# Sensitive signal detection using a feed-forward oscillator network

N. J. M$^C$Cullen* and T. Mullin†

*Manchester Centre for Nonlinear Dynamics, The University of Manchester, Manchester M13 9PL, United Kingdom*

M. Golubitsky‡

*Department of Mathematics, University of Houston, Houston, TX 77204-3008.*
(Dated: May 14, 2007)

We present the results of an experimental investigation of a network of nonlinear coupled oscillators which are coupled in feed forward mode. By exploiting the nonlinear response of each oscillator near its intrinsic Hopf bifurcation point we have found remarkable amplification of small signals over a narrow bandwidth with a large dynamic range. The effect is exploited to extract a small amplitude periodic signal from an input time series which is dominated by noise. Specifically, we have used this relatively simple experimental system to measure responses with a bandwidth of $\sim 1\%$ of the central frequency, amplifications of $\sim 60dB$, a dynamic range of $\sim 80dB$ and can extract signals from a time series with a signal to noise ratio of $\sim -50dB$.

Systems that can detect and amplify signals at specific frequencies are commonplace in the natural world and most notably in the visual and auditory systems of animals [1]. These have excellent filtering characteristics and they operate over a remarkably wide range of levels. Scientists and engineers frequently take inspiration from Nature and signal detection is one area where biology has excelled in producing systems with superior characteristics over man-made devices [2]. A measure which can be used to characterize the quality of a system is the ratio of the minimum to the maximum amplitude of signal that can be usefully detected and this is termed the *dynamic range* of the system. For example, the human auditory system has a quasi-logarithmic amplitude response and a dynamic range covering several orders of magnitude in sensitivity ($> 120dB$) [3]. It has been suggested that obtaining larger amplifications for lower forcing and saturation at large input amplitudes, which is termed *dynamic compression*, could be produced by making use of the nonlinear growth characteristic of Hopf bifurcations [4–6]. Several models exist which employ active elements which mimic the auditory network, where cells are tuned close to a Hopf bifurcation. Cells with properties which are qualitatively similar to van der Pol oscillators are believed to be responsible for amplification in some of these cases [7]. Physiological evidence exists for this *active audition* due to Hopf bifurcations for a range of animals and insect auditory systems [8–11]. Models have also been proposed which make use of coupling between limit cycles, which result from Hopf bifurcations, to produce significant amplification in insect hearing [12, 13].

Current signal detectors used in lock-in amplifiers usually have a small dynamic range compared to those found in the natural world. This is because of their linear amplitude response, necessitating advanced electronic circuitry to extend their range. Previous work on nonlinear amplifiers include using the sensitivity close to pe-
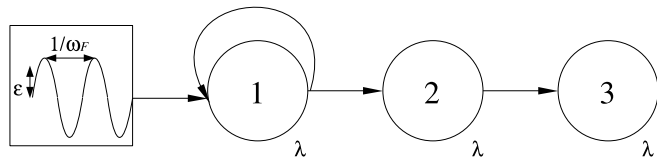


FIG. 1: A schematic of the 3-cell feed-forward network with periodic forcing. Three identical oscillators (or "cells") are coupled unidirectionally as indicated by the arrows. A small forcing signal with amplitude $\varepsilon$ and frequency $\omega_F$ is put into cell-1, triggering oscillation in the system.

riod doubling and Hopf bifurcations [14–16] to provide nonlinear amplification. They found that a period doubling instability was a good candidate for amplification of small signals and tuning closer to the bifurcation point produces greater amplification.

In this Letter we present results of an investigation of a system for small signal detection using a feed-forward network. Specifically, experimental observations are reported for a system of three coupled electronic oscillators, which demonstrate nonlinear amplification over a narrow bandwidth and wide dynamic range. This is achieved using nonlinear processes motivated by those reported for natural systems. Thus advantages are provided over devices currently used in signal processing.

