Determination of singular control in the optimal management of natural resources

Chris Guiver* † Mark R. Opmeer‡

Submitted: January 2025, Revised: March 2025

Abstract

A method is presented to simplify the determination of solutions of certain optimal control problems which commonly arise in natural resource management and bioeconomic contexts. The method, termed the resource-value balance method, essentially leverages an equivalent formulation of the original optimal control problem and, as described, in certain cases the method obviates the need for classical tools from optimal control theory, such as the Pontryagin Principle. Indeed, in these cases the method reduces the original problem to one solvable with elementary calculus techniques. Further, the solution provided by the resource-value balance method is shown to equal the singular solution of an associated (and more commonly considered) input-constrained optimal control problem, providing insight into the nature of singular control in this context. The theory is illustrated with examples from bioeconomics.

Keywords: bioeconomics, natural resource management, optimal control theory, optimal harvesting, Pontryagin Principle, singular control.

MSC (2020): 37N40, 49K15, 49N90, 91B76, 93C28.

1 Introduction

The problem of making decisions which lead to desirable outcomes arises in almost all scientific, social, and economic disciplines, including natural resource management and conservation. The importance of sustainably managing natural resources is well known, has garnered much attention, and nowadays forms one the central pillars of bioeconomics, a wide-ranging term whose meaning is not universally agreed but which broadly includes recognising biophysical limits in economic and social sectors; see, for instance [9] and the discussion therein. The academic literature of optimal natural resource management and conservation is consequently vast, with monographs including [6, 8, 26], from a range of academic perspectives. Mathematics, and here specifically optimal control theory, is one of the key tools for analysing models to provide both quantitative predictions and underlying insight, affording rational guides to decision making in natural resource management contexts.

Here, we propose a novel method for determining the solution of a family of optimal control problems, motivated by examples arising in natural resource management and bioeconomic contexts.

*School of Computing, Engineering & the Built Environment, Edinburgh Napier University, Merchiston Campus, Edinburgh, UK, email: c.guiver@napier.ac.uk, ORCID: 0000-0002-6881-7020

†Corresponding author

‡Department of Mathematical Sciences, University of Bath, Bath, UK, email: m.opmeer@bath.ac.uk, ORCID: 0009-0006-0569-0129

Specifically, consider the problem of maximizing

$$\mathrm{e}^{-\delta t_1} E(t_1, x(t_1)) - \mathrm{e}^{-\delta t_0} E(t_0, x(t_0)) + \int_{t_0}^{t_1} \mathrm{e}^{-\delta t} \frac{\partial E}{\partial x}(t, x(t))\, b(x(t))\, u(t)\, \mathrm{d}t\,, \tag{1.1}$$

subject to the dynamics

$$\dot{x}(t) = f(t, x(t), z(t)) - b(x(t))\, u(t)\,, \tag{1.2a}$$
$$\dot{z}(t) = h(t, x(t), z(t))\,, \tag{1.2b}$$

where, in general terms, x is a dynamic resource of interest, u is the extraction effort, z is an auxiliary variable which interacts with the resource, $t_0 < t_1$ are fixed times, E is the value of the resource and the scalar variable t denotes time. We have that $b(x(t))u(t)$ is the (structured) extraction rate and $\partial E/\partial x$ is the marginal value of the resource. Consequently, the integrand in (1.1) equals the incremental value of the extracted resource (discounted with rate $\delta \geq 0$) and, therefore, the integral equals the total (discounted) value of the amount of resource extracted over the interval $[t_0, t_1]$. The term $\mathrm{e}^{-\delta t_1} E(t_1, x(t_1))$ represents the (discounted) value in the system at time t_1, and similarly for time t_0. As shall become apparent, these terms are essentially included to eliminate boundary effects.

The standard method of solving the optimal control problem (1.1)–(1.2) is to apply the Pontryagin Principle; see, for example [7, 22, 25]. This provides a necessary condition for optimality and asserts that (in the normal case) there exist so-called co-states λ_x and λ_z such that, amongst other properties, the associated Hamiltonian function

$$\mathcal{H}\big(t, (x, z), u, (\lambda_x, \lambda_z)\big) = \begin{pmatrix} \lambda_x^\top & \lambda_z^\top \end{pmatrix} \begin{pmatrix} f(t, x, z) - b(x)u \\ h(t, x, z) \end{pmatrix} + \mathrm{e}^{-\delta t} \frac{\partial E}{\partial x}(t, x)\, b(x)\, u\,, \tag{1.3}$$

is maximized pointwise along an optimal trajectory. The first-order derivative condition $\partial \mathcal{H}/\partial u = 0$ here simplifies to

$$\mathrm{e}^{-\delta t} \frac{\partial E}{\partial x}(t, x(t))\, b(x(t)) = \lambda_x(t)^\top b(x(t)) \quad \forall\, t \in [t_0, t_1]\,. \tag{1.4}$$

In economics, the variable $\lambda_x^\top$ has the interpretation of "shadow price" whereas $(\partial E/\partial x)(t, x(t))$ has the interpretation of net price. In the common case that $b(x)$ is invertible along solutions x of interest, it follows from equation (1.4) that the shadow price should equal the discounted net price for optimality. From an economic perspective, this is a perfectly sensible (even obvious) optimality condition. We refer the reader to, for example, [5, Section 4.3, p. 102] or [30] for economic interpretations of the Pontryagin Principle. However, for those other than economics experts, the notion of shadow price is arguably non-intuitive.

Here, we give an alternative view of the optimal control problem (1.1)–(1.2), and particularly the computation of its solution. The approach is termed the *resource-value balance method* as it is underpinned by the so-called resource-value balance equation. This equation, in essence, rewrites (1.1) in a form that is independent of the original control variable u, and is presented in Theorem 1. Maximizing (1.1) is equivalent to maximizing this equivalent expression and, as we demonstrate, this latter task is often simpler than maximizing (1.1) directly. One consequence is that we obtain an optimality condition which usually has a clear interpretation without using the notion of shadow price. Moreover, the resource-value balance method provides a natural and intuitive connection to the solution of maximizing the integral term in (1.1) only, that is,

$$\int_{t_0}^{t_1} \mathrm{e}^{-\delta t} \frac{\partial E}{\partial x}(t, x(t))\, b(x(t))\, u(t)\, \mathrm{d}t\,, \tag{1.5}$$

subject to the dynamics (1.2) and, additionally, an input constraint. The performance criterion (1.5) appears more commonly across the literature and, from the Pontryagin Principle, the corresponding optimal controls are so-called bang-singular solutions. Recall that singular refers to the situation where the derivative of the Hamiltonian function with respect to the control equals zero. The second main result shows that the optimal control for the problem (1.1)–(1.2) equals the singular control for the problem (1.2) and (1.5), and is presented as Proposition 2.

The resource-value balance method, and the insight and connections it affords, comprise the main contribution and novelty of the present work. The study is motivated by examples from bioeconomics, and examples are drawn from the texts of Clark [5, 6]. As discussed in Remark 3, there is some overlap between the present results and Clark's works [5, 6], although with key distinguishing features. Also, we note that a similar method has been presented by the authors in [16] in the context of renewable energy conversion, where the quantity E denotes energy. However, the overlap with the present work is minimal owing to a somewhat different underlying optimal control problem and the dynamical systems considered.

The remainder of the paper is organised as follows. After gathering mathematical notation and conventions, our main results are presented in Section 2, and examples appear in Section 3. Section 4 contains some concluding remarks.

Notation and conventions For ease of exposition, mathematical notation is kept to a minimum, and the notation used is standard. The symbols $\mathbb{R}^n$ and $\mathbb{R}^n_+$ denote n-dimensional Euclidean space and the nonnegative orthant in $\mathbb{R}^n$, respectively. In particular, $\mathbb{R}_+ := \mathbb{R}^1_+ = [0, \infty)$. We assume throughout that the functions E, f, h and b in (1.1)–(1.2) are sufficiently smooth to ensure: (i) the existence of all derivatives taken; (ii) existence and uniqueness of solutions of initial value problems associated with the differential equations (1.2), and; (iii) existence of at least piecewise continuous optimal controls which maximize (1.1) subject to (1.2). These smoothness assumptions may be stronger than required, but are satisfied in the examples we consider, and determining minimal regularity assumptions on the model data so that the results presented are still valid is not the primary focus.

The variables x, z and u in (1.2) are vector-valued in general, so that f, h and b all have the appropriate domains and codomains which ensure that (1.1) and (1.2) make sense mathematically. We shall selectively suppress the arguments of the functions E, f and h for brevity and clarity. In a slight abuse of notation, we shall identify constants with their corresponding constant functions.

