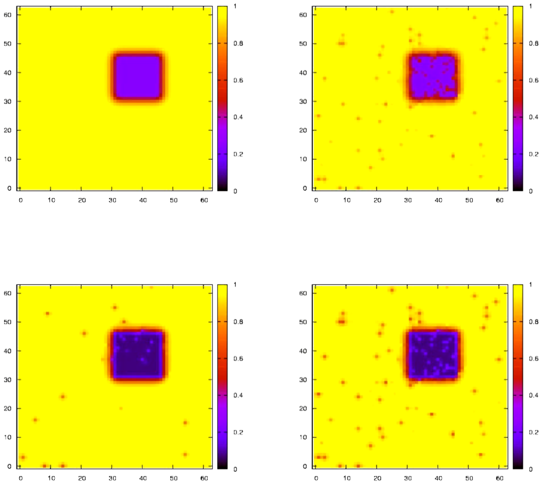
$p = 0.025$



$p = 0.05$



Re-wired patterns



Patterns of Spreading

Current state, $x_i = 0, 1$

- Total *utility* to individual:

$$u_i = \alpha_i p_i + \beta_i s_i + \gamma_i m \quad (1)$$

  - $s_i = \frac{1}{k_i} \sum_{nei(i)} A_{i,j} \mathbf{x_j}$
- p_i, s_i, m: personal, peer-group and societal influence.
- $\alpha_i, \beta_i, \gamma_i$: relative weightings given to each factor,

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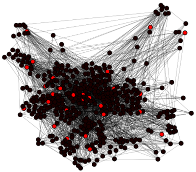
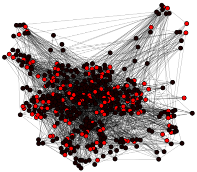
$$\text{future state: } x_i' = \begin{cases} 1 & \text{if } x_i = 1, \\ 1 & \text{if } x_i = 0 \text{ and } u_i > \theta_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