Recently, an interesting set of predictions arose out of a theoretical and numerical investigation of a set of identical coupled ODEs which we will call a *coupled cell system*. In this the elements were coupled identically in a specific way to form a *network* [17]. The system comprised a 3-cell feed-forward linear array of identical oscillators as shown schematically in Figure 1 which also includes external forcing. In order to maintain symmetry of the equations additional external self coupling of the first cell is added. This system can be expressed in the

form:

$$\begin{aligned}
\dot{x}_1 &= f(x_1, x_1, \lambda), \\
\dot{x}_2 &= f(x_2, x_1, \lambda), \\
\dot{x}_3 &= f(x_3, x_2, \lambda),
\end{aligned} \qquad (1)$$

where $\lambda$ is a bifurcation parameter which is identical in all three cells. A Hopf bifurcation exists in the system at a particular value of $\lambda = \lambda_c$ such that stable periodic solutions existed in cells 2 and 3 above $\lambda_c$ while cell 1 remained at a stationary fixed point. One surprising result was that the growth in amplitude of the limit cycle in cell 3 was found to scale as $\lambda^{\frac{1}{6}}$ rather than the usual $\lambda^{\frac{1}{2}}$ dependence for the growth after a standard Hopf bifurcation [18, 19]. This suggests that resonant forcing of the third cell is more important than for a simple Hopf bifurcation.

The idea here is to use such a network to selectively amplify an input signal at a specific frequency, the one which arises at the Hopf bifurcation of the system $\omega_H$. The investigation was focused on the configuration shown in Figure 1, with small harmonic forcing with an amplitude $\varepsilon$ and variable frequency $\omega_F$. The system can then be modeled by:

$$\begin{aligned}
\dot{x}_1 &= f(x_1, x_1 + \varepsilon \cos(\omega_F t), \lambda), \\
\dot{x}_2 &= f(x_2, x_1, \lambda), \\
\dot{x}_3 &= f(x_3, x_2, \lambda).
\end{aligned} \qquad (2)$$

Experimental investigations were carried out using a set of coupled electronic circuits. The individual "cells" of the array were modified van der Pol autonomous oscillators. Each element consisted of an LCR loop in parallel with a chain of diodes which provided a nonlinear element [20]. Details of the circuit and the report of an extensive investigation of its dynamics can be found in [21, 22]. A modification to the original design was used in the present circuit in that a solid-state gyrator replaced the LCR loop element of the circuit. This had the effect of reducing the effects of external noise since this is known to produce additional dynamical effects near Hopf bifurcation points [23]. Coupling was achieved using high gain operational amplifiers to unidirectionally connect the circuits without feedback [24, 25]. Although our experimental system was manufactured from individual elements, the design is such that it should be possible to construct it entirely on a single chip in the future. The coupling strength between the cells was kept fixed at $\sim 10\%$ i.e. the connecting circuits between the oscillators had a set reduction of $\sim 90\%$ in amplitude. A water cooled copper heat-sink and a thermally insulating enclosure was used to stabilize the temperature environment of the devices to within $0.02^oC$. This was required since it is known that nonlinear oscillators are both sensitive to noise close to bifurcation points and thermal fluctuations can influence the nonlinear elements which causes the Hopf bifurcation point to "drift" in $\lambda$ [23].
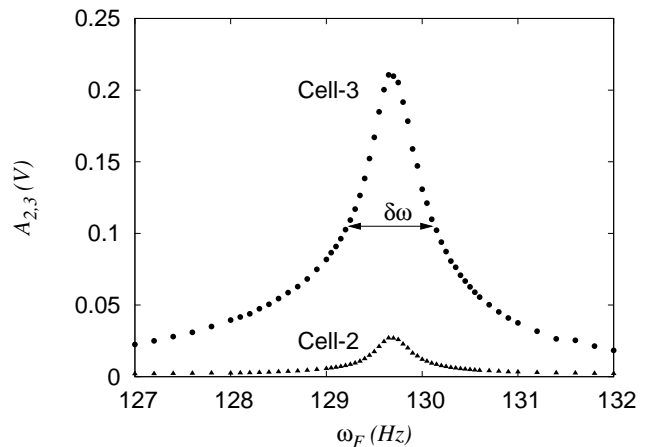


FIG. 2: Responses of the 3-cell feed-forward network, on variation of the drive frequency $\omega$. The responses of cells 2 and 3 are plotted separately to show the enhancement in amplitude obtained by adding a third cell. A sharp peak can be seen when the forcing frequency $\omega_F$ is around $\omega_F = \omega_H$, the natural frequency of the system at the Hopf bifurcation.

The initial investigation was focused on the response of cell 3 as a function of $\omega$ when forcing near $\omega_H$. The amplitude responses of cells 2 and 3, $A_2$ and $A_3$, are shown in Figure 2 for a forcing amplitude of $\varepsilon \sim 5 \times 10^{-4} V$. A sharp response can be seen around $\omega_F = \omega_H$ in cell-3, decaying quickly as $|\omega_F - \omega_H|$ increases. The amplification enhancement from cell-2 to cell-3 can clearly be seen. The bandwidth of the frequency–amplitude curve, measured as the full width at half maximum (FWHM) frequency spread, is $\delta\omega \approx 1\%$ of the central frequency. This corresponds to a *quality factor* $Q \approx 100$, demonstrating that the system has a narrow passband.

