New results on the jerky crack growth in elastoplastic materials

Gianni Dal Maso, SISSA, Trieste, Italy
and
Rodica Toader, University of Udine, Italy

One World PDE Seminar, May 18, 2021

Brittle fracture in elastic materials

- Many mathematical models for the quasistatic growth of brittle cracks in elastic material have been developed in recent years. They are based on the ideas of the seminal work by Francfort and Marigo (1998), who revisited Griffith's theory (1920) of brittle fracture and made a connection with energy minimization problems.
- All these models can be formulated in the framework of Mielke's variational approach to rate-independent evolution problems: at each time the state of the system satisfies a minimality property and an energy-dissipation balance.
- In is well known that in real materials the crack front is surrounded by a plastic zone. Therefore the models of crack growth in elastic materials describe only the limit case in which the plastic zone is negligible. More realistic models should consider crack growth in elastoplastic materials.

Brittle fracture in elastic materials

- Many mathematical models for the quasistatic growth of brittle cracks in elastic material have been developed in recent years. They are based on the ideas of the seminal work by Francfort and Marigo (1998), who revisited Griffith's theory (1920) of brittle fracture and made a connection with energy minimization problems.
- All these models can be formulated in the framework of Mielke's variational approach to rate-independent evolution problems: at each time the state of the system satisfies a minimality property and an energy-dissipation balance.
- In is well known that in real materials the crack front is surrounded by a plastic zone. Therefore the models of crack growth in elastic materials describe only the limit case in which the plastic zone is negligible. More realistic models should consider crack growth in elastoplastic materials.

Brittle fracture in elastic materials

- Many mathematical models for the quasistatic growth of brittle cracks in elastic material have been developed in recent years. They are based on the ideas of the seminal work by Francfort and Marigo (1998), who revisited Griffith's theory (1920) of brittle fracture and made a connection with energy minimization problems.
- All these models can be formulated in the framework of Mielke's variational approach to rate-independent evolution problems: at each time the state of the system satisfies a minimality property and an energy-dissipation balance.
- In is well known that in real materials the crack front is surrounded by a plastic zone. Therefore the models of crack growth in elastic materials describe only the limit case in which the plastic zone is negligible. More realistic models should consider crack growth in elastoplastic materials.

Cracks in elastoplastic materials

- The quasistatic evolution problem for linearly elastic-perfectly plastic materials (without cracks) was studied in 1981 by Suquet, who obtained the existence of a solution in the space $BD(\Omega)$ of functions with bounded deformation, together with a uniqueness result for the stress.
- These results were revisited in 2006 by De Simone, Mora, and me in the framework of Mielke's variational approach to rate-independent evolution problems.
- A model of crack growth in elastic-perfectly plastic materials was studied by Toader and me in 2010 in the same framework. An existence result was obtained, but the main properties of the solutions remained obscure.
- In particular we were not able to answer the following questions. Can the crack growth in elastoplastic materials be continuous in time? Or rather, does every solution have an intermittent crack growth, with jumps followed by intervals where the crack is constant?

Cracks in elastoplastic materials

- The quasistatic evolution problem for linearly elastic-perfectly plastic materials (without cracks) was studied in 1981 by Suquet, who obtained the existence of a solution in the space $BD(\Omega)$ of functions with bounded deformation, together with a uniqueness result for the stress.
- These results were revisited in 2006 by De Simone, Mora, and me in the framework of Mielke's variational approach to rate-independent evolution problems.
- A model of crack growth in elastic-perfectly plastic materials was studied by Toader and me in 2010 in the same framework. An existence result was obtained, but the main properties of the solutions remained obscure.
- In particular we were not able to answer the following questions. Can the crack growth in elastoplastic materials be continuous in time? Or rather, does every solution have an intermittent crack growth, with jumps followed by intervals where the crack is constant?

Cracks in elastoplastic materials

- The quasistatic evolution problem for linearly elastic-perfectly plastic materials (without cracks) was studied in 1981 by Suquet, who obtained the existence of a solution in the space $BD(\Omega)$ of functions with bounded deformation, together with a uniqueness result for the stress.
- These results were revisited in 2006 by De Simone, Mora, and me in the framework of Mielke's variational approach to rate-independent evolution problems.
- A model of crack growth in elastic-perfectly plastic materials was studied by Toader and me in 2010 in the same framework. An existence result was obtained, but the main properties of the solutions remained obscure.
- In particular we were not able to answer the following questions. Can the crack growth in elastoplastic materials be continuous in time? Or rather, does every solution have an intermittent crack growth, with jumps followed by intervals where the crack is constant?

Cracks in elastoplastic materials

- The quasistatic evolution problem for linearly elastic-perfectly plastic materials (without cracks) was studied in 1981 by Suquet, who obtained the existence of a solution in the space $BD(\Omega)$ of functions with bounded deformation, together with a uniqueness result for the stress.
- These results were revisited in 2006 by De Simone, Mora, and me in the framework of Mielke's variational approach to rate-independent evolution problems.
- A model of crack growth in elastic-perfectly plastic materials was studied by Toader and me in 2010 in the same framework. An existence result was obtained, but the main properties of the solutions remained obscure.
- In particular we were not able to answer the following questions. Can the crack growth in elastoplastic materials be continuous in time? Or rather, does every solution have an intermittent crack growth, with jumps followed by intervals where the crack is constant?

Answer imposing a prescribed crack path

- In this talk I will present a model for the quasistatic crack growth in elastic-perfectly plastic materials, with a prescribed crack path, for which we can answer the previous question with a mathematical result.
- In this model the crack growth is always jerky. In other words, the crack length is a pure jump monotone function. Note that this happens even if the material is homogeneous.
- This agrees with the recent numerical results obtained by Brach, Tanné, Bourdin, and Bhattacharya for the quasistatic growth, and with many experimental results in the dynamic regime. As far as I know, no mathematical proof of this phenomenon was known.

Answer imposing a prescribed crack path

- In this talk I will present a model for the quasistatic crack growth in elastic-perfectly plastic materials, with a prescribed crack path, for which we can answer the previous question with a mathematical result.
- In this model the crack growth is always jerky. In other words, the crack length is a pure jump monotone function. Note that this happens even if the material is homogeneous.
- This agrees with the recent numerical results obtained by Brach, Tanné, Bourdin, and Bhattacharya for the quasistatic growth, and with many experimental results in the dynamic regime. As far as I know, no mathematical proof of this phenomenon was known.

Answer imposing a prescribed crack path

- In this talk I will present a model for the quasistatic crack growth in elastic-perfectly plastic materials, with a prescribed crack path, for which we can answer the previous question with a mathematical result.
- In this model the crack growth is always jerky. In other words, the crack length is a pure jump monotone function. Note that this happens even if the material is homogeneous.
- This agrees with the recent numerical results obtained by Brach, Tanné, Bourdin, and Bhattacharya for the quasistatic growth, and with many experimental results in the dynamic regime. As far as I know, no mathematical proof of this phenomenon was known.

Reference configuration and cracks

- The reference configuration Ω is a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$.
- To simplify the exposition, in this talk the crack path, in the reference configuration, is a segment of the form

$$\Gamma := \{(x_1, 0) : 0 \leq x_1 \leq L\} \subset \overline{\Omega},$$

  with $\Gamma \cap \partial\Omega = \{(0,0), (L,0)\}$.
- For every $0 \leq s_1 \leq s_2 \leq L$ we set $\Gamma_{s_1}^{s_2} := \{(x_1, 0) : s_1 \leq x_1 \leq s_2\}$. We assume that at each time the crack, in the reference configuration, is of the form $\Gamma_0^{s(t)}$ for some $0 \leq s(t) \leq L$. The crack tip is $x(t) := (s(t), 0)$. The energy spent to produce it is equal to its length $s(t)$.
- For every $0 \leq s \leq L$ we set $\Omega_s := \Omega \backslash \Gamma_0^s$ and $\widehat{\Omega}_s := \overline{\Omega} \backslash \Gamma_0^s$.

- The reference configuration Ω is a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$.
- To simplify the exposition, in this talk the crack path, in the reference configuration, is a segment of the form

$$\Gamma := \{(x_1, 0) : 0 \leq x_1 \leq L\} \subset \overline{\Omega},$$

with $\Gamma \cap \partial\Omega = \{(0,0),(L,0)\}$.

- For every $0 \leq s_1 \leq s_2 \leq L$ we set $\Gamma_{s_1}^{s_2} := \{(x_1, 0) : s_1 \leq x_1 \leq s_2\}$. We assume that at each time the crack, in the reference configuration, is of the form $\Gamma_0^{s(t)}$ for some $0 \leq s(t) \leq L$. The crack tip is $x(t) := (s(t), 0)$. The energy spent to produce it is equal to its length $s(t)$.
- For every $0 \leq s \leq L$ we set $\Omega_s := \Omega \setminus \Gamma_0^s$ and $\widehat{\Omega}_s := \overline{\Omega} \setminus \Gamma_0^s$.

Reference configuration and cracks

- The reference configuration Ω is a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$.
- To simplify the exposition, in this talk the crack path, in the reference configuration, is a segment of the form

$$\Gamma := \{(x_1, 0) : 0 \le x_1 \le L\} \subset \overline{\Omega},$$

  with $\Gamma \cap \partial\Omega = \{(0,0), (L,0)\}$.
- For every $0 \le s_1 \le s_2 \le L$ we set $\Gamma_{s_1}^{s_2} := \{(x_1, 0) : s_1 \le x_1 \le s_2\}$. We assume that at each time the crack, in the reference configuration, is of the form $\Gamma_0^{s(t)}$ for some $0 \le s(t) \le L$. The crack tip is $x(t) := (s(t), 0)$. The energy spent to produce it is equal to its length $s(t)$.
- For every $0 \le s \le L$ we set $\Omega_s := \Omega \setminus \Gamma_0^s$ and $\widehat{\Omega}_s := \overline{\Omega} \setminus \Gamma_0^s$.

Reference configuration and cracks

- The reference configuration Ω is a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$.
- To simplify the exposition, in this talk the crack path, in the reference configuration, is a segment of the form

$$\Gamma := \{(x_1, 0) : 0 \le x_1 \le L\} \subset \overline{\Omega},$$

  with $\Gamma \cap \partial\Omega = \{(0,0), (L,0)\}$.
- For every $0 \le s_1 \le s_2 \le L$ we set $\Gamma_{s_1}^{s_2} := \{(x_1, 0) : s_1 \le x_1 \le s_2\}$. We assume that at each time the crack, in the reference configuration, is of the form $\Gamma_0^{s(t)}$ for some $0 \le s(t) \le L$. The crack tip is $x(t) := (s(t), 0)$. The energy spent to produce it is equal to its length $s(t)$.
- For every $0 \le s \le L$ we set $\Omega_s := \Omega \backslash \Gamma_0^s$ and $\widehat{\Omega}_s := \overline{\Omega} \backslash \Gamma_0^s$.