Since our examples relate to quantities which are necessarily nonnegative (denoting abundance and so forth), we recall some terminology and an invariance result related to positive dynamical systems. We refer the reader to [17, 1, 20] for further background. First, recall from, for example, [17, Definition 2.1 and p.13] that a function $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is called *essentially nonnegative* if, for all $t \geq 0$, $F_i(t, x) \geq 0$ for all $x \in \mathbb{R}^n_+$ with $x_i = 0$, for all $1 \leq i \leq n$. We recall that a nonempty set $M \subseteq \mathbb{R}^n$ is called *positively invariant* with respect to the differential equation

$$\dot{x} = F(t, x(t)) \,, \tag{1.6}$$

if $x(t) \in M$ for all $t \geq 0$ whenever $x(0) \in M$. Second, it follows from [17, Proposition 2.2] that if F is essentially nonnegative, continuous in x and piecewise continuous in t, then $\mathbb{R}^n_+$ is positively invariant with respect to the differential equation (1.6).

Finally, by convention, derivatives of scalar-valued functions with respect to vectors (gradients) are identified with row vectors, and the superscript $\top$ denotes matrix and vector transposition.

2 The resource-value balance method

The following theorem is our first main result.

Theorem 1. *Consider the cost function* (1.1) *subject to* (1.2). *We have the following "resource-*

value balance" equation:

$$\mathrm{e}^{-\delta t_1} E(t_1, x(t_1)) - \mathrm{e}^{-\delta t_0} E(t_0, x(t_0)) + \int_{t_0}^{t_1} \mathrm{e}^{-\delta t} \frac{\partial E}{\partial x} b(x(t))\, u(t)\, \mathrm{d}t$$
$$= \int_{t_0}^{t_1} \mathrm{e}^{-\delta t} \Big(-\delta E + \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} f(t, x(t), z(t)) \Big)\, \mathrm{d}t\,. \tag{2.1}$$

Proof. To establish (2.1), the fundamental theorem of calculus yields that

$$\mathrm{e}^{-\delta t_1} E(t_1, x(t_1)) - \mathrm{e}^{-\delta t_0} E(t_0, x(t_0)) = \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} \big(\mathrm{e}^{-\delta t} E(t, x(t)) \big)\, \mathrm{d}t\,. \tag{2.2}$$

Invoking the dynamics (1.2a), and the product and chain rules for differentiation, we now compute that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \big(\mathrm{e}^{-\delta t} E \big) &= \mathrm{e}^{-\delta t} \Big(-\delta E + \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} \dot{x} \Big) \\ &= \mathrm{e}^{-\delta t} \Big(-\delta E + \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} \big(f(t, x, z) - b(x)\, u \big) \Big)\,. \end{aligned} \tag{2.3}$$

Inserting equation (2.3) into equation (2.2), and rearranging the resulting expression yields (2.1). □

It follows from equation (2.1) that the original problem of maximizing (1.1) subject to (1.2) is equivalent to maximizing the right-hand side of (2.1), namely,

$$\int_{t_0}^{t_1} \mathrm{e}^{-\delta t} \Big(-\delta E + \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} f(t, x(t), z(t)) \Big)\, \mathrm{d}t\,, \tag{2.4}$$

subject to (1.2) which, note, does not involve the control u. We now change perspective and view x in (2.4) as the control variable; z subject to the differential equation (1.2b) as a state variable, and; u as being *determined* in terms of x and z via (1.2a). This is not restrictive, indeed, in the usual case that $b(x)$ is invertible along a maximizer x of (2.4), u is uniquely determined by

$$u(t) = b(x(t))^{-1} \big(f(t, x(t), z(t)) - \dot{x}(t) \big)\,. \tag{2.5}$$

In many problems z is, in fact, absent. From this perspective, determining the optimal initial condition $x(t_0)$ is part of the optimization problem.

We call the process of maximizing (2.1) via maximizing (2.4) the *resource-value balance method*. Before discussing the method further, we comment on the resource-value function E. From knowledge of $\partial E/\partial x$, the integrand in (1.1), the function E is determined uniquely up to a function of t, that is, with E_0 a fixed anti-derivative of E with respect to x, all anti-derivatives are given by $E = E_0 + \varepsilon(t)$ for some function ε. The objective (1.1) compared to that for E_0 then changes by $\mathrm{e}^{-\delta t_1}\varepsilon(t_1) - \mathrm{e}^{-\delta t_0}\varepsilon(t_0)$, which is a constant. Therefore, the choice of anti-derivative does not alter the maximizer and, consequently, there is choice of an additive constant in the definition of E.

To proceed, consider the following two exhaustive cases.

CASE 1: z is absent. Now $f(t, x, z) = f(t, x)$ and the function (2.4) to be maximized is subject to no differential equation constraints. Therefore, maximization is achieved by pointwise maximization of the integrand which, appealingly, may often be achieved by elementary algebraic or calculus methods. In other words, for given $t \in [t_0, t_1]$, it suffices to maximize:

$$x \mapsto -\delta E(t, x) + \frac{\partial E}{\partial t}(t, x) + \frac{\partial E}{\partial x}(t, x)\, f(t, x)\,. \tag{2.6}$$

This to-be-maximized function often has a natural interpretation, as shall be discussed in several of the examples considered. The first-order (derivative) necessary condition for the maximization in (2.6) is:

$$-\delta\,\frac{\partial E}{\partial x}^{\top} + \frac{\partial^2 E}{\partial x \partial t}^{\top} + \frac{\partial^2 E}{\partial x^2}\, f(t,x) + \frac{\partial f}{\partial x}(t,x)^{\top} \frac{\partial E}{\partial x}^{\top} = 0\,. \tag{2.7}$$

Developing (2.7) further requires additional information of E and f, and is done in the examples considered later.

Case 2: z is present. Now the function (2.4) to be maximized involves the dynamic state-variable z and, hence, the Pontryagin Principle is suitable. Suppressing function arguments, the Pontryagin Principle yields that there should exist a co-state $\lambda = \lambda_z$ such that, for all $t \in [t_0, t_1]$, the Hamiltonian function

$$x \mapsto \mathrm{e}^{-\delta t}\Big(-\delta E + \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x}\, f\Big) + \lambda^{\top} h\,,$$

is maximized, the first-order condition for which is

$$\mathrm{e}^{-\delta t}\Big(-\delta\,\frac{\partial E}{\partial x}^{\top} + \frac{\partial^2 E}{\partial x \partial t}^{\top} + \frac{\partial^2 E}{\partial x^2}\, f + \frac{\partial f}{\partial x}^{\top} \frac{\partial E}{\partial x}^{\top}\Big) + \frac{\partial h}{\partial x}^{\top} \lambda = 0\,, \tag{2.8}$$

and the co-state equation

$$-\dot{\lambda}(t) = \mathrm{e}^{-\delta t} \frac{\partial f}{\partial z}^{\top} \frac{\partial E}{\partial x}^{\top} + \frac{\partial h}{\partial z}^{\top} \lambda(t), \qquad \lambda(t_1) = 0\,, \tag{2.9}$$

holds. The transversality final-state condition $\lambda(t_1) = 0$ from (2.9), when substituted into (2.8) gives

$$\Big(-\delta\,\frac{\partial E}{\partial x}^{\top} + \frac{\partial^2 E}{\partial x \partial t}^{\top} + \frac{\partial^2 E}{\partial x^2}\, f + \frac{\partial f}{\partial x}^{\top} \frac{\partial E}{\partial x}^{\top}\Big)\Big|_{t=t_1} = 0\,, \tag{2.10}$$

which can be interpreted as a necessary "destination curve" in (x, z)-space for the optimal state and associated auxiliary variable to reach.