The system was found to exhibit much greater amplification for lower forcing amplitudes which highlights the nonlinear nature of the response. An amplitude response curve which was measured over a range of $\varepsilon$ at fixed $\omega_F \approx \omega_H$ is shown in Figure 3. Values of the amplitude of the third cell $A_3$ for a range of forcing amplitudes $\varepsilon$ are given in $dB$, relative to the noise level $\nu$. The system can be seen to have a wide dynamic range, measured to be $\gtrsim 80dB$, which compares well to the figure of $120dB$ given for biological systems [3]. It can be proved that for sufficiently small epsilon the response of the system varies linearly with $\varepsilon$ [5] and the data in Figure 3 are in accord with this as indicated by the line of slope 1. Montgomery, Silber, and Solla [26] find a "phase transition" in a related system, where for larger $\varepsilon$ a high power response is seen, which is qualitatively similar to the results reported here. This is an indication of the dynamic compression arising out of the nonlinear response of the system.

An indication that the device has good filtering characteristics can be gleaned from Figure 3 where the data
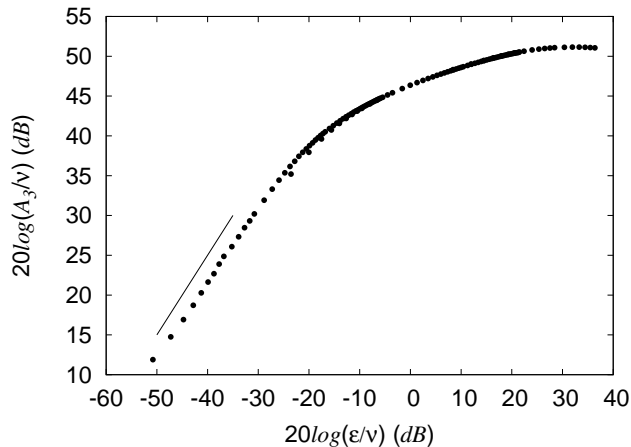
FIG. 3: The amplitude found in cell-3 relative to the noise floor is shown on a log scale, plotted as a function of the signal-noise ratio. The line has a slope of one which indicates that the growth in amplitude is approximately linear for small $\varepsilon$. The decrease in gradient for large values of $\varepsilon$ demonstrate the dynamic compression which arises from the nonlinear response of the system.

suggests that periodic signals with a low signal to noise ratio can be detected. We illustrate this by showing the results of an investigation of extracting a small amplitude periodic signal from an input which was dominated by broadband noise. An example of the time sequence used is shown in the inset of Fig. 4 (a). We also show the averaged power spectrum of the input and output signals in Fig. 4 plotted on (a) on a linear and (b) on a log scale. The response frequency $\omega_R$ remained locked to the input ($\omega_F$) because of frequency entrainment.

A striking feature of the original unforced model is the prediction of $\lambda^{\frac{1}{6}}$ amplitude growth for the third cell, as compared with the $\lambda^{\frac{1}{2}}$ amplitude growth for the second cell [17, 19]. This result motivates the investigation of the signal amplification of the forced system in cells two and three. The growth rates in the unforced system suggest that when the periodic forcing is near the Hopf frequency there should be substantial amplitude growth in the third cell when compared to that of the second cell. This can be seen in Figure 2.