Elastic and plastic strains

- For every open set $U \subset \mathbb{R}^2$ the space $BD(U)$ of functions of bounded deformation is defined as the space of functions $u \in L^1(U;\mathbb{R}^2)$ such that the symmetric part of the gradient $Eu := \frac{1}{2}(Du + (Du)^T)$ is a bounded Radon measure with values in the space $\mathbb{R}^{2\times 2}_{sym}$ of symmetric 2×2 matrices.
- At each time $t \in [0,T]$ the displacement $u(t)$ belongs to $BD(\Omega_{s(t)})$.
- Its strain $Eu(t)$ is additively decomposed as $Eu(t) = e(t) + p(t)$. The elastic part $e(t)$ belongs to $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$, while the plastic part $p(t)$ belongs to $\mathcal{M}_b(\widehat{\Omega}_{s(t)};\mathbb{R}^{2\times 2}_{sym})$, the space of bounded Radon measures on $\widehat{\Omega}_{s(t)}$ with values in $\mathbb{R}^{2\times 2}_{sym}$.
- The possible singular part of the measure $p(t)$ accounts for concentrated strains, which may occur in $\Omega_{s(t)}$ and also on $\partial\Omega$, where it will be interpreted as a mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition.

Elastic and plastic strains

- For every open set $U \subset \mathbb{R}^2$ the space $BD(U)$ of functions of bounded deformation is defined as the space of functions $u \in L^1(U;\mathbb{R}^2)$ such that the symmetric part of the gradient $Eu := \frac{1}{2}(Du + (Du)^T)$ is a bounded Radon measure with values in the space $\mathbb{R}^{2\times 2}_{sym}$ of symmetric 2×2 matrices.
- At each time $t \in [0,T]$ the displacement $u(t)$ belongs to $BD(\Omega_{s(t)})$.
- Its strain $Eu(t)$ is additively decomposed as $Eu(t) = e(t) + p(t)$. The elastic part $e(t)$ belongs to $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$, while the plastic part $p(t)$ belongs to $\mathcal{M}_b(\widehat{\Omega}_{s(t)};\mathbb{R}^{2\times 2}_{sym})$, the space of bounded Radon measures on $\widehat{\Omega}_{s(t)}$ with values in $\mathbb{R}^{2\times 2}_{sym}$.
- The possible singular part of the measure $p(t)$ accounts for concentrated strains, which may occur in $\Omega_{s(t)}$ and also on $\partial\Omega$, where it will be interpreted as a mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition.

Elastic and plastic strains

- For every open set $U \subset \mathbb{R}^2$ the space $BD(U)$ of functions of bounded deformation is defined as the space of functions $u \in L^1(U;\mathbb{R}^2)$ such that the symmetric part of the gradient $Eu := \frac{1}{2}(Du + (Du)^T)$ is a bounded Radon measure with values in the space $\mathbb{R}^{2\times 2}_{sym}$ of symmetric 2×2 matrices.
- At each time $t \in [0,T]$ the displacement $u(t)$ belongs to $BD(\Omega_{s(t)})$.
- Its strain $Eu(t)$ is additively decomposed as $Eu(t) = e(t) + p(t)$. The elastic part $e(t)$ belongs to $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$, while the plastic part $p(t)$ belongs to $\mathcal{M}_b(\widehat{\Omega}_{s(t)};\mathbb{R}^{2\times 2}_{sym})$, the space of bounded Radon measures on $\widehat{\Omega}_{s(t)}$ with values in $\mathbb{R}^{2\times 2}_{sym}$.
- The possible singular part of the measure $p(t)$ accounts for concentrated strains, which may occur in $\Omega_{s(t)}$ and also on $\partial\Omega$, where it will be interpreted as a mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition.

Elastic and plastic strains

- For every open set $U \subset \mathbb{R}^2$ the space $BD(U)$ of functions of bounded deformation is defined as the space of functions $u \in L^1(U;\mathbb{R}^2)$ such that the symmetric part of the gradient $Eu := \frac{1}{2}(Du + (Du)^T)$ is a bounded Radon measure with values in the space $\mathbb{R}^{2\times 2}_{sym}$ of symmetric 2×2 matrices.
- At each time $t \in [0,T]$ the displacement $u(t)$ belongs to $BD(\Omega_{s(t)})$.
- Its strain $Eu(t)$ is additively decomposed as $Eu(t) = e(t) + p(t)$. The elastic part $e(t)$ belongs to $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$, while the plastic part $p(t)$ belongs to $\mathcal{M}_b(\widehat{\Omega}_{s(t)};\mathbb{R}^{2\times 2}_{sym})$, the space of bounded Radon measures on $\widehat{\Omega}_{s(t)}$ with values in $\mathbb{R}^{2\times 2}_{sym}$.
- The possible singular part of the measure $p(t)$ accounts for concentrated strains, which may occur in $\Omega_{s(t)}$ and also on $\partial\Omega$, where it will be interpreted as a mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition.

Boundary conditions

- The evolution in the time interval $[0,T]$ is driven by a time-dependent Dirichlet boundary condition $u(t) = w(t)$ imposed on $\partial\Omega$. As usual, we assume that $w \in AC([0,T]; H^1(\Omega))$.
- In general the desired equality $u(t) = w(t)$ cannot be obtained on the whole of $\partial\Omega$, since concentrated strains may occur at the boundary.
- The weak formulation of the Dirichlet boundary condition is

$$p(t) = (w(t) - u(t)) \odot \nu_\Omega \mathcal{H}^1 \quad \text{as measures on } \partial\Omega\,,$$

  where ν_Ω is the outer unit normal to $\partial\Omega$ and $a \odot b$ is the symmetrized tensor product between two vectors $a, b \in \mathbb{R}^2$, i.e., the symmetric matrix with entries $(a_i b_j + a_j b_i)/2$.
- The restriction of $p(t)$ to $\partial\Omega$ accounts for the mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition, given by the trace of $w(t)$.

Boundary conditions

- The evolution in the time interval $[0,T]$ is driven by a time-dependent Dirichlet boundary condition $u(t) = w(t)$ imposed on $\partial\Omega$. As usual, we assume that $w \in AC([0,T]; H^1(\Omega))$.
- In general the desired equality $u(t) = w(t)$ cannot be obtained on the whole of $\partial\Omega$, since concentrated strains may occur at the boundary.
- The weak formulation of the Dirichlet boundary condition is

$$p(t) = (w(t) - u(t)) \odot \nu_\Omega \mathcal{H}^1 \quad \text{as measures on } \partial\Omega,$$

where ν_Ω is the outer unit normal to $\partial\Omega$ and $a \odot b$ is the symmetrized tensor product between two vectors $a, b \in \mathbb{R}^2$, i.e., the symmetric matrix with entries $(a_i b_j + a_j b_i)/2$.
- The restriction of $p(t)$ to $\partial\Omega$ accounts for the mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition, given by the trace of $w(t)$.

Boundary conditions

- The evolution in the time interval $[0,T]$ is driven by a time-dependent Dirichlet boundary condition $u(t) = w(t)$ imposed on $\partial\Omega$. As usual, we assume that $w \in AC([0,T]; H^1(\Omega))$.
- In general the desired equality $u(t) = w(t)$ cannot be obtained on the whole of $\partial\Omega$, since concentrated strains may occur at the boundary.
- The weak formulation of the Dirichlet boundary condition is

$$p(t) = (w(t) - u(t)) \odot \nu_\Omega \mathcal{H}^1 \quad \text{as measures on } \partial\Omega\,,$$

  where ν_Ω is the outer unit normal to $\partial\Omega$ and $a \odot b$ is the symmetrized tensor product between two vectors $a, b \in \mathbb{R}^2$, i.e., the symmetric matrix with entries $(a_i b_j + a_j b_i)/2$.
- The restriction of $p(t)$ to $\partial\Omega$ accounts for the mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition, given by the trace of $w(t)$.

Boundary conditions

- The evolution in the time interval $[0, T]$ is driven by a time-dependent Dirichlet boundary condition $u(t) = w(t)$ imposed on $\partial\Omega$. As usual, we assume that $w \in AC([0, T]; H^1(\Omega))$.
- In general the desired equality $u(t) = w(t)$ cannot be obtained on the whole of $\partial\Omega$, since concentrated strains may occur at the boundary.
- The weak formulation of the Dirichlet boundary condition is

$$p(t) = (w(t) - u(t)) \odot \nu_\Omega \mathcal{H}^1 \quad \text{as measures on } \partial\Omega\,,$$

  where ν_Ω is the outer unit normal to $\partial\Omega$ and $a \odot b$ is the symmetrized tensor product between two vectors $a, b \in \mathbb{R}^2$, i.e., the symmetric matrix with entries $(a_i b_j + a_j b_i)/2$.
- The restriction of $p(t)$ to $\partial\Omega$ accounts for the mismatch between the trace of the displacement $u(t)$ and the prescribed boundary condition, given by the trace of $w(t)$.

Stress and stress constraint

- The stress $\sigma(t)$ at time t belongs to $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$ and depends on the elastic strain $e(t)$ through the linear relation

$$\sigma(t) := \mathbb{C}e(t)\,,$$

where $\mathbb{C}\colon \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$, the elasticity tensor, is symmetric, linear, and

$$\lambda|A|^2 \le \mathbb{C}A{:}A \le \Lambda|A|^2 \quad \text{for every } A \in \mathbb{R}^{2\times 2}_{sym}\,, \text{ with } 0<\lambda\le\Lambda\,.$$

- In plasticity we have a constraint on the stress of the form

$$\sigma(t,x) \in \mathbb{K} \quad \text{for a.e. } x\in\Omega\,,$$

where $\mathbb{K}$ is a prescribed closed and convex set in $\mathbb{R}^{2\times 2}_{sym}$ depending on the material, whose boundary plays the role of yield surface.

- In our model $\mathbb{K}$ (and hence the yield surface) depends also the pressure component of the stress (pressure-sensitive elasto-plastic material). To simplify the exposition in this talk I choose $\mathbb{K} := \{A \in \mathbb{R}^{2\times 2}_{sym} : |A| \le 1\}$.