In the special simplifying case that

$$\delta = 0, \quad \frac{\partial E}{\partial t} = 0, \quad \frac{\partial h}{\partial t} = 0, \quad \text{and} \quad \frac{\partial f}{\partial t} = 0\,, \tag{2.11}$$

so that, in particular, $E(t,x) = E(x)$, two further consequences are afforded. First, the destination curve (2.10) reduces to

$$\Big(\frac{\partial^2 E}{\partial x^2}\, f + \frac{\partial f}{\partial x}^{\top} \frac{\partial E}{\partial x}^{\top}\Big)\Big|_{t=t_1} = 0\,. \tag{2.12}$$

Second, the Hamiltonian is constant along the optimal trajectory: that is, there exists a real constant d such that

$$\frac{\partial E}{\partial x}\, f + \lambda^{\top} h = d\,. \tag{2.13}$$

If z is scalar valued, and hence the function h is as well, then equation (2.13) may be solved for λ. Therefore, λ may be eliminated from (2.8) to obtain

$$\Big(\frac{\partial^2 E}{\partial x^2}\, f + \frac{\partial f}{\partial x} \frac{\partial E}{\partial x}^{\top}\Big) h + \frac{\partial h}{\partial x}^{\top} \Big(d - f^{\top} \frac{\partial E}{\partial x}^{\top}\Big) = 0\,.$$

If, additionally, x is scalar valued and $\partial h/\partial x$ is not identically equal to zero, then we obtain a necessary condition for the optimal state and auxiliary variable, namely, that they are level curves of the function:

$$(x,z) \mapsto \Big(\frac{\partial^2 E}{\partial x^2}(x) f(x,z) + \frac{\partial f}{\partial x}(x,z) \frac{\partial E}{\partial x}(x)\Big) \frac{h(x,z)}{\frac{\partial h}{\partial x}(x,z)} - f(x,z) \frac{\partial E}{\partial x}(x)\,. \tag{2.14}$$

Whilst the value of the level, denoted d above, may be determined by evaluating (2.14) at any $t \in [t_0, t_1]$, including $t = t_0$ which gives d in terms of the initial data $x(t_0)$ and $z(t_0)$, recall that determining $x(t_0)$, and hence the value of the level, are part of the optimization problem in the resource-value balance method. Developing (2.10) and (2.14) further requires bespoke information on E and f. Observe that this method obviates the need to determine the co-state variable λ.

As a further specialisation of Case 2, consider the situation wherein an equilibrium solution is sought, that is, f and h are independent of t, and (constant) x and z are such that

$$0 = f(x, z) - b(x)u \quad \text{and} \quad 0 = h(x, z) \,. \tag{2.15}$$

This situation arises in maximum sustainable yield (MSY) approaches; see, for example [28] and the references therein. Again for simplicity imposing (2.11), the to-be-maximized function

$$(x, z) \mapsto \frac{\partial E}{\partial x} f \,,$$

is now subject to the (algebraic) constraint $h(x, z) = 0$, and is a problem which may be solved via Lagrange multipliers. Assuming that x and z are scalar valued for notational simplicity, we arrive at the first-order condition

$$\frac{\partial^2 E}{\partial x^2} f + \frac{\partial E}{\partial x}\frac{\partial f}{\partial x} + \theta \frac{\partial h}{\partial x} = 0, \quad \frac{\partial E}{\partial x}\frac{\partial f}{\partial z} + \theta \frac{\partial h}{\partial z} = 0, \quad h(x, z) = 0 \,, \tag{2.16}$$

for Lagrange multiplier θ. Assuming that $\partial h/\partial z \neq 0$, then θ may be eliminated from (2.16) to yield

$$\frac{\partial h}{\partial z}\frac{\partial^2 E}{\partial x^2} f + \frac{\partial h}{\partial z}\frac{\partial E}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial E}{\partial x}\frac{\partial f}{\partial z}\frac{\partial h}{\partial x} = 0, \quad h(x, z) = 0 \,, \tag{2.17}$$

that is, two equations in the two unknowns x and z. Once (2.17) is solved, then the (constant) optimal control u is determined by (2.15) as $u = b(x)^{-1} f(x, z)$. Here we are assuming that u is feasible (typically meaning $u \geq 0$ for natural resource management scenarios) and additional inequality constraints such as $b(x)^{-1} f(x, z) \geq 0$ may be required to enforce this. In this case, a more general optimization framework such as the Karush-Kuhn-Tucker (KKT) conditions (see, for instance [3, Section 5.5]) for inequality-constrained optimality are required. This situation is beyond the scope of the present work and not considered further.

2.1 An alternative optimization criterion

Here the relationship between the resource-value balance method for maximizing (1.1) subject to (1.2) and maximizing (1.5) (recall, the integral term in (1.1) only), subject to the dynamics (1.2) and the control constraint

$$u_{\min} \leq u(t) \leq u_{\max} \quad \text{(componentwise inequality)} \quad t_0 \leq t \leq t_1 \,, \tag{2.18}$$

is explored. In (2.18) the constant vectors $u_{\min}$ and $u_{\max}$ are given. It is this latter problem which occurs more frequently across the optimal control literature. The next result is the second main result of the current work.

Proposition 2. *Let (P1) denote the optimal control problem of maximizing* (1.1) *subject to* (1.2)*, and let (P2) denote the optimal control problem of maximizing* (1.5) *subject to the dynamics* (1.2) *and the control constraint* (2.18)*.*

A critical control of (P1) is precisely a singular control of (P2).

Here a *critical control* of (P1) is one such that the derivative of the Hamiltonian function $\mathcal{H}$ with respect to the control variable equals zero, and the conditions of the Pontryagin Principle are satisfied. A *singular control* of (P2) is also a control such that $\partial\mathcal{H}/\partial u = 0$ and the conditions of the Pontryagin Principle are satisfied, but here the additional input constraint (2.18) is present. Informally, this is the situation where pointwise maximisation of the Hamiltonian does not, by itself, determine the control in terms of the state and co-state.

Proof of Proposition 2. We invoke the Pontryagin Principle for (P1) and (P2). The dynamics (1.2) and Hamiltonian $\mathcal{H}$ as in (1.3) are equal for both problems, for co-state variables λ_x and λ_z. Consequently, the co-state equations for both problems coincide as well.

We express $\mathcal{H}$ as

$$\begin{aligned}\mathcal{H}\big(t,(x,z),u,(\lambda_x,\lambda_z)\big) &= \begin{pmatrix}\lambda_x^\top & \lambda_z^\top\end{pmatrix}\begin{pmatrix}f(t,x,z)\\ h(t,x,z)\end{pmatrix} + \Big(\mathrm{e}^{-\delta t}\frac{\partial E}{\partial x}(t,x) - \lambda_x^\top\Big)b(x)\,u \\ &= \begin{pmatrix}\lambda_x^\top & \lambda_z^\top\end{pmatrix}\begin{pmatrix}f(t,x,z)\\ h(t,x,z)\end{pmatrix} + \frac{\partial \mathcal{H}}{\partial u}\,u\,. \end{aligned} \tag{2.19}$$

A critical control for (P1) corresponds to the condition that $\partial\mathcal{H}/\partial u = 0$. For problem (P2), in light of (2.19), pointwise maximisation of the Hamiltonian with respect to u gives the following condition for optimal control u_*:

$$(u_*(t))_i = \begin{cases} u_{\max,i} & (\partial\mathcal{H}/\partial u)_i > 0 \\ \text{singular} & (\partial\mathcal{H}/\partial u)_i = 0 \\ u_{\min,i} & (\partial\mathcal{H}/\partial u)_i < 0\,. \end{cases}$$

In particular, the (any) singular portions of u_* for (P2) occur precisely when $\partial\mathcal{H}/\partial u = 0$. The proof is complete. □

Proposition 2 is not a deep result, but is, we contend, both useful and instructive. As already mentioned, the resource-value balance method pertains to solving problem (P1), and in particular, how the problem (P1) may be simplified. Then, problem (P1) is connected to the more-commonly studied problem (P2) via Proposition 2. The upshot is that maximizing (1.1) gives an intuitively simpler (and, hence, arguably superior) method of determining the singular control/state of (1.5). We do note that since Proposition 2 invokes the Pontryagin Principle, it provides necessary conditions for maximisers which are not sufficient in general. However, in the presence of suitable convexity assumptions (see, for example, the Mangasarian sufficiency conditions in [25, pp.105-106]) the necessary conditions are sufficient. Moreover, as shown above, problem (P1) reduces to a pointwise maximization problem in the case that z is absent, and the Pontryagin Principle is not required. Finally, we comment that in the case that the maximizer of (1.1) is constant, as is the case in several examples considered, then it follows that each component of the optimal bang-singular control of (1.5) takes (at most) three values.

Some last commentary is in order, recorded as a remark.

Remark 3. Whilst to the best of the authors' knowledge, the resource-value balance equation and method (Theorem 1) or the connection between optimal control problems (Proposition 2) are novel, there is some overlap of ideas related to simplifying solving optimal control problems in natural resource management contexts between the present work and those of Clark [5, 6] (and earlier papers such as [4]). These techniques are dotted around [5], including the derivation of [5, equation (2.16), p. 40] via the Euler-Lagrange equations, in [5, pp. 326-327] via integration by parts, and in [5, Section 2.7] which invokes Green's Theorem. The generality of Theorem 1 via the inclusion of the auxiliary variable z and the (potentially) time-varying functions f and h arguably distinguishes the present contribution from these earlier works.