In the experiment we measured the ratio of the amplitude responses of cells two and three over a narrow range of forcing frequencies. The resulting plot of the amplification of $A_2$ and $A_3$ ($\Gamma_2$ and $\Gamma_3$, in $dB$) for data across a range of $\omega_F$ close to the peak response is shown in Figure 5. Also plotted in Figure 5 is a line with slope $m = 1/3$. A linear relationship can be seen, with a least squares fit estimate of the amplitude ratio of $0.3687 \pm 0.001602$. Although theory does not predict a scaling law in the amplitude response of the second cell as a function of $\omega_F - \omega_H$, it is curious that the slope that appears in the amplitude of solutions is the same one in that appears in
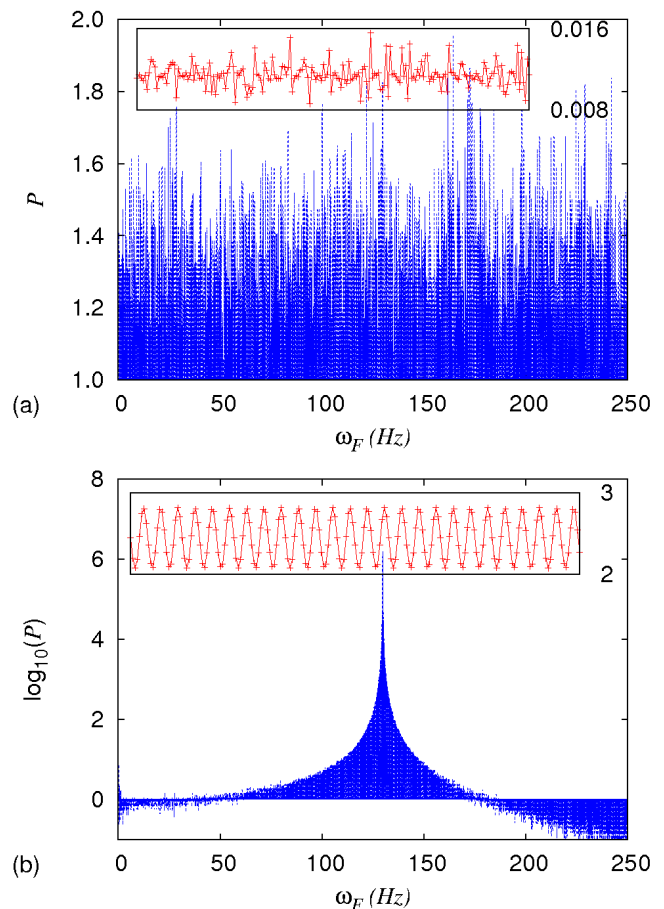


FIG. 4: Plots of typical power spectra of the forcing signal and output of cell-3, respectively. In (a) a linear scale is used and a log scale is used in (b), demonstrating that the output signal is almost purely harmonic. The time series were sampled at 1 kHz for ten seconds to enable good averages for the spectra to be obtained. Insets show example portions of the time sequences of the respective signals.

solutions emanating from a Hopf bifurcation in the unforced system as $\lambda$ is varied. This perhaps suggests that that the 1/3 power amplitude growth between neighboring cells is robust.

In summary, we have demonstrated how a coupled nonlinear oscillator system can be used to detect small periodic signals embedded in large amplitude broadband noise. The exact mechanism underlying the resonant interaction in the chain of coupled oscillators is not well understood at present. Encouraging results have been obtained using a relatively simple experimental set-up to produce significant amplification over a very narrow bandwidth. The system required careful balancing for optimal results but once this was achieved, the theoretically predicted amplitude enhancement in the third cell was found. Crucially, the system has a wide dynamic range and many features comparable to that in natural auditory systems, providing a good model for mech-
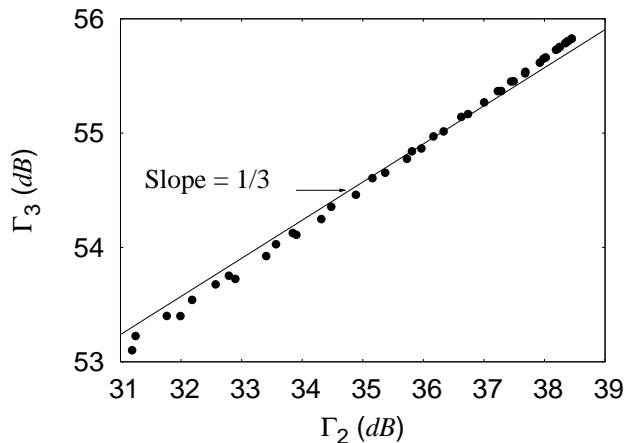
FIG. 5: The amplification found in cells 2 and 3 ($\Gamma_2$ and $\Gamma_3$, respectively) in the range of $\omega_F$ between 129.5 and 130 Hz are shown, in $dB$. The line indicates a gradient of 1/3 over this range of $\omega_F$.

anisms involved in hearing. In this we envisage a bank of such resonant cells arranged in parallel and all tuned to slightly different frequencies to cover a broad spectral range.

These desirable features give the system advantages over existing techniques used in science and industry for signal detection. This prototype system would benefit from further development using high tolerance, well balanced and controlled circuits which could be done using modern chip technology. Indeed, biological systems are believed to achieve the control required close to bifurcation points using a feedback mechanism termed "self-tuned Hopf bifurcation" [27] which has been extensively investigated theoretically [28, 29]. A signal detector based on the principals introduced in this Letter could provide significant advantages over current devices. In addition, not only electronic oscillators could be used as the basic "cells" of the system. The model providing the motivation for the current study is generic to any type of oscillator close to a Hopf bifurcation. Therefore, in principle, other physical systems could be constructed which operate over a wide range of frequencies. Applications using the principles outlined here might include coupled lasers, neural networks or mechanical systems.