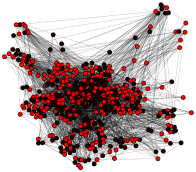
Patterns of Spreading



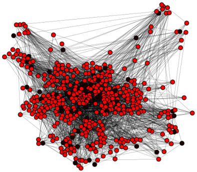
t_1



t_2

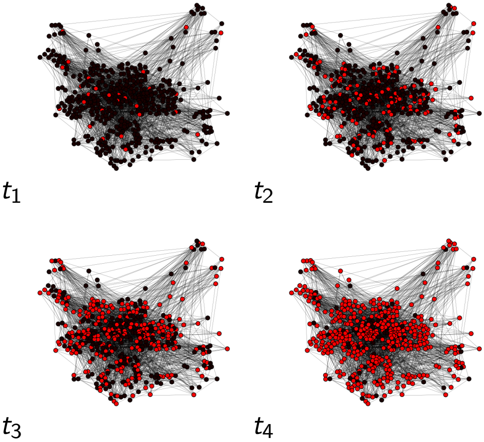


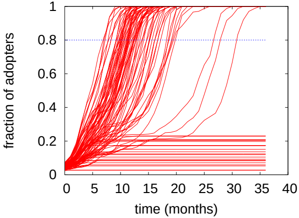
t_3



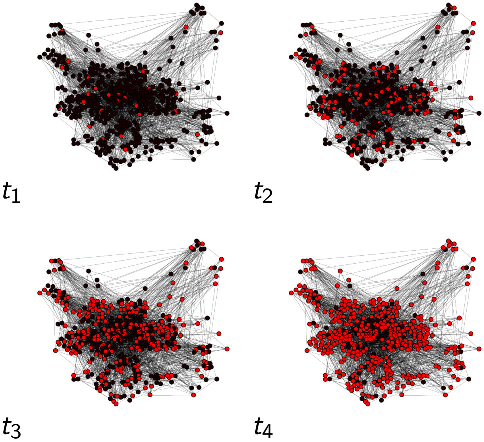
t_4

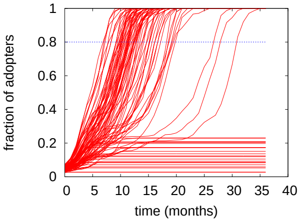
Patterns of Spreading

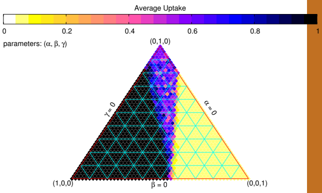




Patterns of Spreading







Neighbourhoods of Individual Nodes

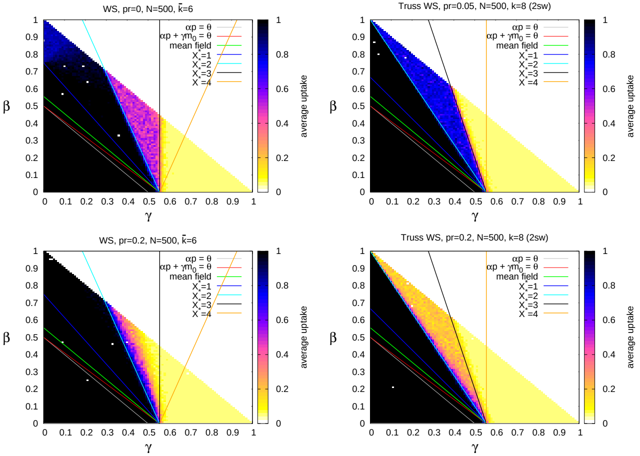
- Require critical fraction of *active* neighbours:

$$s^* = \frac{\theta - \alpha p - \gamma m}{\beta}, \tag{3}$$

- $0 < s^* \leq 1$: required *number* of active contacts:

$$X_i \equiv \sum_j A_{ij} x_j \geq \lceil k_i s^* \rceil \equiv X_i^*, \tag{4}$$

On Small-World Networks



- probability of occurence then gives expectation of success

Turing Patterns on Networks

Reaction-diffusion system:

$$\begin{aligned}\dot{u}_i &= f(u_i, v_i) - D\sum_{j=1}^{N} L_{(i,j)} u_j \\ \dot{v}_i &= g(u_i, v_i) - \sigma D\sum_{j=1}^{N} L_{(i,j)} v_j\end{aligned}$$

- u activator, v inhibitor, $L = (L_{(i,j)})$ network Laplacian.
- Turing instability: stable equilibrium of the reaction kinetics $u_t = f(u, v)$, $v_t = g(u, v)$ is destabilized by increase of bif. parameter σ
- $\Rightarrow$ emergence of alternating activator-rich and activator-low domains (periodic Turing pattern)

Mimura-Murray model

Reaction-diffusion system:

$$\begin{aligned} \dot{u} &= f(u, v) - DLu \\ \dot{v} &= g(u, v) - \sigma DLv \end{aligned}$$

with

$$f(u, v) = \frac{au + bu^2 - u^3}{c} - uv, \quad g(u, v) = uv - v - dv^2$$

at the parameter values $a = 35$, $b = 16$, $c = 9$, $d = 2/5$.

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- the model has an equilibrium at $(\bar{u}, \bar{v}) = (5, 10)$, which undergoes a *supercritical* Turing bifurcation at $\sigma = \sigma_T \approx 15.5$.

Mimura-Murray on networks

Nakao and Mikhailov studied the Turing instability in large scale-free networks and found interesting differences to the continuous case:

- Turing patterns are different from the unstable linear modes
- stable patterns exist before the homogeneous equilibrium becomes unstable (subcritical bifurcation)
- coexistence and multi-stability of a huge variety of patterns