Stress and stress constraint

- The stress $\sigma(t)$ at time t belongs to $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$ and depends on the elastic strain $e(t)$ through the linear relation

$$\sigma(t) := \mathbb{C}e(t)\,,$$

where $\mathbb{C}\colon \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$, the elasticity tensor, is symmetric, linear, and

$$\lambda|A|^2 \le \mathbb{C}A{:}A \le \Lambda|A|^2 \quad \text{for every } A \in \mathbb{R}^{2\times 2}_{sym}\,, \text{ with } 0 < \lambda \le \Lambda\,.$$

- In plasticity we have a constraint on the stress of the form

$$\sigma(t,x) \in \mathbb{K} \quad \text{for a.e. } x \in \Omega\,,$$

where $\mathbb{K}$ is a prescribed closed and convex set in $\mathbb{R}^{2\times 2}_{sym}$ depending on the material, whose boundary plays the role of yield surface.

- In our model $\mathbb{K}$ (and hence the yield surface) depends also the pressure component of the stress (pressure-sensitive elasto-plastic material). To simplify the exposition in this talk I choose $\mathbb{K} := \{A \in \mathbb{R}^{2\times 2}_{sym} : |A| \le 1\}$.

Stress and stress constraint

- The stress $\sigma(t)$ at time t belongs to $L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$ and depends on the elastic strain $e(t)$ through the linear relation

$$\sigma(t) := \mathbb{C}e(t)\,,$$

where $\mathbb{C}\colon \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$, the elasticity tensor, is symmetric, linear, and

$$\lambda|A|^2 \le \mathbb{C}A{:}A \le \Lambda|A|^2 \quad \text{for every } A \in \mathbb{R}^{2\times 2}_{sym}\,, \text{ with } 0 < \lambda \le \Lambda\,.$$

- In plasticity we have a constraint on the stress of the form

$$\sigma(t,x) \in \mathbb{K} \quad \text{for a.e. } x \in \Omega\,,$$

where $\mathbb{K}$ is a prescribed closed and convex set in $\mathbb{R}^{2\times 2}_{sym}$ depending on the material, whose boundary plays the role of yield surface.

- In our model $\mathbb{K}$ (and hence the yield surface) depends also the pressure component of the stress (pressure-sensitive elasto-plastic material). To simplify the exposition in this talk I choose $\mathbb{K} := \{A \in \mathbb{R}^{2\times 2}_{sym} : |A| \le 1\}$.

Elastic energy and dissipation distance

- The stored elastic energy at time t depends only on the elastic strain $e(t)$ and is given by
$$\frac{1}{2}\int_\Omega \sigma(t){:}e(t)\,dx = \int_\Omega Q(e(t))\,dx\,,$$
where $Q(A) := \frac{1}{2}\mathbb{C}A{:}A$ for every $A \in \mathbb{R}^{2\times 2}_{sym}$.
- The energy dissipated in a time interval depends on the evolution of the pair $(p(t), s(t))$ composed of the plastic strain and the (lenght of the) crack. According to the terminology of rate-independent systems, the dissipation distance between two pairs (p_2, s_2) and (p_1, s_1), with $s_i \in [0, L]$ and $p_i \in \mathcal{M}_b(\widehat{\Omega}_{s_i}; \mathbb{R}^{2\times 2}_{sym})$, is given by
$$d((p_2, s_2), (p_1, s_1)) := \begin{cases} |p_2 - p_1|(\widehat{\Omega}_{s_2}) + s_2 - s_1 & \text{if } s_1 \le s_2\,, \\ +\infty & \text{otherwise,} \end{cases}$$
where $|p_2 - p_1|(\widehat{\Omega}_{s_2})$ accounts for the plastic dissipation distance and $s_2 - s_1$ is the energy dissipated to produce the crack increment $\Gamma^{s_2}_{s_1}$.

Elastic energy and dissipation distance

- The stored elastic energy at time t depends only on the elastic strain $e(t)$ and is given by $$\frac{1}{2}\int_{\Omega}\sigma(t){:}e(t)\,dx=\int_{\Omega}Q(e(t))\,dx\,,$$ where $Q(A):=\frac{1}{2}\mathbb{C}A{:}A$ for every $A\in\mathbb{R}^{2\times2}_{sym}$.
- The energy dissipated in a time interval depends on the evolution of the pair $(p(t),s(t))$ composed of the plastic strain and the (lenght of the) crack. According to the terminology of rate-independent systems, the dissipation distance between two pairs (p_2,s_2) and (p_1,s_1), with $s_i\in[0,L]$ and $p_i\in\mathcal{M}_b(\widehat{\Omega}_{s_i};\mathbb{R}^{2\times2}_{sym})$, is given by

$$d((p_2,s_2),(p_1,s_1)):=\begin{cases}|p_2-p_1|(\widehat{\Omega}_{s_2})+s_2-s_1 & \text{if } s_1\le s_2\,,\\ +\infty & \text{otherwise,}\end{cases}$$

where $|p_2-p_1|(\widehat{\Omega}_{s_2})$ accounts for the plastic dissipation distance and s_2-s_1 is the energy dissipated to produce the crack increment $\Gamma^{s_2}_{s_1}$.

Discrete-time incremental formulation

- Given $s \in [0, L]$ and $w \in H^1(\Omega; \mathbb{R}^2)$, let $\mathcal{A}(w, s)$ (admissible triples) be the set of (u, e, p), with $u \in BD(\Omega_s)$, $e \in L^2(\Omega; \mathbb{R}^{2\times 2}_{sym})$, $p \in \mathcal{M}_b(\widehat{\Omega}_s; \mathbb{R}^{2\times 2}_{sym})$, which satisfy the weak kinematic admissibility conditions

$$Eu = e + p \quad \text{as measures in } \Omega_s\,,$$

$$p = (w - u) \odot \nu_\Omega \mathcal{H}^1 \quad \text{as measures on } \partial\Omega\,.$$

- Given a subdivision $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ of $[0, T]$, for $i = 1, \ldots, n$ let (u_i, e_i, p_i, s_i) be a solution of the incremental minimum problem for the quadruple (u, e, p, s):

$$\min_{\substack{s \in [s_{i-1}, L] \\ (u,e,p) \in \mathcal{A}(w(t_i), s)}} \Big(\int_\Omega Q(e)dx + |p - p(t_{i-1})|(\widehat{\Omega}_s) + s - s_{i-1} \Big)\,.$$

- As in our 2010 paper we can prove that, passing to a subsequence, the piecewise constant interpolation of (u_i, e_i, p_i, s_i) converges, as the fineness of the subdivision tends to zero, to a continuous-time quasistatic evolution, according to the definition given in the next slide.

Discrete-time incremental formulation

- Given $s \in [0, L]$ and $w \in H^1(\Omega;\mathbb{R}^2)$, let $\mathcal{A}(w,s)$ (admissible triples) be the set of (u, e, p), with $u \in BD(\Omega_s)$, $e \in L^2(\Omega;\mathbb{R}^{2\times 2}_{sym})$, $p \in \mathcal{M}_b(\widehat{\Omega}_s;\mathbb{R}^{2\times 2}_{sym})$, which satisfy the weak kinematic admissibility conditions

$$Eu = e + p \quad \text{as measures in } \Omega_s\,,$$
$$p = (w-u)\odot \nu_\Omega \mathcal{H}^1 \quad \text{as measures on } \partial\Omega\,.$$

- Given a subdivision $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ of $[0, T]$, for $i = 1, \ldots, n$ let (u_i, e_i, p_i, s_i) be a solution of the incremental minimum problem for the quadruple (u, e, p, s):

$$\min_{\substack{s\in[s_{i-1},L] \\ (u,e,p)\in\mathcal{A}(w(t_i),s)}} \Big(\int_\Omega Q(e)dx + |p - p(t_{i-1})|(\widehat{\Omega}_s) + s - s_{i-1}\Big)\,.$$

- As in our 2010 paper we can prove that, passing to a subsequence, the piecewise constant interpolation of (u_i, e_i, p_i, s_i) converges, as the fineness of the subdivision tends to zero, to a continuous-time quasistatic evolution, according to the definition given in the next slide.

Definition of quasistatic evolution

Definition. A quasistatic evolution with boundary value $t \mapsto w(t)$ on $\partial\Omega$ is a function $t \mapsto (u(t), e(t), p(t), s(t))$, with $s(t) \in [0, L]$, $u(t) \in BD(\Omega_{s(t)})$, $e(t) \in L^2(\Omega; \mathbb{R}^{2\times 2}_{sym})$, and $p(t) \in \mathcal{M}_b(\widehat{\Omega}_{s(t)}; \mathbb{R}^{2\times 2}_{sym})$, which satisfies the following conditions:

- *(irreversibility)* $t \mapsto s(t)$ is nondecreasing;
- *(equilibrium)* for every t we have $(u(t), e(t), p(t)) \in \mathcal{A}(w(t), s(t))$ and

$$\int_\Omega Q(e(t))dx \le \int_\Omega Q(\tilde{e})dx + |\tilde{p} - p(t)|(\widehat{\Omega}_{\tilde{s}}) + \tilde{s} - s(t),$$

  for every $\tilde{s} \in [s(t), L]$ and every $(\tilde{u}, \tilde{e}, \tilde{p}) \in \mathcal{A}(w(t), \tilde{s})$;
- *(energy-dissipation inequality)* for every $t_1 < t_2$ it is

$$\int_\Omega Q(e(t_2))dx + |p(t_2) - p(t_1)|(\widehat{\Omega}_{s(t_2)}) + s(t_2) - s(t_1)$$

$$\le \int_\Omega Q(e(t_1))dx + \int_{t_1}^{t_2} \Big(\int_\Omega \sigma(\tau) : E\dot{w}(\tau)dx \Big) d\tau .$$

Definition of quasistatic evolution

Definition. A quasistatic evolution with boundary value $t \mapsto w(t)$ on $\partial\Omega$ is a function $t \mapsto (u(t), e(t), p(t), s(t))$, with $s(t) \in [0, L]$, $u(t) \in BD(\Omega_{s(t)})$, $e(t) \in L^2(\Omega; \mathbb{R}^{2\times 2}_{sym})$, and $p(t) \in \mathcal{M}_b(\widehat{\Omega}_{s(t)}; \mathbb{R}^{2\times 2}_{sym})$, which satisfies the following conditions:

- *(irreversibility)* $t \mapsto s(t)$ is nondecreasing;
- *(equilibrium)* for every t we have $(u(t), e(t), p(t)) \in \mathcal{A}(w(t), s(t))$ and

$$\int_\Omega Q(e(t))dx \le \int_\Omega Q(\tilde{e})dx + |\tilde{p} - p(t)|(\widehat{\Omega}_{\tilde{s}}) + \tilde{s} - s(t),$$

  for every $\tilde{s} \in [s(t), L]$ and every $(\tilde{u}, \tilde{e}, \tilde{p}) \in \mathcal{A}(w(t), \tilde{s})$;
- *(energy-dissipation inequality)* for every $t_1 < t_2$ it is

$$\int_\Omega Q(e(t_2))dx + |p(t_2) - p(t_1)|(\widehat{\Omega}_{s(t_2)}) + s(t_2) - s(t_1)$$
$$\le \int_\Omega Q(e(t_1))dx + \int_{t_1}^{t_2} \Big(\int_\Omega \sigma(\tau):E\dot{w}(\tau)dx \Big) d\tau.$$