3 Examples

We apply the resource-value balance method of Section 2 to three examples from bioeconomics. By way of notation, $(u_{\mathrm{s}}, x_{\mathrm{s}})$ shall denote a critical trajectory to (1.1), subject to dynamics (1.2), with corresponding z_{s} if included. Further, (u_*, x_*) shall denote an optimal trajectory to the alternative problem (1.5), subject to dynamics (1.2) and control constraint (2.18). With this notation, the conclusion of Proposition 2 may be expressed as $(u_{\mathrm{s}}, x_{\mathrm{s}})$ is the (any) singular state/control portion of (u_*, x_*), hence explaining the choice of notation. We comment that for $(u_{\mathrm{s}}, x_{\mathrm{s}})$ to be feasible as part of an optimal trajectory of (1.5), (1.2) and (2.18) requires that there are $t \in [t_0, t_1]$ such that $u_{\min} \le u_{\mathrm{s}}(t) \le u_{\max}$.

3.1 Harvesting of a single population

Consider the classical problem of optimally harvesting a single population modelled by a scalar differential equation. This model is discussed in, for example, the texts [5, Chapter 1] and [24, Section 1.6] where a number of bibliographic details given. In a bioeconomic context, the following optimal control problem is also called the Gordon-Schaefer logistic model [23], although the migration term we consider is not usually included. The uncontrolled (that is, unharvested) dynamics are assumed to be governed by the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) + v\,,$$

where x denotes local population abundance, the positive constants r and K denote the intrinsic growth rate and natural carrying capacity, respectively, and v denotes a migration term. The inclusion of harvesting as a control action leads to the equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - qxu + v\,, \tag{3.1}$$

where the positive constant q denotes harvesting efficacy, and the control variable u denotes harvesting effort which is nonnegative valued. The equation (3.1) is of the form (1.2) with

$$f(t,x,z) = f(t,x) := rx\Big(1 - \frac{x}{K}\Big) + v(t), \quad b(x) := qx\,,$$

and where the variable z is absent. It is clear that $(t,x) \mapsto f(t,x) - qxu(t)$ is essentially nonnegative when $v(t) \ge 0$, with the upshot that $x(t) \ge 0$ whenever $x(0) \ge 0$. Moreover, x is bounded when v is, and so there exists $B = B(\|v\|_{L^\infty}) > 0$ such that the interval $[0, B]$ is positively invariant for (3.1). Define the resource-value function

$$E(x) := px + \frac{c}{q}\big(\ln(B) - \ln(x)\big) \quad x > 0\,, \tag{3.2}$$

where the positive constants p and c denote the price per unit harvested and cost per unit of control effort, respectively. Note that E is independent of t, and the choice of additive constant available in E ensures that $E(x(t)) > 0$ along non-zero solutions of (3.1) from $x(0) \in [0, B]$. The resource-value function E in (3.2) shall appear across the examples considered, and we revisit it in Section 3.4. Presently, the following integral in (1.1) is obtained:

$$\int_{t_0}^{t_1} \mathrm{e}^{-\delta t} \frac{\partial E}{\partial x}(t, x(t))\, b(x(t))\, u(t)\, \mathrm{d}t = \int_{t_0}^{t_1} \mathrm{e}^{-\delta t}\, (pqx(t) - c)\, u(t)\, \mathrm{d}t\,.$$

Thus, the terms $pqxu$ and $-cu$ denote the revenue of harvested quantity and cost in doing so, respectively, over the time interval $[t_0, t_1]$. As usual, $\delta \ge 0$ is a discounting rate.

To apply the resource-value balance method of Section 2, we note that we are in CASE 1 as z is absent. Therefore, the function to be maximized according to (2.6) is (with $\delta = 0$ and $c = 0$ first for simplicity of interpretation)

$$x \mapsto pf(t,x) = prx\left(1 - \frac{x}{K}\right) + pv(t)\,.$$

Since $v(t)$ is assumed to be independent of x, in words, the above condition means that we want to be in the state x that maximizes the natural growth rate of the population. In the specific case considered presently of logistic growth, this state is given by $x_{\mathrm{s}} := K/2$. The corresponding (constant) optimal control effort u_{s} is obtained by substituting (the constant) x_{s} into the differential equation (3.1) to obtain

$$\begin{aligned} u_{\mathrm{s}}(t) &= \frac{1}{qx_{\mathrm{s}}}\Big(rx_{\mathrm{s}}\left(1 - \frac{x_{\mathrm{s}}}{K}\right) + v(t)\Big) - \frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{s}} \\ &= \frac{r}{2q} + \frac{2v(t)}{qK}\,. \end{aligned} \tag{3.3}$$

In words, in the absence of harvest cost and discounting, the optimal control effort u_{s} keeps the state at its constant value $x_{\mathrm{s}} = K/2$. Observe that knowledge of v is required to compute u_{s}, which may not be known with certainty in practice. We comment that the above conclusions are well-known in the case $v = 0$; see, for instance [24, equations (1.45)–(1.47), p. 31].

For general $\delta, c \neq 0$, the first derivative necessary condition (2.7) becomes

$$-\delta x\,(pqx - c) + c\Big(rx\Big(1 - \frac{x}{K}\Big) + v(t)\Big) + rx\Big(1 - \frac{2x}{K}\Big)\,(pqx - c) = 0\,, \tag{3.4}$$

which is a cubic equation in the variable x. Gathering powers of x together in (3.4) yields

$$-\frac{2pqr}{K}x^3 + \Big(pq(r - \delta) + \frac{cr}{K}\Big)x^2 + c\delta x + cv(t) = 0\,.$$

Since $v(t) \geq 0$ by assumption, there is a single sign change in the coefficients. Therefore, from Descartes rule of signs, there is at most a single positive root of (3.4), denoted $x_{\mathrm{s}}(t)$. It is clear from the above equality that x_{s} is an increasing function of $v(t)$. The corresponding optimal control $u_{\mathrm{s}}(t)$ is again obtained from (3.3).

For a numerical example we take model data

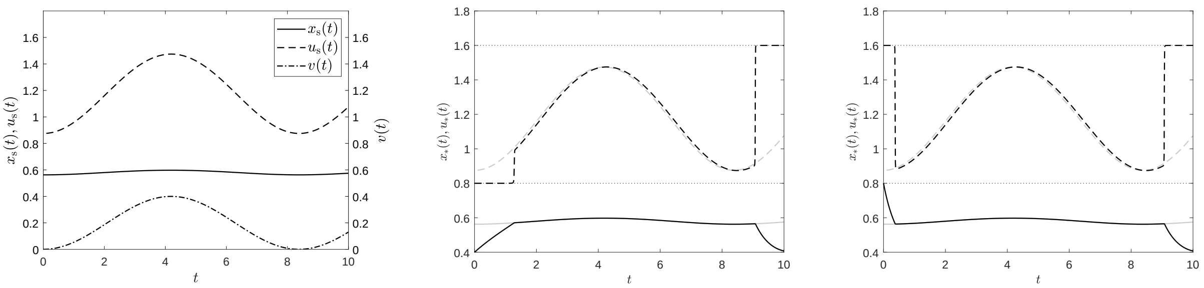
$$r = 2, \quad K = q = p = 1, \quad c = 0.125, \quad t_0 = 0, \quad t_1 = 10, \quad \delta = 0, \quad v(t) = 0.2(1 - \cos(0.75t))\,.$$

The terms x_{s} and u_{s} are computed by solving (3.4) numerically and (3.3), respectively, and graphs of these functions, along with v, are contained in Figure 3.1a.