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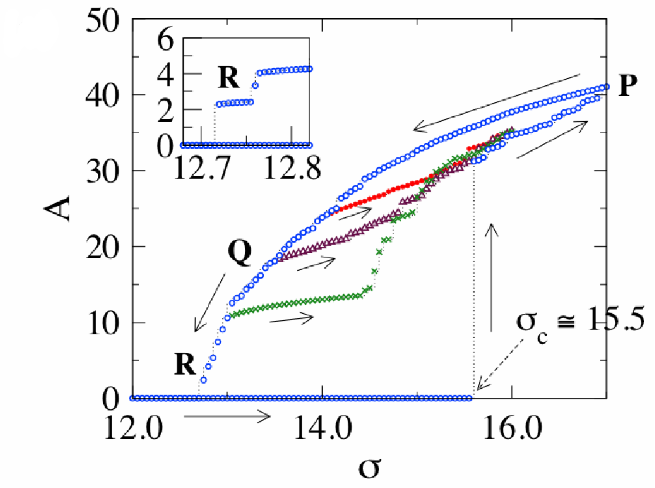
Nakao, H. and Mikhailov, A.S., "Turing patterns in network-organized activator-inhibitor systems", *Nature Physics* (2010).

Mimura-Murray on networks



Nakao, H. and Mikhailov, A.S., "Turing patterns in network-organized activator-inhibitor systems", *Nature Physics* (2010).

Single Node Dynamics

Consider the dynamics of node k under the assumption that all other nodes are fixed at the stable equilibrium $(\bar{u}, \bar{v})$

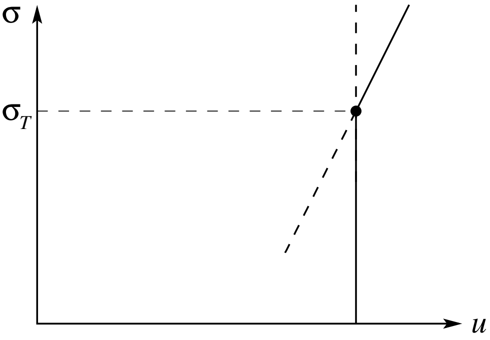
$$\begin{aligned}\dot{u}_k &= f(u_k, v_k) + \beta(\bar{u} - u_k) \\ \dot{v}_k &= g(u_k, v_k) + \sigma\beta(\bar{v} - v_k)\end{aligned}$$

where $\beta = d_k D$ (d_k is the degree of node k).

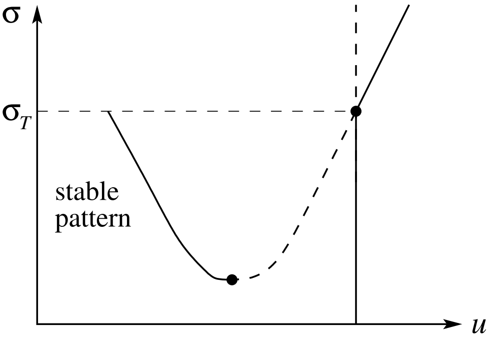
- $(\bar{u}, \bar{v})$ is an equilibrium, which is stable for $\sigma < \sigma_T$ and unstable for $\sigma > \sigma_T$.

Wolfrum, M. "The Turing bifurcation in network systems: Collective patterns and single differentiated nodes", *Physica D: Nonlinear Phenomena*, (2012).