Definition of quasistatic evolution

Definition. A quasistatic evolution with boundary value $t \mapsto w(t)$ on $\partial\Omega$ is a function $t \mapsto (u(t), e(t), p(t), s(t))$, with $s(t) \in [0, L]$, $u(t) \in BD(\Omega_{s(t)})$, $e(t) \in L^2(\Omega; \mathbb{R}^{2\times 2}_{sym})$, and $p(t) \in \mathcal{M}_b(\widehat{\Omega}_{s(t)}; \mathbb{R}^{2\times 2}_{sym})$, which satisfies the following conditions:

- *(irreversibility)* $t \mapsto s(t)$ is nondecreasing;
- *(equilibrium)* for every t we have $(u(t), e(t), p(t)) \in \mathcal{A}(w(t), s(t))$ and

$$\int_\Omega Q(e(t))dx \le \int_\Omega Q(\tilde{e})dx + |\tilde{p} - p(t)|(\widehat{\Omega}_{\tilde{s}}) + \tilde{s} - s(t)\,,$$

  for every $\tilde{s} \in [s(t), L]$ and every $(\tilde{u}, \tilde{e}, \tilde{p}) \in \mathcal{A}(w(t), \tilde{s})$;
- *(energy-dissipation inequality)* for every $t_1 < t_2$ it is

$$\int_\Omega Q(e(t_2))dx + |p(t_2) - p(t_1)|(\widehat{\Omega}_{s(t_2)}) + s(t_2) - s(t_1)$$

$$\le \int_\Omega Q(e(t_1))dx + \int_{t_1}^{t_2}\Big(\int_\Omega \sigma(\tau){:}E\dot{w}(\tau)dx\Big)d\tau\,.$$

Definition of quasistatic evolution

Definition. A quasistatic evolution with boundary value $t \mapsto w(t)$ on $\partial\Omega$ is a function $t \mapsto (u(t), e(t), p(t), s(t))$, with $s(t) \in [0, L]$, $u(t) \in BD(\Omega_{s(t)})$, $e(t) \in L^2(\Omega; \mathbb{R}^{2\times 2}_{sym})$, and $p(t) \in \mathcal{M}_b(\widehat{\Omega}_{s(t)}; \mathbb{R}^{2\times 2}_{sym})$, which satisfies the following conditions:

- *(irreversibility)* $t \mapsto s(t)$ is nondecreasing;
- *(equilibrium)* for every t we have $(u(t), e(t), p(t)) \in \mathcal{A}(w(t), s(t))$ and

$$\int_\Omega Q(e(t))dx \leq \int_\Omega Q(\tilde{e})dx + |\tilde{p} - p(t)|(\widehat{\Omega}_{\tilde{s}}) + \tilde{s} - s(t)\,,$$

  for every $\tilde{s} \in [s(t), L]$ and every $(\tilde{u}, \tilde{e}, \tilde{p}) \in \mathcal{A}(w(t), \tilde{s})$;
- *(energy-dissipation inequality)* for every $t_1 < t_2$ it is

$$\int_\Omega Q(e(t_2))dx + |p(t_2) - p(t_1)|(\widehat{\Omega}_{s(t_2)}) + s(t_2) - s(t_1)$$

$$\leq \int_\Omega Q(e(t_1))dx + \int_{t_1}^{t_2} \Big(\int_\Omega \sigma(\tau) : E\dot{w}(\tau)dx \Big) d\tau\,.$$

The Euler condition

- Using a suitable notion of dissipation, which I will never use in this talk, we can prove that the three conditions in the definition of quasistatic evolution imply also an energy-dissipation balance.
- For every $t \in [0, T]$ the equilibrium condition, applied with $\tilde{s} = s(t)$, implies that $(u(t), e(t), p(t))$ is the solution of the minimum problem

$$\min_{(\tilde{u},\tilde{e},\tilde{p}) \in \mathcal{A}(w(t),s(t))} \int_\Omega Q(\tilde{e})dx + |\tilde{p} - p_0|(\widehat{\Omega}_{s(t)}),$$

with $p_0 = p(t)$.
- The corresponding Euler conditions are $\mathrm{div}\sigma(t) = 0$ in $\Omega_{s(t)}$, $\sigma(t)\nu = 0$ on $\Gamma_0^{s(t)}$, and $\|\sigma(t)\|_\infty \leq 1$ (stress constraint).

The Euler condition

- Using a suitable notion of dissipation, which I will never use in this talk, we can prove that the three conditions in the definition of quasistatic evolution imply also an energy-dissipation balance.
- For every $t \in [0, T]$ the equilibrium condition, applied with $\tilde{s} = s(t)$, implies that $(u(t), e(t), p(t))$ is the solution of the minimum problem

$$\min_{(\tilde{u},\tilde{e},\tilde{p}) \in \mathcal{A}(w(t),s(t))} \int_\Omega Q(\tilde{e})dx + |\tilde{p} - p_0|(\widehat{\Omega}_{s(t)}),$$

with $p_0 = p(t)$.

- The corresponding Euler conditions are $\operatorname{div}\sigma(t) = 0$ in $\Omega_{s(t)}$, $\sigma(t)\nu = 0$ on $\Gamma_0^{s(t)}$, and $\|\sigma(t)\|_\infty \leq 1$ (stress constraint).

The Euler condition

- Using a suitable notion of dissipation, which I will never use in this talk, we can prove that the three conditions in the definition of quasistatic evolution imply also an energy-dissipation balance.
- For every $t \in [0,T]$ the equilibrium condition, applied with $\tilde{s} = s(t)$, implies that $(u(t), e(t), p(t))$ is the solution of the minimum problem

$$\min_{(\tilde{u},\tilde{e},\tilde{p}) \in \mathcal{A}(w(t),s(t))} \int_\Omega Q(\tilde{e})dx + |\tilde{p} - p_0|(\widehat{\Omega}_{s(t)}),$$

  with $p_0 = p(t)$.
- The corresponding Euler conditions are $\operatorname{div}\sigma(t) = 0$ in $\Omega_{s(t)}$, $\sigma(t)\nu = 0$ on $\Gamma_0^{s(t)}$, and $\|\sigma(t)\|_\infty \le 1$ (stress constraint).

The main result

Definition

We set $s(t\pm) := \lim_{\tau\to t\pm} s(\tau)$ for $t \in (0,T)$, with the convention $s(0-) := s(0)$ and $s(T+) := s(T)$. We can write $s = s^{cont} + s^{jump}$, where s^{cont} is continuous and s^{jump} is the pure jump component of s, defined by

$$s^{jump}(t) = s(t) - s(t-) + \sum_{\tau\in J_s, \tau<t} (s(\tau+) - s(\tau-)) \quad \text{for every } t \in [0,T],$$

where J_s is the (at most countable) set of jump points of $t \mapsto s(t)$.

Theorem (DM-Toader 2020)

Let (u,e,p,s) be a quasistatic evolution with boundary value w on $\partial\Omega$, with $w \in AC([0,T];H^1(\Omega))$. Then s^{cont} is constant on the interval $[0,T]$.

I shall present only the ideas of the proof of the following partial result: if $t \mapsto s(t)$ is continuous in $[0,T]$ and $|p(t_2) - p(t_1)|(\Omega_{s(t_2)}) \leq C(t_2 - t_1)$ for every $t_1 < t_2$, then $t \mapsto s(t)$ is constant. The general case can be reduced to this one, but requires much more technicalities.

The main result

Definition

We set $s(t\pm) := \lim_{\tau\to t\pm} s(\tau)$ for $t \in (0,T)$, with the convention $s(0-) := s(0)$ and $s(T+) := s(T)$. We can write $s = s^{cont} + s^{jump}$, where s^{cont} is continuous and s^{jump} is the pure jump component of s, defined by

$$s^{jump}(t) = s(t) - s(t-) + \sum_{\tau\in J_s, \tau<t} (s(\tau+) - s(\tau-)) \quad \text{for every } t \in [0,T],$$

where J_s is the (at most countable) set of jump points of $t \mapsto s(t)$.

Theorem (DM-Toader 2020)

Let (u,e,p,s) be a quasistatic evolution with boundary value w on $\partial\Omega$, with $w \in AC([0,T]; H^1(\Omega))$. Then s^{cont} is constant on the interval $[0,T]$.

I shall present only the ideas of the proof of the following partial result: if $t \mapsto s(t)$ is continuous in $[0,T]$ and $|p(t_2) - p(t_1)|(\Omega_{s(t_2)}) \le C(t_2 - t_1)$ for every $t_1 < t_2$, then $t \mapsto s(t)$ is constant. The general case can be reduced to this one, but requires much more technicalities.

The main result

Definition

We set $s(t\pm) := \lim_{\tau\to t\pm} s(\tau)$ for $t \in (0,T)$, with the convention $s(0-) := s(0)$ and $s(T+) := s(T)$. We can write $s = s^{cont} + s^{jump}$, where s^{cont} is continuous and s^{jump} is the pure jump component of s, defined by

$$s^{jump}(t) = s(t) - s(t-) + \sum_{\tau\in J_s, \tau<t} (s(\tau+) - s(\tau-)) \quad \text{for every } t \in [0,T],$$

where J_s is the (at most countable) set of jump points of $t \mapsto s(t)$.

Theorem (DM-Toader 2020)

Let (u,e,p,s) be a quasistatic evolution with boundary value w on $\partial\Omega$, with $w \in AC([0,T];H^1(\Omega))$. Then s^{cont} is constant on the interval $[0,T]$.

I shall present only the ideas of the proof of the following partial result: if $t \mapsto s(t)$ is continuous in $[0,T]$ and $|p(t_2) - p(t_1)|(\Omega_{s(t_2)}) \le C(t_2 - t_1)$ for every $t_1 < t_2$, then $t \mapsto s(t)$ is constant. The general case can be reduced to this one, but requires much more technicalities.