To illustrate Proposition 2 numerically, we further take

$$x(0) \in \{0.4, 0.8\}, \quad u_{\min} = 0.8, \quad u_{\max} = 1.6\,.$$

An optimal trajectory (u_*, x_*) for (1.5) subject to the control constraint (2.18) was computed in the open-source optimal control toolbox Bocop [27] for $x(0) = 0.4$ and $x(0) = 0.8$. Figure 3.1b and 3.1c plot $u_*(t)$ and $x_*(t)$, both against t, for $x(0) = 0.4$ and $x(0) = 0.8$, respectively. In each case, a bang-singular-bang solution is observed, as expected from the Pontryagin Principle, and the singular part coincides with the trajectory $(u_{\mathrm{s}}, x_{\mathrm{s}})$, shown in faint lines, as expected from the resource-value balance method. Note that the first portion of bang control is determined by the relative size of $x(0)$ compared to $x_{\mathrm{s}}(0)$. By way of further commentary, recall that the term *turnpike* in optimal control problems refers, in broad terms, to the property that for varying initial conditions and time horizons, optimal solutions remain near a specific steady state for a substantial part of the time horizon. For optimal control problems in the context of economics, the turnpike property dates back to the foundational work [10]. For more background and recent results on turnpikes in optimal control problems, we refer the reader to, for example, [12, 14]. The connection between turnpikes and dissipativity has been explored in [15]. In fact, turnpikes with respect to time-varying solutions (or, more generally, to manifolds) have recently been studied in, for example, [13] and references therein. Observe in Figures 3.1b and 3.1c that the optimal states x_* demonstrate a so-called exact (time-varying) turnpike property, that is, for a major portion of the time-horizon $x_*(t)$ is equal to $x_{\mathrm{s}}(t)$.



(a) (b) (c)

Figure 3.1: Numerical simulations from Example 3.1. (a) Graph of $x_{\mathrm{s}}(t)$, $u_{\mathrm{s}}(t)$ and $v(t)$ against t. (b) and (c) Graphs of $u_*(t)$ (dashed line) and $x_*(t)$ (solid line) against t, respectively. Black dotted lines are input constraints, and faint lines are $x_{\mathrm{s}}(t)$, $u_{\mathrm{s}}(t)$.

We conclude this section by commenting on further simplifying cases. If $v = 0$, then equation (3.4) reduces to a quadratic equation in x and has the unique positive solution

$$\frac{cr + pqK(r-\delta) + K\Big(\big(cr/K + pq(r-\delta)\big)^2 + 8c\delta pqr/K\Big)^{1/2}}{4pqr}.$$

Observe that when $\delta = 0$, the solution above simplifies further to

$$x_\dagger := \frac{c + pqK}{2pq} = \frac{K}{2} + \frac{c}{2pq} > K/2\,,$$

and yields the corresponding optimal control

$$u_\dagger := r\Big(1 - \frac{x_\dagger}{K}\Big) = \frac{r}{q}\Big(\frac{pqK - c}{2pqK}\Big) < \frac{r}{2q}\,.$$

These expressions are readily seen to collapse further to $K/2$ and (3.3), respectively, when $c = 0$. Moreover, for $u_\dagger$ to be biologically meaningful requires that $pqK - c > 0$. The upshot is that the presence of a cost of harvesting in the model increases the optimal population size and reduces the optimal harvesting effort, both by a constant proportional to the ratio c/p of cost to price per unit harvested.

3.2 An inshore-offshore model

Consider the inshore-offshore model from [5, equation (10.48), p. 337], augmented to include migration terms. Specifically, denote the inshore biomass by x_1 and the offshore biomass by x_2. These are modelled by coupled scalar differential equations with intrinsic growth rates given by the non-linear functions f_1 and f_2, respectively. The parameter $\sigma > 0$ determines movement between the inshore and offshore biomasses. There are at most two control variables u_1 and u_2 which denote the harvesting effort of the inshore and offshore biomasses, respectively, and are both nonnegative valued. Combined, the dynamics are thus given by

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1) + \sigma(x_2 - x_1) - x_1 u_1 + v_1 \\ \dot{x}_2 &= f_2(x_2) + \sigma(x_1 - x_2) - x_2 u_2 + v_2 \,. \end{aligned}\right\} \tag{3.5}$$

The function

$$(t, x) \mapsto \begin{pmatrix} f_1(x_1) + \sigma(x_2 - x_1) - x_1 u_1(t) + v_1(t) \\ f_2(x_2) + \sigma(x_1 - x_2) - x_2 u_2(t) + v_2(t) \end{pmatrix} ,$$

is essentially nonnegative when both f_1 and f_2 are and $v_i(t) \geq 0$, so that solutions x_i of (3.5) satisfy $x_1(t), x_2(t) \geq 0$ for all $t \geq 0$ whenever both $x_1(0)$ and $x_2(0)$ are nonnegative, and u and v_i are piecewise continuous. A number of further possible mild assumptions on the f_i yield that nonnegative solutions x_i of (3.5) are bounded, again when the v_i are.

Analogously to (3.2), the resource-value function E is defined by

$$E(x) := p_1 x_1 + c_1\big(\ln(B_1) - \ln(x_1)\big) + p_2 x_2 + c_2\big(\ln(B_2) - \ln(x_2)\big), \quad x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

where the positive constants B_i are upper bounds for x_i, and the interpretation of the positive constants p_i and c_i is as in Section 3.1. Here the harvesting efficacy of both populations is assumed to equal 1, as compared to the situation considered in Section 3.1 where it was represented by $q = q_i$.

The model (3.5) fits the framework (1.2), albeit differently depending on whether one- or both- of the populations are harvested. These situations are considered separately, beginning with the former.

Harvesting the inshore population only

Assume that $u_2 = 0$ in (3.5), and so set $p_2 = c_2 = 0$ and relabel

$$p_1 = p, \quad c_1 = c, \quad u_1 = u, \quad x_1 = x, \quad x_2 = z \,,$$

so that (3.5) is of the form (1.2) with

$$\begin{aligned} f(t, x, z) = f(t, x) &:= f_1(x) + \sigma(z - x) + v_1(t), \quad b(x) := x, \\ h(t, x, z) &:= f_2(z) + \sigma(x - z) + v_2(t) \,, \end{aligned}$$

and E now coincides with that in (3.2). We further assume that both f_1 and f_2 are logistic, that is,

$$f_i(x) = r_i x\left(1 - \frac{x}{K_i}\right) , \tag{3.6}$$

for constants $r_i, K_i > 0$.

Since the variable z is present, this example falls into Case 2 of the resource-value balance method. However, as both x and z are scalar valued with

$$\frac{\partial h}{\partial z} = \sigma \neq 0 \,,$$

computing the co-state λ can be avoided in the special case (2.11), if we further assume that

$$\delta = 0, \quad \text{and} \quad v_1,\, v_2 \text{ are equal to zero (hence constant).} \tag{3.7}$$

Consequently, the necessary conditions from the Pontryagin Principle reduce to an optimal trajectory $(x_\mathrm{s}, z_\mathrm{s})$ satisfying the destination curve (2.12) at $t = t_1$ and are level curves of (2.14).

Routine differentiation gives that the destination curve condition becomes

$$\left(pr_1\left(1-\frac{2x}{K_1}\right)+\frac{cr_1}{K_1}+\frac{c\sigma z}{x^2}-\sigma p\right)\Big|_{t=t_1}=0\,,$$

which rearranges to the cubic equation

$$z_{\rm s}(t_1)=\frac{1}{c\sigma}\left(\sigma p-\frac{cr_1}{K_1}-pr_1\left(1-\frac{2x_{\rm s}(t_1)}{K_1}\right)\right)x_{\rm s}^2(t_1)\,. \tag{3.8}$$

After calculation, the function for the level curves is

$$\begin{aligned}\left(pr_1\left(1-\frac{2x}{K_1}\right)+\frac{cr_1}{K_1}+\frac{c\sigma z}{x^2}-\sigma p\right)\left(r_2 z\left(1-\frac{z}{K_2}\right)+\sigma(x-z)\right)\\ -\sigma\left(r_1 x\left(1-\frac{x}{K_1}\right)+\sigma(z-x)\right)\left(p-\frac{c}{x}\right).\end{aligned} \tag{3.9}$$

The above expression admits some simplification, but seemingly not substantially in general. In light of the differential equation for z,

$$\dot z=r_2 z\left(1-\frac{z}{K_2}\right)+\sigma(x-z)=-\frac{r_2 z^2}{K_2}+(r_2-\sigma)z+\sigma x\,,$$

the level curves in the (x,z)-plane are traversed according to sign of the quadratic

$$-\frac{r_2 z^2}{K_2}+(r_2-\sigma_2)z+\sigma x\,. \tag{3.10}$$

Since $\dot x$ and $\dot z$ are bounded, very roughly speaking, the level curve to-be-traversed depends on the length of the time interval $[t_0,t_1]$, capturing the time available to reach the destination curve. The current problem is time-invariant, and so t_0 may be chosen equal to 0.

In this optimal control problem the initial value $z(0)$ is given, and $x_{\rm s}(0)$ is to be determined. Thus, we arrive at the following qualitative description of the optimal state/auxiliary variable trajectory, namely:

1. Begin on a level curve as determined by $z(0)$ and time duration t_1-t_0.
2. Traverse the level curve (3.9) according to sign of (3.10).
3. Arrive at the destination curve (3.8) at time $t=t_1$.