\* Electronic address: n.mccullen@bath.ac.uk
† Electronic address: tom@reynolds.ph.man.ac.uk
‡ Electronic address: mg@uh.edu

[1] L. E. Kinsler, *Fundamentals of acoustics* (Wiley, 1982), chap. 11, 3rd ed.
[2] J. Watkinson, *The Art of Digital Audio* (Focal Press, 1994), chap. 3, 2nd ed.
[3] B. C. J. Moore, *An introduction to the psychology of hearing* (London : Elsevier Science, 2003), chap. 4, 5th ed.
[4] J. Harte, S. Elliott, and H. Rice, The Journal of the Acoustical Society of America **117**, 3777 (2005).
[5] V.M. Eguíluz, M. Ospeck, Y. Choe, A. Hudspeth, and M. Magnasco, Phys. Rev. Lett. **84(22)**, 5232 (2000).
[6] T. Gold and R. Pumphrey, Proceedings of the Royal Society of London. Series B, Biological Sciences **135**, 462 (1948).
[7] R. Stoop, A. Kern, M. Göpfert, D. Smirnov, T. Dikanev, and B. Bezrucko, European Biophysics Journal **35**, 511 (2006).
[8] F. Jülicher, C. R. Acad. Sci. Paris Sér. IV pp. 849–860 (2001).
[9] M. C. Göpfert and D. Robert, **100 (9)**, 5514 (2003).
[10] P. Martin and A. J. Hudspeth, PNAS **96**, 14306 (1999), http://www.pnas.org/cgi/reprint/96/25/14306.pdf, URL http://www.pnas.org/cgi/content/abstract/96/25/14306.
[11] M. Ospeck, V. Eguiluz, and M. Magnasco, Biophysical Journal **80**, 2597 (2001).
[12] R. Stoop, J.-J. V. Vyver, and A. Kern, Microscopy Res. and Tech. **63**, 400 (2004).
[13] A. Kern and R. Stoop, Physical Review Letters **91**, 128101 (2003).
[14] K. Wiesenfeld and B. McNamara, Phys. Rev. Lett. **55**, 13 (1985).
[15] K. Wiesenfeld, A. R. Bulsara, and M. E. Inchiosa, Phys. Rev. B **62 (14)**, R9232 (2000).
[16] B. Derighetti, M. Ravani, R. Stoop, P. F. Meier, E. Brun, and R. Badii, Phys. Rev. Lett. **55 (17)**, 1746 (1985).
[17] M. Golubitsky, M. Nicol, and I. Stewart, J. Nonlin. Sci. **14(2)**, 207 (2004).
[18] S. H. Strogatz, *Nonlinear dynamics and chaos* (Westview press, 1994).
[19] T. Elmhirst and M. Golubitsky, SIAM J. Appl. Dynam. Sys. **5**, 205 (2006).
[20] M. Shinriki, M. Yamamoto, and S. Mori, Proc. I.E.E.E. **69**, 394 (1981).
[21] J. J. Healey, D. S. Broomhead, K. A. Cliffe, R. Jones, and T. Mullin, Physica D **48**, 332 (1991).
[22] Paul Glendinning, J. Abshagen, and T. Mullin, Phys. Rev. E **64**, 036208 (2001).
[23] A. Juel, A. G. Darbyshire, and T. Mullin, Proc. R. Soc. Lond. A **453**, 2627 (1997).
[24] N. J. McCullen and P. Moresco, Phys. Rev. E. **73**, 46203 (2006).
[25] P. Kerr-Delworth, Ph.D. thesis, University of Manchester (1999).
[26] K. A. Montgomery, M. Silber, and S. A. Solla, *Amplification in the auditory periphery: the effect of coupling tuning mechanisms* (2007), URL http://www.citebase.org/abstract?id=oai:arXiv.org:nlin/070102%2
[27] S. Camelet, T. Duke, F. Jülicher, and J. Prost, Proc. Natl. Acad. Sci. USA **97(7)**, 3183 (2000).
[28] L. Moreau and E. D. Sontag, Phys. Rev. E. **68(2)**, 020901 (2003).
[29] L. Moreau, E. D. Sontag, and M. Arcak, Syst. Cont. Lett. **50(3)**, 229 (2003).