Using the Euler conditions

- At the beginning of the proof we fix $0 \le t_1 < t_2 \le T$. To simplify the notation we set, for $i = 1, 2$, $u_i = u(t_i)$, $e_i = e(t_i)$, $p_i = p(t_i)$, $\sigma_i = \sigma(t_i)$, $w_i = w(t_i)$, and $s_i = s(t_i)$.
- Let $(\varphi, \eta, q) \in \mathcal{A}(0, s_2)$. Assuming regularity, and using the fact that $\operatorname{div}\sigma_1 = 0$ in Ω_{s_1} and $\sigma_1 \nu = 0$ on Γ_{s_1}, we obtain

$$\int_\Omega \sigma_1 \cdot \eta \, dx + \int_{\Omega_{s_2}} \sigma_1 \cdot dq = \int_{\Omega_{s_2}} \sigma_1 \cdot E\varphi \, dx = \int_{\Gamma_{s_1}^{s_2}} \sigma_1 \nu [\varphi] \, d\mathcal{H}^1 ,$$

where $[\varphi]$ denotes the jump of φ. Using $\|\sigma_1\|_\infty \le 1$ we get

$$- \int_\Omega \sigma_1 \cdot \eta \, dx \le |q|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[\varphi]| \, d\mathcal{H}^1 .$$

By approximation this holds also without regularity.
- With $\varphi = u_2 - u_1 - w_2 + w_1$, $\eta = e_2 - e_1 - Ew_2 + Ew_1$, and $q = p_2 - p_1$, we get

$$-\int_\Omega \sigma_1 : (e_2 - e_1) dx + \int_\Omega \sigma_1 : (Ew_2 - Ew_1) dx \le |p_2 - p_1|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1$$

Using the Euler conditions

- At the beginning of the proof we fix $0 \le t_1 < t_2 \le T$. To simplify the notation we set, for $i = 1, 2$, $u_i = u(t_i)$, $e_i = e(t_i)$, $p_i = p(t_i)$, $\sigma_i = \sigma(t_i)$, $w_i = w(t_i)$, and $s_i = s(t_i)$.
- Let $(\varphi, \eta, q) \in \mathcal{A}(0, s_2)$. Assuming regularity, and using the fact that $\mathrm{div}\sigma_1 = 0$ in Ω_{s_1} and $\sigma_1\nu = 0$ on Γ_{s_1}, we obtain
$$\int_\Omega \sigma_1 \cdot \eta \, dx + \int_{\Omega_{s_2}} \sigma_1 \cdot dq = \int_{\Omega_{s_2}} \sigma_1 \cdot E\varphi \, dx = \int_{\Gamma_{s_1}^{s_2}} \sigma_1 \nu [\varphi] \, d\mathcal{H}^1 ,$$
where $[\varphi]$ denotes the jump of φ. Using $\|\sigma_1\|_\infty \le 1$ we get
$$-\int_\Omega \sigma_1 \cdot \eta \, dx \le |q|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[\varphi]| \, d\mathcal{H}^1 .$$
By approximation this holds also without regularity.
- With $\varphi = u_2 - u_1 - w_2 + w_1$, $\eta = e_2 - e_1 - Ew_2 + Ew_1$, and $q = p_2 - p_1$, we get
$$-\int_\Omega \sigma_1 : (e_2 - e_1) dx + \int_\Omega \sigma_1 : (Ew_2 - Ew_1) dx \le |p_2 - p_1|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1$$

Using the Euler conditions

- At the beginning of the proof we fix $0 \le t_1 < t_2 \le T$. To simplify the notation we set, for $i = 1, 2$, $u_i = u(t_i)$, $e_i = e(t_i)$, $p_i = p(t_i)$, $\sigma_i = \sigma(t_i)$, $w_i = w(t_i)$, and $s_i = s(t_i)$.
- Let $(\varphi, \eta, q) \in \mathcal{A}(0, s_2)$. Assuming regularity, and using the fact that $\mathrm{div}\sigma_1 = 0$ in Ω_{s_1} and $\sigma_1\nu = 0$ on Γ_{s_1}, we obtain

$$\int_\Omega \sigma_1 \cdot \eta \, dx + \int_{\Omega_{s_2}} \sigma_1 \cdot dq = \int_{\Omega_{s_2}} \sigma_1 \cdot E\varphi \, dx = \int_{\Gamma_{s_1}^{s_2}} \sigma_1 \nu [\varphi] \, d\mathcal{H}^1 ,$$

where $[\varphi]$ denotes the jump of φ. Using $\|\sigma_1\|_\infty \le 1$ we get

$$-\int_\Omega \sigma_1 \cdot \eta \, dx \le |q|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[\varphi]| \, d\mathcal{H}^1 .$$

By approximation this holds also without regularity.
- With $\varphi = u_2 - u_1 - w_2 + w_1$, $\eta = e_2 - e_1 - Ew_2 + Ew_1$, and $q = p_2 - p_1$, we get

$$-\int_\Omega \sigma_1 : (e_2 - e_1) dx + \int_\Omega \sigma_1 : (Ew_2 - Ew_1) dx \le |p_2 - p_1|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1$$

Using the energy-dissipation inequality

- Since $Q(e_i) = \frac{1}{2}\int_\Omega \sigma_i{:}e_i dx$, from energy-dissipation we obtain

$$\frac{1}{2}\int_\Omega \sigma_2{:}e_2 dx - \frac{1}{2}\int_\Omega \sigma_1{:}e_1 dx + |p_2 - p_1|(\widehat{\Omega}_{s_2}) + s_2 - s_1 \le \int_{t_1}^{t_2}\Big(\int_\Omega \sigma(\tau){:}E\dot{w}(\tau)dx\Big)d\tau\,.$$

- From the inequality obtained from the Euler condition we get

$$-\int_\Omega \sigma_1{:}(e_2 - e_1)dx \le |p_2 - p_1|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 - \int_\Omega \sigma_1{:}(Ew_2 - Ew_1)dx\,.$$

- Adding these inequalities and using $Ew_2 - Ew_1 = \int_{t_1}^{t_2} E\dot{w}(\tau)d\tau$ we obtain

$$\frac{1}{2}\int_\Omega (\sigma_2 - \sigma_1){:}(e_2 - e_1)dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 + \omega_{1,2}\,,$$

where $\omega_{1,2} = \omega(t_1, t_2) := \int_{t_1}^{t_2}\Big(\int_\Omega (\sigma(\tau) - \sigma_1){:}E\dot{w}(\tau)dx\Big)d\tau.$

- Since $(\sigma_2 - \sigma_1){:}(e_2 - e_1) = \mathbb{C}(e_2 - e_1){:}(e_2 - e_1)$, using coerciveness we get

$$\frac{\lambda}{2}\int_\Omega |e_2 - e_1|^2 dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 + \omega_{1,2}\,.$$

Using the energy-dissipation inequality

- Since $Q(e_i) = \frac{1}{2}\int_\Omega \sigma_i{:}e_i dx$, from energy-dissipation we obtain

$$\frac{1}{2}\int_\Omega \sigma_2{:}e_2 dx - \frac{1}{2}\int_\Omega \sigma_1{:}e_1 dx + |p_2 - p_1|(\widehat{\Omega}_{s_2}) + s_2 - s_1 \le \int_{t_1}^{t_2}\Big(\int_\Omega \sigma(\tau){:}E\dot{w}(\tau)dx\Big)d\tau\,.$$

- From the inequality obtained from the Euler condition we get

$$-\int_\Omega \sigma_1{:}(e_2 - e_1)dx \le |p_2 - p_1|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]|d\mathcal{H}^1 - \int_\Omega \sigma_1{:}(Ew_2 - Ew_1)dx\,.$$

- Adding these inequalities and using $Ew_2 - Ew_1 = \int_{t_1}^{t_2} E\dot{w}(\tau)d\tau$ we obtain

$$\frac{1}{2}\int_\Omega (\sigma_2 - \sigma_1){:}(e_2 - e_1)dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]|d\mathcal{H}^1 + \omega_{1,2}\,,$$

where $\omega_{1,2} = \omega(t_1, t_2) := \int_{t_1}^{t_2}\Big(\int_\Omega (\sigma(\tau) - \sigma_1){:}E\dot{w}(\tau)dx\Big)d\tau$.

- Since $(\sigma_2 - \sigma_1){:}(e_2 - e_1) = \mathbb{C}(e_2 - e_1){:}(e_2 - e_1)$, using coerciveness we get

$$\frac{\lambda}{2}\int_\Omega |e_2 - e_1|^2 dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]|d\mathcal{H}^1 + \omega_{1,2}\,.$$

Using the energy-dissipation inequality

- Since $Q(e_i) = \frac{1}{2}\int_\Omega \sigma_i{:}e_i dx$, from energy-dissipation we obtain

$$\frac{1}{2}\int_\Omega \sigma_2{:}e_2 dx - \frac{1}{2}\int_\Omega \sigma_1{:}e_1 dx + |p_2 - p_1|(\widehat{\Omega}_{s_2}) + s_2 - s_1 \le \int_{t_1}^{t_2}\Big(\int_\Omega \sigma(\tau){:}E\dot{w}(\tau)dx\Big)d\tau\,.$$

- From the inequality obtained from the Euler condition we get

$$-\int_\Omega \sigma_1{:}(e_2 - e_1)dx \le |p_2 - p_1|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]|\,d\mathcal{H}^1 - \int_\Omega \sigma_1{:}(Ew_2 - Ew_1)dx\,.$$

- Adding these inequalities and using $Ew_2 - Ew_1 = \int_{t_1}^{t_2} E\dot{w}(\tau)d\tau$ we obtain

$$\frac{1}{2}\int_\Omega (\sigma_2 - \sigma_1){:}(e_2 - e_1)dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]|\,d\mathcal{H}^1 + \omega_{1,2}\,,$$

where $\omega_{1,2} = \omega(t_1, t_2) := \int_{t_1}^{t_2}\Big(\int_\Omega (\sigma(\tau) - \sigma_1){:}E\dot{w}(\tau)dx\Big)d\tau.$

- Since $(\sigma_2 - \sigma_1){:}(e_2 - e_1) = \mathbb{C}(e_2 - e_1){:}(e_2 - e_1)$, using coerciveness we get

$$\frac{\lambda}{2}\int_\Omega |e_2 - e_1|^2 dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]|\,d\mathcal{H}^1 + \omega_{1,2}\,.$$

Using the energy-dissipation inequality

- Since $Q(e_i) = \frac{1}{2}\int_\Omega \sigma_i{:}e_i dx$, from energy-dissipation we obtain

$$\frac{1}{2}\int_\Omega \sigma_2{:}e_2 dx - \frac{1}{2}\int_\Omega \sigma_1{:}e_1 dx + |p_2 - p_1|(\widehat{\Omega}_{s_2}) + s_2 - s_1 \le \int_{t_1}^{t_2}\Big(\int_\Omega \sigma(\tau){:}E\dot{w}(\tau)dx\Big)d\tau\,.$$

- From the inequality obtained from the Euler condition we get

$$-\int_\Omega \sigma_1{:}(e_2 - e_1)dx \le |p_2 - p_1|(\widehat{\Omega}_{s_2}) + \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 - \int_\Omega \sigma_1{:}(Ew_2 - Ew_1)dx\,.$$

- Adding these inequalities and using $Ew_2 - Ew_1 = \int_{t_1}^{t_2} E\dot{w}(\tau)d\tau$ we obtain

$$\frac{1}{2}\int_\Omega (\sigma_2 - \sigma_1){:}(e_2 - e_1)dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 + \omega_{1,2}\,,$$

where $\omega_{1,2} = \omega(t_1, t_2) := \int_{t_1}^{t_2}\Big(\int_\Omega (\sigma(\tau) - \sigma_1){:}E\dot{w}(\tau)dx\Big)d\tau.$

- Since $(\sigma_2 - \sigma_1){:}(e_2 - e_1) = \mathbb{C}(e_2 - e_1){:}(e_2 - e_1)$, using coerciveness we get

$$\frac{\lambda}{2}\int_\Omega |e_2 - e_1|^2 dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 + \omega_{1,2}\,.$$

A crucial estimate

- The trace estimate in BD gives

$$\int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| \, d\mathcal{H}^1 \le c_0 |Eu_2 - Eu_1| (B_{s_2-s_1}(x_2) \cap \Omega_{s_2}) \,,$$

where $x_2 := x(t_2) := (s_2, 0)$ is the crack tip at time t_2 and c_0 is independent of s_1 and s_2.