For a numerical example, we take model data:

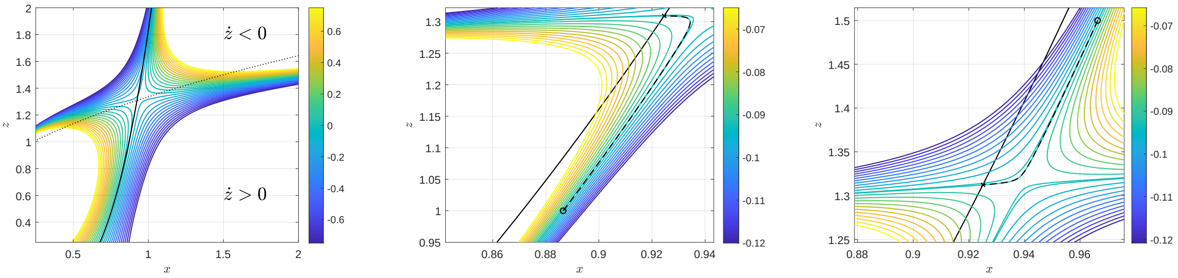
$$r_1=1.5,\quad r_2=2.3,\quad K_1=1,\quad K_2=1.5,\quad \sigma=p=1,\quad c=0.75,\quad t_0=0,\quad t_1=2\,, \tag{3.11}$$

chosen somewhat arbitrarily to illustrate our results.

Numerical simulation results are plotted in Figure 3.2. Sample level curves and the destination curve in the (x,z)-plane are plotted in Figure 3.2a. Panels (b) and (c) contain level curves and the destination curve, with optimal trajectories computed in Bocop additionally plotted, varying by choice of $z(0)=1$ and $z(0)=1.5$, respectively. As expected from the resource-value balance method, we see that the optimal curves are level curves of (3.9) and terminate at the destination curve (3.8). The actual control variable $u_{\rm s}$ is determined (at least numerically) from $(x_{\rm s},z_{\rm s})$ from the first equation in (3.5).

Harvesting the inshore population only — equilibrium solutions

Still under assumption (3.7), now assume that a (constant) equilibrium optimal solution is sought, so that equation (2.15) holds.



(a) (b) (c)

Figure 3.2: Numerical simulations from the inshore harvesting model only of Example 3.2. (a) Level curves (3.9) (coloured) and destination curve (3.8) (black) in (x, z)-plane. The black dotted line is the zero contour of (3.10), determining the sign of $\dot{z}$. (b) and (c) Optimal trajectories $(x_{\mathrm{s}}, z_{\mathrm{s}})$ from black circle to black cross, subject to $z(0) = 1$ and $z(0) = 1.5$, respectively.

Here the constraint $h(x, z) = 0$ reads

$$h(x, z) = r_2 z\Big(1 - \frac{z}{K_2}\Big) + \sigma(x - z)\,,$$

and the first-order condition for optimality becomes, after some simplification,

$$\begin{aligned} M(x, z) = {}& c\Big(r_1 x\Big(1 - \frac{x}{K_1}\Big) + \sigma(z - x)\Big)\Big(r_2\big(1 - \frac{2z}{K_2}\big) - \sigma\Big) \\ & + \big(px^2 - c\big)\Big(r_1 r_2\big(1 - \frac{2x}{K_1}\big)\big(1 - \frac{2z}{K_1}\big) - \sigma r_1\big(1 - \frac{2x}{K_1}\big) - \sigma r_2\big(1 - \frac{2z}{K_2}\big)\Big) = 0\,. \end{aligned} \tag{3.12}$$

With model data (3.11), simulation results are plotted in Figure 3.3. Figure 3.3a contains the zero contours of $h(x, z)$ and $M(x, z)$ which, observe, appear as the centre of the hyperbolae-like of the level curves (3.9). Their intersection is the optimal equilibrium solution, denoted $(x_{\mathrm{c}}, z_{\mathrm{c}})$. Figure 3.3b compares solutions $(x_{\mathrm{s}}, z_{\mathrm{s}})$ and $(x_{\mathrm{c}}, z_{\mathrm{c}})$. Observe that these solutions are distinct, but that the time evolution of $(x_{\mathrm{s}}, z_{\mathrm{s}})$ is "somewhat close" to $(x_{\mathrm{c}}, z_{\mathrm{c}})$ — see Figure 3.3c — another illustration of the so-called turnpike property in optimal control. This effect becomes more pronounced as the length of the time interval increases.

Harvesting both inshore and offshore populations

Now assume that both controls in u_1 and u_2 in (3.5) are available. Thus, the model (3.5) is of the form (1.2) with

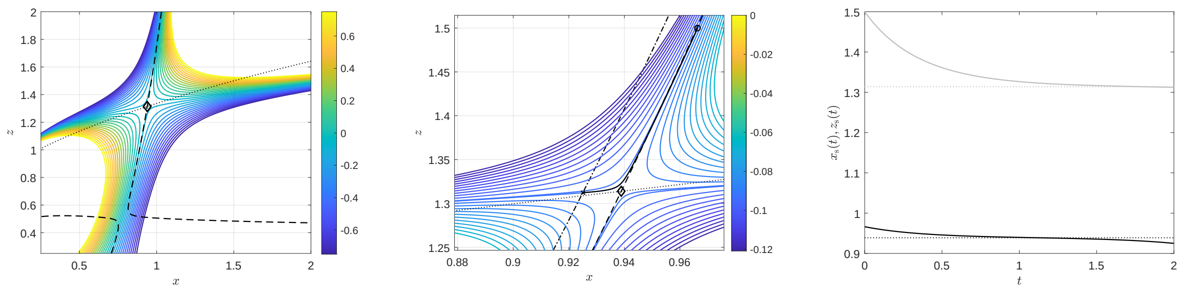
$$f(t, x) = f(t, x, z) := \begin{pmatrix} f_1(x_1) + \sigma(x_2 - x_1) + v_1(t) \\ f_2(x_2) + \sigma(x_1 - x_2) + v_2(t) \end{pmatrix}, \quad b(x) := \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix},$$

and where the variable z is absent, so that Case 1 of the resource-value balance method is applicable. Consequently, the to-be-maximized function (2.6) is given by

$$\begin{aligned} (x_1, x_2) \mapsto {}& - \delta E + \Big(p_1 - \frac{c_1}{x_1}\Big)\Big(f_1(x_1) + \sigma(x_2 - x_1) + v_1(t)\Big) \\ & + \Big(p_2 - \frac{c_2}{x_2}\Big)\Big(f_2(x_2) + \sigma(x_1 - x_2) + v_2(t)\Big)\,. \end{aligned} \tag{3.13}$$

According to (2.7), under the simplifying assumption that $p_1 = p_2 = p$, the first-order necessary condition for a maximum of (3.13) is

$$\left.\begin{aligned} -\delta\Big(p - \frac{c_1}{x_1}\Big) + \frac{c_1}{x_1^2}\big(f_1(x_1) + v_1(t)\big) + \Big(p - \frac{c_1}{x_1}\Big) f_1'(x_1) + \sigma\frac{c_1 x_2^2 - c_2 x_1^2}{x_2 x_1^2} = 0\,, \\ -\delta\Big(p - \frac{c_2}{x_2}\Big) + \frac{c_2}{x_2^2}\big(f_2(x_2) + v_2(t)\big) + \Big(p - \frac{c_2}{x_2}\Big) f_2'(x_2) + \sigma\frac{c_2 x_1^2 - c_1 x_2^2}{x_1 x_2^2} = 0\,, \end{aligned}\right\} \tag{3.14}$$



(a) (b) (c)

Figure 3.3: Numerical simulations from the equilibrium-solutions inshore harvesting model only of Example 3.2. (a) Zero contours of h (black dotted) and M (black dashed) with intersection $(x_{\mathrm{c}}, z_{\mathrm{c}})$ marked with the black diamond. (c) Optimal trajectory $(x_{\mathrm{s}}, z_{\mathrm{s}})$ (black solid), from circle to cross, equilibrium solution in black diamond. Black dotted, dashed and dashed-dotted lines are zero contours of h, M, and destination curve, respectively. (c) Graphs of $x_{\mathrm{s}}(t)$ (black) and $z_{\mathrm{s}}(t)$ (grey) as in panel (b) against t, with x_{c}, z_{c} plotted in dotted lines.