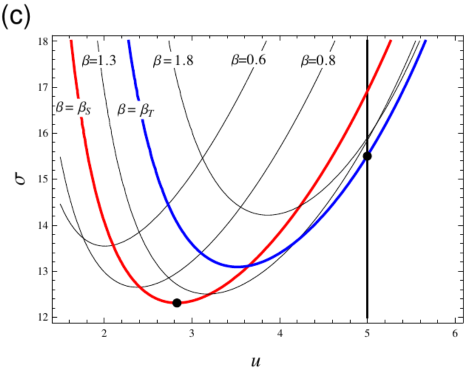
Transcritical bifurcation



Transcritical bifurcation



Bifurcation curves in Mimura-Murray



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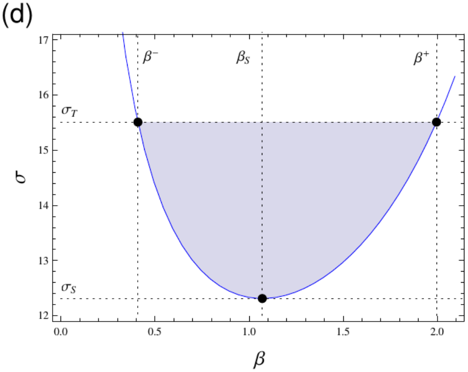
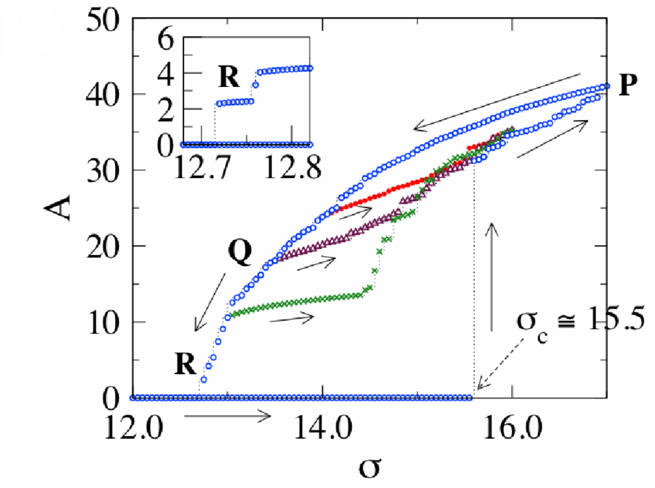
Existence region for SDNs



Wolfrum, M. "The Turing bifurcation in network systems: Collective patterns and single differentiated nodes", *Physica D: Nonlinear Phenomena*, (2012).

Snakes originating from a SDN?



Nakao, H. and Mikhailov, A.S., "Turing patterns in network-organized activator-inhibitor systems", *Nature Physics* (2010).

Catching the Snakes in a Net

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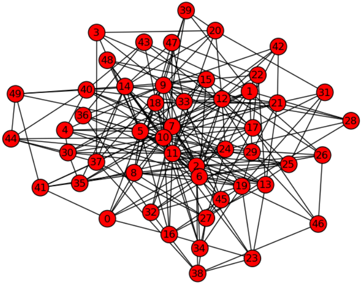
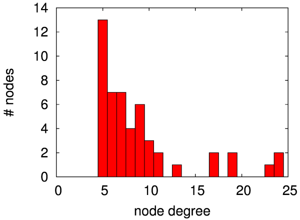
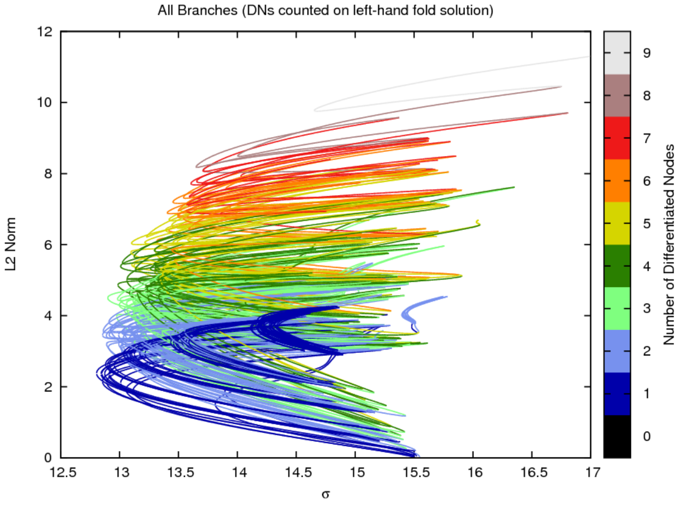
Scale-free network:

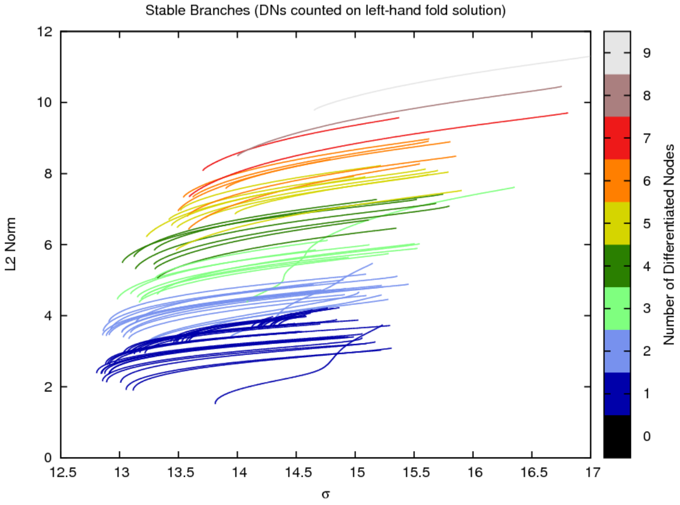




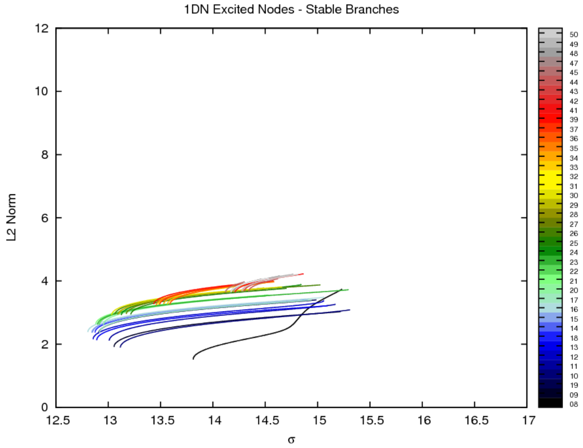
Subcritical emergence of patterns



Stable patterns

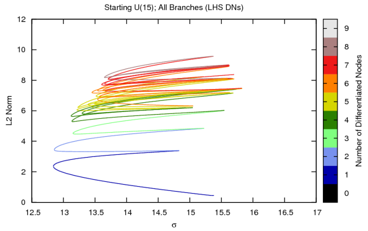


Stable SDNs

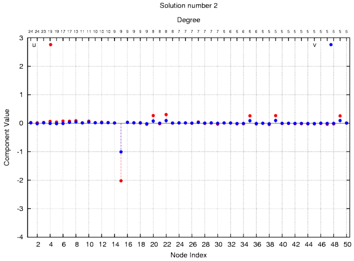


⇒ good agreement with analytical region of existence

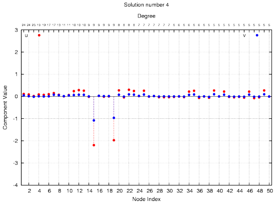
A Snake!



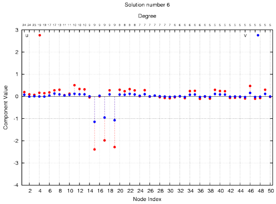
Starting U(15); All Branches (LHS DNs)
L2 Norm
σ
Number of Differentiated Nodes



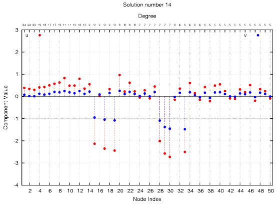
Solution number 2
Degree
Component Value
Node Index
u
v



Solution number 4
Degree
Component Value
Node Index

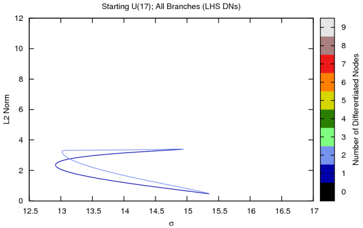


Solution number 6
Degree
Component Value
Node Index



Solution number 14
Degree
Component Value
Node Index

A zoo of bifurcation curves



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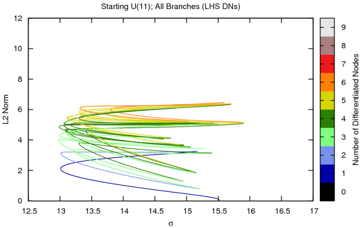

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Conclusions

- networks very interesting
  - lots of real-world applications
  - reaction-diffusion systems
- two main approaches:
  1. global patterns: look at analytical properties of connectivity matrices:
     - linear analysis: eigenvalues, eigenvectors
  2. local behaviour and emergence of pattern and spreading