- Hence

$$\frac{\lambda}{2} \int_\Omega |e_2 - e_1|^2 dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 + \omega_{1,2}$$

$$\le c_0 \int_{B_{s_2-s_1}(x_2) \cap \Omega_{s_2}} |e_2 - e_1| dx + c_0 |p_2 - p_1| (B_{s_2-s_1}(x_2) \cap \Omega_{s_2}) + \omega_{1,2} \,.$$

- Using the Cauchy inequality we find that for $\eta > 0$ small we have

$$\frac{1}{2}(s_2 - s_1) \le c_0 |p_2 - p_1| (B_{s_2-s_1}(x_2) \cap \Omega_{s_2}) + \omega_{1,2} \le c_0 |p_2 - p_1| (B_\eta(x_2) \cap \Omega_{s_2}) + \omega_{1,2}$$

for $s_2 - s_2 < \eta$.

A crucial estimate

- The trace estimate in BD gives

$$\int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]|\, d\mathcal{H}^1 \leq c_0 |Eu_2 - Eu_1|(B_{s_2-s_1}(x_2) \cap \Omega_{s_2}),$$

where $x_2 := x(t_2) := (s_2, 0)$ is the crack tip at time t_2 and c_0 is independent of s_1 and s_2.

- Hence $$\frac{\lambda}{2}\int_\Omega |e_2-e_1|^2 dx + s_2-s_1 \leq \int_{\Gamma_{s_1}^{s_2}} |[u_2-u_1]| d\mathcal{H}^1 + \omega_{1,2}$$

$$\leq c_0 \int_{B_{s_2-s_1}(x_2)\cap\Omega_{s_2}} |e_2-e_1| dx + c_0|p_2-p_1|(B_{s_2-s_1}(x_2)\cap\Omega_{s_2}) + \omega_{1,2}\,.$$

- Using the Cauchy inequality we find that for $\eta > 0$ small we have

$$\frac{1}{2}(s_2-s_1) \leq c_0|p_2-p_1|(B_{s_2-s_1}(x_2)\cap\Omega_{s_2})+\omega_{1,2} \leq c_0|p_2-p_1|(B_\eta(x_2)\cap\Omega_{s_2})+\omega_{1,2}$$

for $s_2 - s_2 < \eta$.

A crucial estimate

- The trace estimate in BD gives

$$\int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| \, d\mathcal{H}^1 \le c_0 |Eu_2 - Eu_1|(B_{s_2 - s_1}(x_2) \cap \Omega_{s_2}) \, ,$$

where $x_2 := x(t_2) := (s_2, 0)$ is the crack tip at time t_2 and c_0 is independent of s_1 and s_2.

- Hence

$$\frac{\lambda}{2} \int_\Omega |e_2 - e_1|^2 dx + s_2 - s_1 \le \int_{\Gamma_{s_1}^{s_2}} |[u_2 - u_1]| d\mathcal{H}^1 + \omega_{1,2}$$

$$\le c_0 \int_{B_{s_2-s_1}(x_2) \cap \Omega_{s_2}} |e_2 - e_1| dx + c_0 |p_2 - p_1|(B_{s_2-s_1}(x_2) \cap \Omega_{s_2}) + \omega_{1,2} \, .$$

- Using the Cauchy inequality we find that for $\eta > 0$ small we have

$$\frac{1}{2}(s_2 - s_1) \le c_0 |p_2 - p_1|(B_{s_2-s_1}(x_2) \cap \Omega_{s_2}) + \omega_{1,2} \le c_0 |p_2 - p_1|(B_\eta(x_2) \cap \Omega_{s_2}) + \omega_{1,2}$$

for $s_2 - s_2 < \eta$.

Use of the Lipschitz continuity

- Let $0 = t_0 < t_1 < \cdots < t_m = T$ with $s(t_j) - s(t_{j-1}) < \eta$. Applying the inequality of the previous step to $[t_{j-1}, t_j]$ we obtain

$$s(T)-s(0) \leq 2c_0 \sum_{j=1}^{m} |p(t_j)-p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) + 2 \sum_{j=1}^{m} \omega(t_{j-1}, t_j)\,,$$

where $x(t) := (s(t), 0)$ is the crack tip at time t.

- By the Lipschitz continuity we have

$$|p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \leq \int_{t_{j-1}}^{t_j} |\dot{p}|(B_\eta(x(t_j)) \cap \Omega_{s(\tau)}) d\tau$$

- Since $B_\eta(x(t_j)) \subset B_{2\eta}(x(\tau))$ for $\tau \in [t_{j-1}, t_j]$ we have

$$|p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \leq \int_{t_{j-1}}^{t_j} |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau\,,$$

hence

$$\sum_{j=1}^{m} |p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \leq \int_0^T |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau\,.$$

- Let $0 = t_0 < t_1 < \cdots < t_m = T$ with $s(t_j) - s(t_{j-1}) < \eta$. Applying the inequality of the previous step to $[t_{j-1}, t_j]$ we obtain

$$s(T)-s(0) \leq 2c_0 \sum_{j=1}^{m} |p(t_j)-p(t_{j-1})|(B_\eta(x(t_j))\cap\Omega_{s(t_j)})+2\sum_{j=1}^{m}\omega(t_{j-1},t_j)\,,$$

where $x(t) := (s(t), 0)$ is the crack tip at time t.

- By the Lipschitz continuity we have

$$|p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \leq \int_{t_{j-1}}^{t_j} |\dot{p}|(B_\eta(x(t_j)) \cap \Omega_{s(\tau)})d\tau$$

- Since $B_\eta(x(t_j)) \subset B_{2\eta}(x(\tau))$ for $\tau \in [t_{j-1}, t_j]$ we have

$$|p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \leq \int_{t_{j-1}}^{t_j} |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)})d\tau\,,$$

hence

$$\sum_{j=1}^{m} |p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \leq \int_0^T |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)})d\tau\,.$$

Use of the Lipschitz continuity

- Let $0 = t_0 < t_1 < \cdots < t_m = T$ with $s(t_j) - s(t_{j-1}) < \eta$. Applying the inequality of the previous step to $[t_{j-1}, t_j]$ we obtain

$$s(T) - s(0) \le 2c_0 \sum_{j=1}^{m} |p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) + 2 \sum_{j=1}^{m} \omega(t_{j-1}, t_j)\,,$$

where $x(t) := (s(t), 0)$ is the crack tip at time t.

- By the Lipschitz continuity we have

$$|p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \le \int_{t_{j-1}}^{t_j} |\dot{p}|(B_\eta(x(t_j)) \cap \Omega_{s(\tau)}) d\tau$$

- Since $B_\eta(x(t_j)) \subset B_{2\eta}(x(\tau))$ for $\tau \in [t_{j-1}, t_j]$ we have

$$|p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \le \int_{t_{j-1}}^{t_j} |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau\,,$$

hence

$$\sum_{j=1}^{m} |p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) \le \int_{0}^{T} |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau\,.$$

- Therefore

$$s(T)-s(0)\le 2c_0\sum_{j=1}^{m}|p(t_j)-p(t_{j-1})|(B_\eta(x(t_j))\cap\Omega_{s(t_j)})+2\sum_{j=1}^{m}\omega(t_{j-1},t_j)$$
$$\le 2c_0\int_0^T|\dot p|(B_{2\eta}(x(\tau))\cap\Omega_{s(\tau)})d\tau+2\sum_{j=1}^{m}\omega(t_{j-1},t_j)\,.$$

- Since $B_{2\eta}(x(\tau))\cap\Omega_{s(\tau)}\to\emptyset$ as $\eta\to 0$, we have

$$2c_0\int_0^T(|\dot p|(B_{2\eta}(x(\tau))\cap\Omega_{s(\tau)})d\tau\to 0 \text{ as } \eta\to 0\,.$$

- Let us fix $\varepsilon>0$. Then there exists $\eta>0$ such that

$$s(T)-s(0)\le\varepsilon+2\sum_{j=1}^{m}\omega(t_{j-1},t_j)\quad \text{if } s(t_j)-s(t_{j-1})<\eta \text{ for } j=1,\dots,m\,.$$

- To conclude the proof of the theorem it is enough to show that $\sum_{j=1}^{m}\omega(t_{j-1},t_j)<\varepsilon$ if we choose a suitable subdivision.
- This would give $s(T)-s(0)<3\varepsilon$, which leads to the conclusion.

- Therefore

$$s(T)-s(0)\le 2c_0\sum_{j=1}^m |p(t_j)-p(t_{j-1})|(B_\eta(x(t_j))\cap\Omega_{s(t_j)})+2\sum_{j=1}^m \omega(t_{j-1},t_j)$$

$$\le 2c_0\int_0^T |\dot p|(B_{2\eta}(x(\tau))\cap\Omega_{s(\tau)})d\tau+2\sum_{j=1}^m \omega(t_{j-1},t_j)\,.$$

- Since $B_{2\eta}(x(\tau))\cap\Omega_{s(\tau)}\to\emptyset$ as $\eta\to 0$, we have

$$2c_0\int_0^T (|\dot p|(B_{2\eta}(x(\tau))\cap\Omega_{s(\tau)})d\tau\to 0 \text{ as } \eta\to 0\,.$$

- Let us fix $\varepsilon>0$. Then there exists $\eta>0$ such that

$$s(T)-s(0)\le\varepsilon+2\sum_{j=1}^m \omega(t_{j-1},t_j)\quad \text{if } s(t_j)-s(t_{j-1})<\eta \text{ for } j=1,\dots,m\,.$$

- To conclude the proof of the theorem it is enough to show that $\sum_{j=1}^m \omega(t_{j-1},t_j)<\varepsilon$ if we choose a suitable subdivision.
- This would give $s(T)-s(0)<3\varepsilon$, which leads to the conclusion.