which coincides with [5, equation (10.50), p. 338] when $v_1 = v_2 = 0$, obtained there by integration by parts in (1.5). In general, developing the condition (3.14) further requires bespoke assumptions on f_1 and f_2. However, under the assumption that, for each $v_1(t)$, $v_2(t)$, equation (3.13) has a unique componentwise positive maximum x_{s}, equation (3.5) is used as a definition of the resulting (in general not constant) optimal control, that is,

$$\left.\begin{aligned} u_{\mathrm{s},1} &:= \frac{f_1(x_{\mathrm{s},1}) + \sigma(x_{\mathrm{s},2} - x_{\mathrm{s},1}) + v_1 - \dot{x}_{\mathrm{s},1}}{x_{\mathrm{s},1}} \\ \text{and} \quad u_{\mathrm{s},2} &:= \frac{f_2(x_{\mathrm{s},2}) + \sigma(x_{\mathrm{s},1} - x_{\mathrm{s},2}) + v_2 - \dot{x}_{\mathrm{s},2}}{x_{\mathrm{s},2}} \,. \end{aligned}\right\} \tag{3.15}$$

Again note that knowledge of v_i is required to determine u_{s}. As a numerical illustration, assume that the f_i are logistic and that the inshore and offshore habitats give rise to the same dynamics, that is, f_i are of the form (3.6) with $r = r_1 = r_2$ and $K = K_1 = K_2$. By rescaling both x_i and time, we may assume that

$$r = 1, \quad K = 1 \quad \text{so that} \quad f_1(y) = f_2(y) = y(1-y)\,. \tag{3.16a}$$

Assuming that the price for both populations is equal, and that the offshore population is more expensive to catch, we consequently set

$$\sigma = 0.01, \quad p_1 = p_2 = 1, \quad c_1 = 0.4, \quad c_2 = 0.6\,, \tag{3.16b}$$

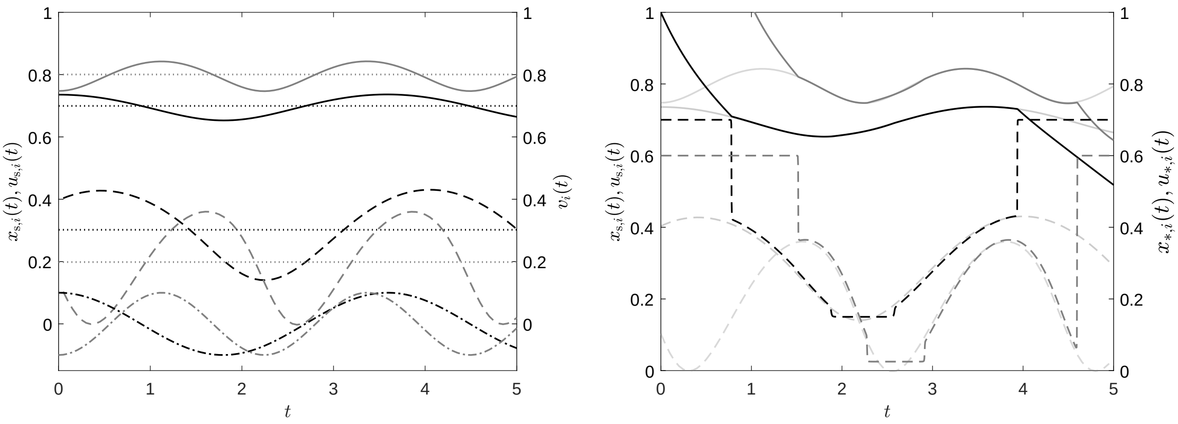
and, finally, set

$$\delta = 0, \quad t_0 = 0, \quad t_1 = 5, \quad v_1(t) = 0.1\cos(1.75t) \quad \text{and} \quad v_2(t) = -0.1\cos(2.8t)\,. \tag{3.16c}$$

Observe that the v_i are not nonnegative valued, but this is only used to *a priori* ensure nonnegativity of state trajectories. Here we compute numerically that (3.13) has a (componentwise) positive maximum, and that the resulting u_{s} is nonnegative valued. Graphs of $x_{\mathrm{s}}(t), u_{\mathrm{s}}(t)$ and $v(t)$ against t are are contained in Figure 3.4a. The functions x_{s} were computed using `fmincon` in MATLAB.

In the case that $v_i = 0$, then x_{s} and u_{s} are constant, and are here given by

$$x_{\mathrm{s}} = \begin{pmatrix} 0.6995 \\ 0.8004 \end{pmatrix} \quad \text{and} \quad u_{\mathrm{s}} = \begin{pmatrix} 0.3019 \\ 0.1983 \end{pmatrix}\,, \tag{3.17}$$



(a) (b)

Figure 3.4: Numerical simulations from the two-harvesting model (3.5) of Example 3.2 subject to model data (3.16). (a) Graphs of $x_{\mathrm{s}}(t)$ (solid), $u_{\mathrm{s}}(t)$ (dashed) and $v(t)$ (dashed-dotted) against t. Black and grey lines correspond to $i = 1$ and $i = 2$, respectively. Dotted lines are respective constant x_{s}, u_{s} components corresponding to $v_i = 0$. (b) Graphs of $x_*(t)$ (solid), $u_*(t)$ (dashed) against t. Fainter lines are x_{s}, u_{s} from panel (a).

which are also plotted in Figure 3.4a. To illustrate Proposition 2, for the alternative optimal control problem of maximising (1.5) additionally subject to the control constraint (2.18), we take

$$u_{\min} = \begin{pmatrix} 0.15 \\ 0.025 \end{pmatrix} \quad \text{and} \quad u_{\max} = \begin{pmatrix} 0.7 \\ 0.6 \end{pmatrix}. \tag{3.18}$$

Recall that in this problem the initial state $x(0) = x_*(0)$ is assumed given. An optimal trajectory (u_*, x_*) was computed in Bocop for $x(0) = (1, 4)$, and graphs of $u_*(t)$ and $x_*(t)$ against t are plotted in Figure 3.4b. A bang-singular trajectory is observed with singular portions coinciding with $(u_{\mathrm{s}}, x_{\mathrm{s}})$, as predicted by the resource-value balance method.

The example is concluded with commentary on how the above deliberations inform, at least qualitatively, the solution to the alternative optimization criterion (1.5), with the additional control constraint (2.18) in the case $v_1 = v_2 = 0$, so that x_{s} and u_{s} as in (3.17) are constant and, moreover, $(u_{\mathrm{s}}, x_{\mathrm{s}})$ is a feasible trajectory.

Anticipating the qualitative description of the optimal trajectory from the Pontryagin Principle, we note that there are eight combinations of bang-bang/singular controls, for ease denoted:

$$\left.\begin{array}{l} w_1 = u_{\min}, \quad w_2 = (u_{\max,1}, u_{\min,2}), \quad w_3 = (u_{\min,1}, u_{\max,2}), \quad w_4 = u_{\max}, \\ w_5 = (u_{\min,1}, u_{\mathrm{s},2}), \quad w_6 = (u_{\max,1}, u_{\mathrm{s},2}), \quad w_7 = (u_{\mathrm{s},1}, u_{\min,2}), \quad w_8 = (u_{\mathrm{s},1}, u_{\max,2}). \end{array}\right\} \tag{3.19}$$

We partition the (x_1, x_2) phase portrait[1] into eight regions, those enclosed by the solutions of the final value problem (3.5) subject to $u = w_i$ and final value x_{s}. These curves are denoted by γ_i. Since the problem (3.5) subject to constant controls is time-invariant, the choice of final time is unimportant for the construction of γ_i, provided that it is sufficiently large.

The Pontryagin Principle yields that the optimal control is bang-singular. For the current problem, we further expect that the first portion of an optimal trajectory steers $x_*(0)$ to x_{s}, at which point u_* switches to u_{s}. Since the switching times of the components are generically different, the switch to u_{s} is from one of the four bang-singular controls w_5, w_6, w_7, w_8. In other words, generically, optimal trajectories reach x_{s} along one the bang-singular curves $\gamma_5, \dots \gamma_8$.

Thus, the following qualitative description of the first portion of optimal trajectories is proposed:

[1] restricting attention to the phase portrait intersected with the nonnegative orthant $\mathbb{R}^2_+$

(I) solutions with $x_*(t_0) \in \gamma_i$ traverse them to x_{s};

(II) solutions with $x_*(t_0) \notin \gamma_i$ are subject to one of the bang-bang controls $w_1, \dots w_4$, determined by the region of the phase portrait, until the trajectory intersects one of the bang-singular curves $\gamma_5, \dots \gamma_8$. The solution then traverses this curve to x_{s}.