Towards the conclusion

- Therefore

$$s(T) - s(0) \le 2c_0 \sum_{j=1}^{m} |p(t_j) - p(t_{j-1})|(B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) + 2\sum_{j=1}^{m} \omega(t_{j-1}, t_j)$$

$$\le 2c_0 \int_0^T |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau + 2\sum_{j=1}^{m} \omega(t_{j-1}, t_j)\,.$$

- Since $B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)} \to \emptyset$ as $\eta \to 0$, we have

$$2c_0 \int_0^T (|\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau \to 0 \text{ as } \eta \to 0\,.$$

- Let us fix $\varepsilon > 0$. Then there exists $\eta > 0$ such that

$$s(T) - s(0) \le \varepsilon + 2\sum_{j=1}^{m} \omega(t_{j-1}, t_j) \quad \text{if } s(t_j) - s(t_{j-1}) < \eta \text{ for } j = 1, \dots, m\,.$$

- To conclude the proof of the theorem it is enough to show that $\sum_{j=1}^{m} \omega(t_{j-1}, t_j) < \varepsilon$ if we choose a suitable subdivision.
- This would give $s(T) - s(0) < 3\varepsilon$, which leads to the conclusion.

Towards the conclusion

- Therefore

$$s(T) - s(0) \le 2c_0 \sum_{j=1}^{m} |p(t_j) - p(t_{j-1})| (B_\eta(x(t_j)) \cap \Omega_{s(t_j)}) + 2 \sum_{j=1}^{m} \omega(t_{j-1}, t_j)$$

$$\le 2c_0 \int_0^T |\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau + 2 \sum_{j=1}^{m} \omega(t_{j-1}, t_j) \,.$$

- Since $B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)} \to \emptyset$ as $\eta \to 0$, we have

$$2c_0 \int_0^T (|\dot{p}|(B_{2\eta}(x(\tau)) \cap \Omega_{s(\tau)}) d\tau \to 0 \text{ as } \eta \to 0 \,.$$

- Let us fix $\varepsilon > 0$. Then there exists $\eta > 0$ such that

$$s(T) - s(0) \le \varepsilon + 2 \sum_{j=1}^{m} \omega(t_{j-1}, t_j) \quad \text{if } s(t_j) - s(t_{j-1}) < \eta \text{ for } j = 1, \dots, m \,.$$

- To conclude the proof of the theorem it is enough to show that $\sum_{j=1}^{m} \omega(t_{j-1}, t_j) < \varepsilon$ if we choose a suitable subdivision.
- This would give $s(T) - s(0) < 3\varepsilon$, which leads to the conclusion.

Looking for a suitable subdivision

- Since $t \mapsto s(t)$ is continuous, for every $\eta > 0$ there exists $\delta > 0$ such that $t_j - t_{j-1} < \delta \Rightarrow s(t_j) - s(t_{j-1}) < \eta$.
- It remains to prove that, given $\varepsilon > 0$ and $\delta > 0$, we can find a subdivision $0 = t_0 < t_1 < \cdots < t_m = T$ such that $t_j - t_{j-1} < \delta$ for every $1 \le j \le m$ and $\sum_{j=1}^{m} \omega(t_{j-1}, t_j) < \varepsilon$.
- Recall that $\omega(t_{j-1}, t_j) := \int_{t_{j-1}}^{t_j} \Big(\int_\Omega (\sigma(\tau) - \sigma(t_{j-1})) : E\dot{w}(\tau)\, dx \Big) d\tau$.
- Therefore, the result about ω holds if and only if

$$\int_0^T \Big(\int_\Omega \sigma(t) : E\dot{w}(t)\, dx \Big) d\tau - \sum_{j=1}^{m} \int_\Omega \sigma(t_{j-1}) : (Ew(t_j) - Ew(t_{j-1}))\, dx < \varepsilon .$$

Looking for a suitable subdivision

- Since $t \mapsto s(t)$ is continuous, for every $\eta > 0$ there exists $\delta > 0$ such that $t_j - t_{j-1} < \delta \Rightarrow s(t_j) - s(t_{j-1}) < \eta$.
- It remains to prove that, given $\varepsilon > 0$ and $\delta > 0$, we can find a subdivision $0 = t_0 < t_1 < \cdots < t_m = T$ such that $t_j - t_{j-1} < \delta$ for every $1 \leq j \leq m$ and $\sum_{j=1}^{m} \omega(t_{j-1}, t_j) < \varepsilon$.
- Recall that $\omega(t_{j-1}, t_j) := \int_{t_{j-1}}^{t_j} \Big(\int_\Omega (\sigma(\tau) - \sigma(t_{j-1})) : E\dot{w}(\tau) dx \Big) d\tau$.
- Therefore, the result about ω holds if and only if

$$\int_0^T \Big(\int_\Omega \sigma(t) : E\dot{w}(t) dx \Big) d\tau - \sum_{j=1}^{m} \int_\Omega \sigma(t_{j-1}) : (Ew(t_j) - Ew(t_{j-1})) dx < \varepsilon .$$

Looking for a suitable subdivision

- Since $t \mapsto s(t)$ is continuous, for every $\eta > 0$ there exists $\delta > 0$ such that $t_j - t_{j-1} < \delta \Rightarrow s(t_j) - s(t_{j-1}) < \eta$.
- It remains to prove that, given $\varepsilon > 0$ and $\delta > 0$, we can find a subdivision $0 = t_0 < t_1 < \cdots < t_m = T$ such that $t_j - t_{j-1} < \delta$ for every $1 \le j \le m$ and $\sum_{j=1}^m \omega(t_{j-1}, t_j) < \varepsilon$.
- Recall that $\omega(t_{j-1}, t_j) := \int_{t_{j-1}}^{t_j} \Big(\int_\Omega (\sigma(\tau) - \sigma(t_{j-1})) : E\dot{w}(\tau)\, dx \Big) d\tau$.
- Therefore, the result about ω holds if and only if

$$\int_0^T \Big(\int_\Omega \sigma(t) : E\dot{w}(t)\, dx \Big) d\tau - \sum_{j=1}^m \int_\Omega \sigma(t_{j-1}) : (Ew(t_j) - Ew(t_{j-1}))\, dx < \varepsilon .$$

Looking for a suitable subdivision

- Since $t \mapsto s(t)$ is continuous, for every $\eta > 0$ there exists $\delta > 0$ such that $t_j - t_{j-1} < \delta \Rightarrow s(t_j) - s(t_{j-1}) < \eta$.
- It remains to prove that, given $\varepsilon > 0$ and $\delta > 0$, we can find a subdivision $0 = t_0 < t_1 < \cdots < t_m = T$ such that $t_j - t_{j-1} < \delta$ for every $1 \leq j \leq m$ and $\sum_{j=1}^m \omega(t_{j-1}, t_j) < \varepsilon$.
- Recall that $\omega(t_{j-1}, t_j) := \int_{t_{j-1}}^{t_j} \Big(\int_\Omega (\sigma(\tau) - \sigma(t_{j-1})) : E\dot{w}(\tau) dx \Big) d\tau$.
- Therefore, the result about ω holds if and only if

$$\int_0^T \Big(\int_\Omega \sigma(t) : E\dot{w}(t) dx \Big) d\tau - \sum_{j=1}^m \int_\Omega \sigma(t_{j-1}) : (Ew(t_j) - Ew(t_{j-1})) dx < \varepsilon \,.$$

Riemann sums for Lebesgue integrals

The inequality

$$\int_0^T \Big(\int_\Omega \sigma(t) : E\dot{w}(t)\, dx \Big) d\tau - \sum_{j=1}^m \int_\Omega \sigma(t_{j-1}) : (Ew(t_j) - Ew(t_{j-1}))\, dx < \varepsilon$$

can be easily obtained from the following well known result about Riemann sums for Lebesgue integrals.

Theorem (Hahn 1914)

Let $f \colon [0,T] \to \mathbb{R}$ be Lebesgue integrable. For every $\varepsilon > 0$ and $\delta > 0$ there exists a subdivision $0 = t_0 < t_1 < \cdots < t_m = T$ such that $t_j - t_{j-1} < \delta$ for every $1 \le j \le m$ and $\displaystyle\int_0^T f(t)\, dt - \sum_{j=1}^m f(t_{j-1})(t_j - t_{j-1}) < \varepsilon$.

An elegant proof, based only on Fubini Theorem, can be found in Doob: Stochastic Processes, page 63.

Riemann sums for Lebesgue integrals

The inequality

$$\int_0^T \Big(\int_\Omega \sigma(t){:}E\dot{w}(t)dx \Big) d\tau - \sum_{j=1}^m \int_\Omega \sigma(t_{j-1}){:}(Ew(t_j) - Ew(t_{j-1}))dx < \varepsilon$$

can be easily obtained from the following well known result about Riemann sums for Lebesgue integrals.

Theorem (Hahn 1914)

Let $f\colon [0,T] \to \mathbb{R}$ be Lebesgue integrable. For every $\varepsilon > 0$ and $\delta > 0$ there exists a subdivision $0 = t_0 < t_1 < \cdots < t_m = T$ such that $t_j - t_{j-1} < \delta$ for every $1 \le j \le m$ and $\int_0^T f(t)dt - \sum_{j=1}^m f(t_{j-1})(t_j - t_{j-1}) < \varepsilon\,.$

An elegant proof, based only on Fubini Theorem, can be found in Doob: Stochastic Processes, page 63.

Riemann sums for Lebesgue integrals

The inequality

$$\int_0^T \Big(\int_\Omega \sigma(t) : E\dot{w}(t)\,dx \Big) d\tau - \sum_{j=1}^m \int_\Omega \sigma(t_{j-1}) : (Ew(t_j) - Ew(t_{j-1}))\,dx < \varepsilon$$

can be easily obtained from the following well known result about Riemann sums for Lebesgue integrals.

Theorem (Hahn 1914)

Let $f \colon [0, T] \to \mathbb{R}$ *be Lebesgue integrable. For every* $\varepsilon > 0$ *and* $\delta > 0$ *there exists a subdivision* $0 = t_0 < t_1 < \cdots < t_m = T$ *such that* $t_j - t_{j-1} < \delta$ *for every* $1 \le j \le m$ *and* $\int_0^T f(t)\,dt - \sum_{j=1}^m f(t_{j-1})(t_j - t_{j-1}) < \varepsilon$.

An elegant proof, based only on Fubini Theorem, can be found in Doob: Stochastic Processes, page 63.

Does the crack really grow?

- All results proved so far are compatible with a crack that is constant in the whole interval $[0, T]$.
- This is a serious issue. In a similar problem, the dynamic evolution of cracks in viscoelastic materials, the crack cannot grow if we consider the Kelvin-Voigt model of viscoelasticity. This phenomenon is known and is called "viscoelastic paradox" in the mechanical literature.
- In our model of crack growth in elastoplastic materials, are there boundary conditions for which the crack really grows?
- The answer is: yes. If we replace $|p_2 - p_1|$ by $\beta|p_2 - p_1|$ in the definition of the dissipation distance, we obtain that the stress constraint becomes $\|\sigma(t)\|_\infty \leq \beta$, and we can prove that the limit of the quasistatic evolutions as $\beta \to +\infty$ is the quasistatic evolution for brittle cracks in elastic materials (without plasticity), for which we know that the crack can grow.

Does the crack really grow?

- All results proved so far are compatible with a crack that is constant in the whole interval $[0, T]$.
- This is a serious issue. In a similar problem, the dynamic evolution of cracks in viscoelastic materials, the crack cannot grow if we consider the Kelvin-Voigt model of viscoelasticity. This phenomenon is known and is called "viscoelastic paradox" in the mechanical literature.
- In our model of crack growth in elastoplastic materials, are there boundary conditions for which the crack really grows?
- The answer is: yes. If we replace $|p_2 - p_1|$ by $\beta|p_2 - p_1|$ in the definition of the dissipation distance, we obtain that the stress constraint becomes $\|\sigma(t)\|_\infty \leq \beta$, and we can prove that the limit of the quasistatic evolutions as $\beta \to +\infty$ is the quasistatic evolution for brittle cracks in elastic materials (without plasticity), for which we know that the crack can grow.

Does the crack really grow?

- All results proved so far are compatible with a crack that is constant in the whole interval $[0, T]$.
- This is a serious issue. In a similar problem, the dynamic evolution of cracks in viscoelastic materials, the crack cannot grow if we consider the Kelvin-Voigt model of viscoelasticity. This phenomenon is known and is called "viscoelastic paradox" in the mechanical literature.
- In our model of crack growth in elastoplastic materials, are there boundary conditions for which the crack really grows?
- The answer is: yes. If we replace $|p_2 - p_1|$ by $\beta|p_2 - p_1|$ in the definition of the dissipation distance, we obtain that the stress constraint becomes $\|\sigma(t)\|_\infty \le \beta$, and we can prove that the limit of the quasistatic evolutions as $\beta \to +\infty$ is the quasistatic evolution for brittle cracks in elastic materials (without plasticity), for which we know that the crack can grow.

Does the crack really grow?

- All results proved so far are compatible with a crack that is constant in the whole interval $[0, T]$.
- This is a serious issue. In a similar problem, the dynamic evolution of cracks in viscoelastic materials, the crack cannot grow if we consider the Kelvin-Voigt model of viscoelasticity. This phenomenon is known and is called "viscoelastic paradox" in the mechanical literature.
- In our model of crack growth in elastoplastic materials, are there boundary conditions for which the crack really grows?
- The answer is: yes. If we replace $|p_2 - p_1|$ by $\beta|p_2 - p_1|$ in the definition of the dissipation distance, we obtain that the stress constraint becomes $\|\sigma(t)\|_\infty \leq \beta$, and we can prove that the limit of the quasistatic evolutions as $\beta \to +\infty$ is the quasistatic evolution for brittle cracks in elastic materials (without plasticity), for which we know that the crack can grow.

A simplified model

- We also studied in a simplified model with antiplane displacement and with plastic strain constrained to be supported on the segment Γ. For this case obtain a stronger result: the function $t \mapsto s(t)$ has a finite number of jumps.
- Moreover we have a direct proof of the fact that the crack is not constant in a specific example.
- Numerical solutions of further examples in this simplified model have been obtained in collaboration with Luca Heltai.

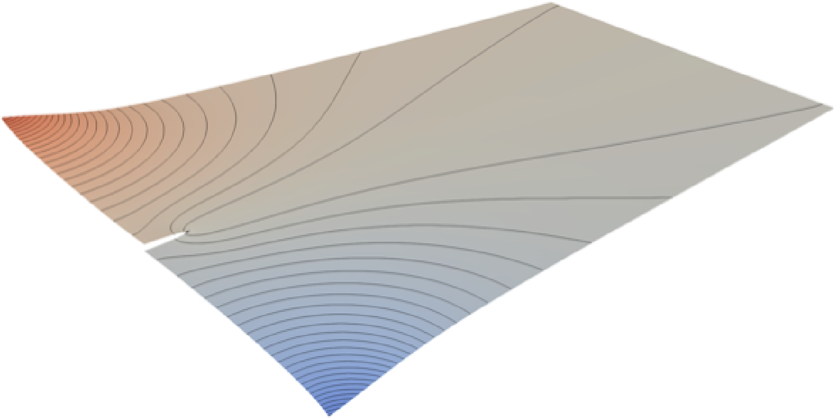
A simplified model

- We also studied in a simplified model with antiplane displacement and with plastic strain constrained to be supported on the segment Γ. For this case obtain a stronger result: the function $t \mapsto s(t)$ has a finite number of jumps.
- Moreover we have a direct proof of the fact that the crack is not constant in a specific example.
- Numerical solutions of further examples in this simplified model have been obtained in collaboration with Luca Heltai.

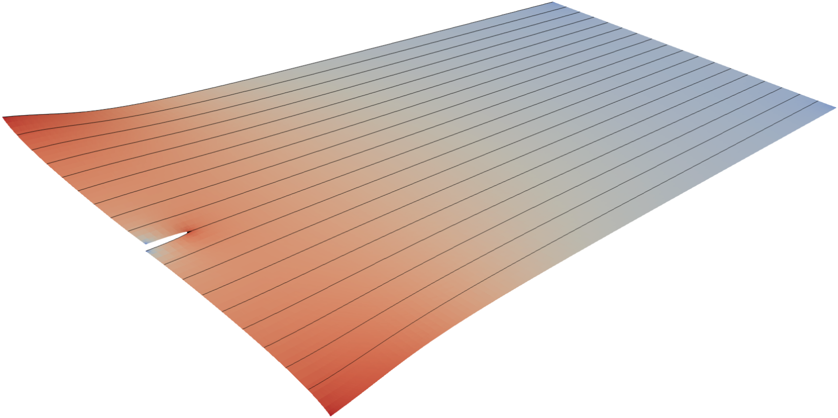
A simplified model

- We also studied in a simplified model with antiplane displacement and with plastic strain constrained to be supported on the segment Γ. For this case obtain a stronger result: the function $t \mapsto s(t)$ has a finite number of jumps.
- Moreover we have a direct proof of the fact that the crack is not constant in a specific example.
- Numerical solutions of further examples in this simplified model have been obtained in collaboration with Luca Heltai.

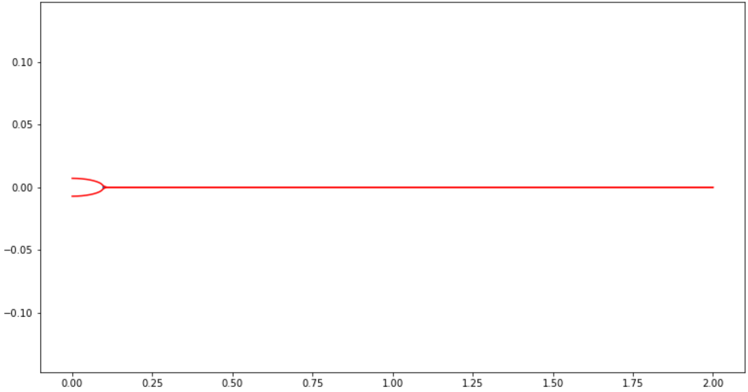
$\beta = 20$, elevation



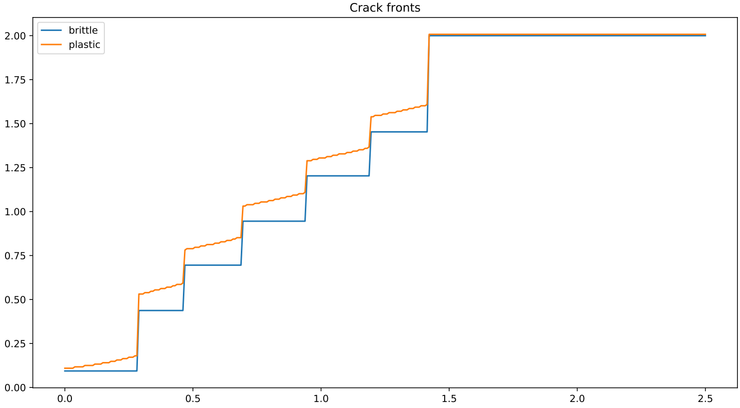
$\beta = 20$, stress



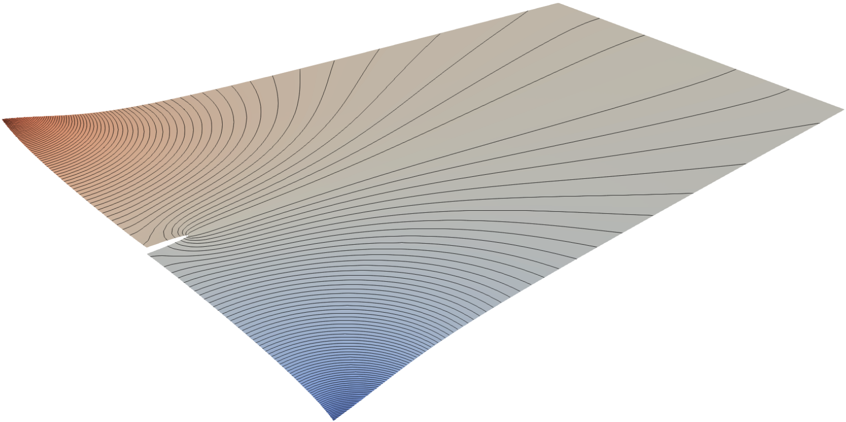
$\beta = 20$, crack and plastic opening on Γ



β = 20, crack and plastic fronts as functions of time



$\beta = 80$, elevation



THANK YOU FOR YOUR ATTENTION!