We expect the optimal control to switch to one of the maximal bang-bang/singular controls (w_4, w_6, w_8) near the final $t = t_1$, a phenomenon which is not predicted by the present resource-value balance method. We also comment that we have not rigorously ruled out multiple switches between bang-bang controls before an optimal trajectory intersects one of the curves $\gamma_5, \dots \gamma_8$. However, we have not observed such multiple switches occurring in this example, and it seems implausible in this context.

Simulation results are plotted in Figures 3.5. The curves γ_i are plotted in coloured lines. The nine black dotted lines are optimal trajectories for a range of initial conditions, as computed by Bocop. Each trajectory is observed to reach x_{s}. The solutions plotted have been truncated once they reach x_{s}. The qualitative behaviour predicted in (I) and (II) above is observed — eight of the nine solutions (those starting at the crosses) are subject to bang-bang controls $w_1, \dots, w_4$ before the state-trajectory intersects one of $\gamma_5, \dots \gamma_8$. The optimal state-trajectory then traverse these curves to x_{s}. The ninth solution has been chosen to satisfy $x_*(0) \in \gamma_4$, and is seen to traverse γ_4 to x_{s}.

As an illustration of item (II), Figure 3.6 contains phase portraits of (3.5) subject to the four bang-bang controls $w_1, \dots, w_4$. These were computed using the 2D Phase Portrait Plotter function [31] for MATLAB. The bang-singular curves $\gamma_5, \dots, \gamma_8$ as in Figure 3.5 are also shown. Each phase portrait is applicable for initial conditions $x_*(0)$ which belong to coloured regions shown. It is observed that the phase lines intersect the bang-singular curves $\gamma_5, \dots \gamma_8$, at which point the control switches to the corresponding bang-singular control.

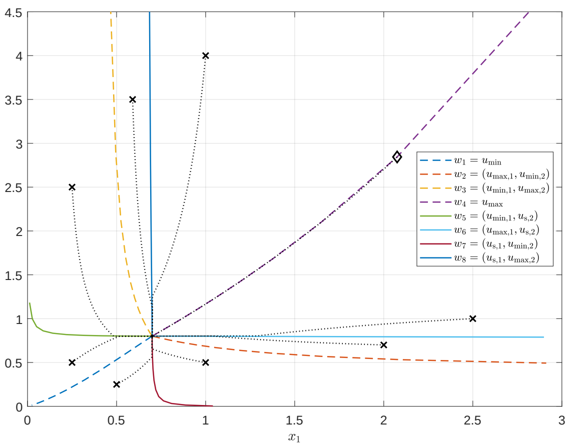


Figure 3.5: Numerical simulations from the two-harvesting model (3.5) of Example 3.2 subject to model data (3.16) and (3.18). The graph is in (x_1, x_2)-space. Solid coloured lines are the curves γ_i, the solutions x of (3.5) subject to the four combinations of bang-singular controls $w_5, \dots, w_8$ in (3.19) and the final value condition x_{s}. Dashed coloured lines are the curves γ_i corresponding to four combinations of bang-bang controls $w_1, \dots, w_4$ in (3.19). The black dotted lines are numerically-computed optimal trajectories x_* subject to different initial conditions, marked with cross or diamond. The optimal controls for initial states not on γ_i are bang-bang until the corresponding state x intersects the curves γ_i, at which point the solutions traverse these curves to x_{s}.

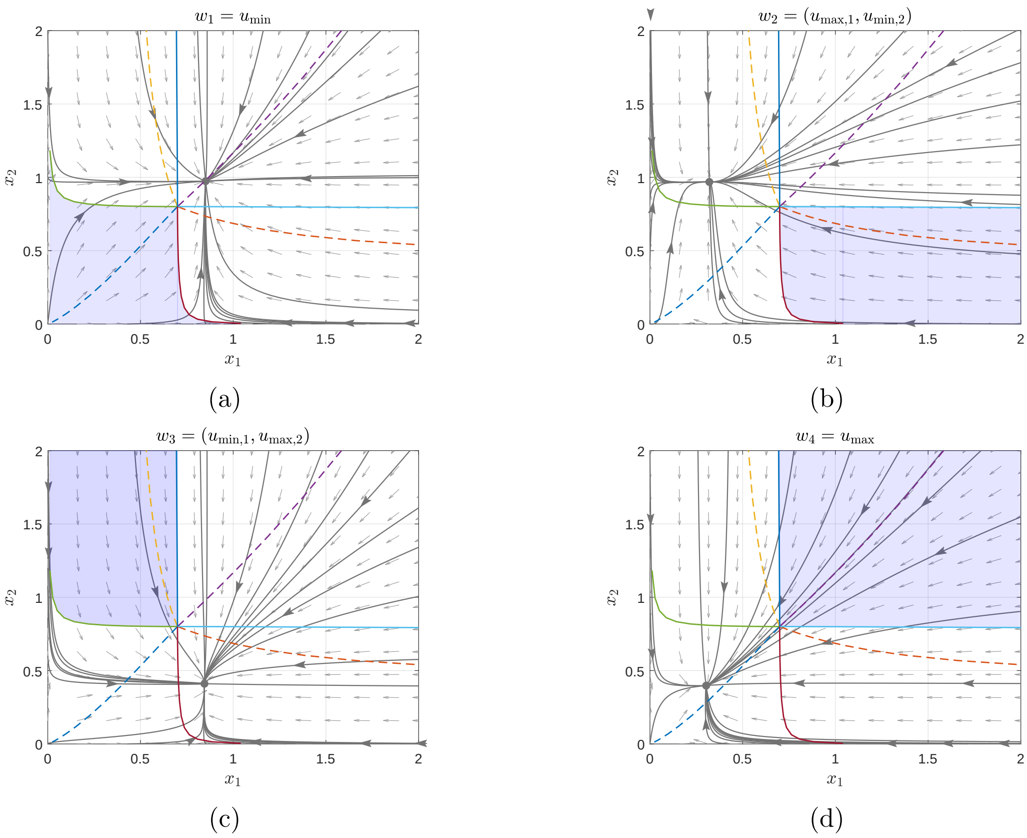


Figure 3.6: Numerical simulations from the two-harvesting model (3.5) of Example 3.2 subject to model data (3.16) and (3.18). The coloured (solid, dashed) lines are the curves γ_i as in Figure 3.5 and described in main text, and which intersect at x_s. Each panel gives the phase portrait of (3.5) subject to constant bang-bang control (see titles). The combination of the blue coloured regions is a phase portrait for the optimally-controlled system. The phase and quiver lines are plotted in grey. In each panel, it is observed from the phase/quiver lines that trajectories starting in each coloured region intersect one of the solid lines $\gamma_5, \ldots, \gamma_8$, at which point the control switches to bang-singular, and the solution traverses these curves to x_s. Solutions which begin on $\gamma_1, \ldots, \gamma_4$ traverse these to x_s.

3.3 A cohort fisheries model

A cohort fisheries model and corresponding optimal control problems are presented in detail in [6, Section 6.2, pp. 233–243], and also in the second edition [5, Sections 9.2 and 9.3] where it is termed the Beverton-Holt commercial fisheries model, after [2]. We approach the problem from the resource-value balance perspective. This example contains a resource-value function with explicit t dependence, distinguishing it from the examples considered hitherto.

Let x denote the abundance of a single local population with biomass per individual denoted w — a given function. The following model and resource-value function are assumed:

$$\dot{x} = -(r + qu)x, \quad E(t, x) := p(t)w(t)x(t) + \frac{c}{q}\big(\ln(x(0)) - \ln(x)\big)\,, \tag{3.20}$$

where the positive constant r denotes natural mortality, and c, $p = p(t)$, and q are as in Section 3.1, although p is now permitted to depend on time. Here u denotes fishing mortality. In particular, wx denotes the total biomass of the population and, consequently, the value of the resource-value E depends explicitly on time t through the functions p and w. The differential equation in (3.20) is of the form (1.1) with

$$f(t, x, z) = f(x) := -rx, \quad b(x) := qx\,.$$

It is clear that solutions x to the differential equation (3.20) are nonnegative from nonnegative initial conditions and are bounded by $x(0)$ when $u \geq 0$. The positive initial condition $x(0)$ is called the recruitment in [5, p. 276].

With the above value function, the integral in the performance criterion (1.1) equals

$$\int_{t_0}^{t_1} \mathrm{e}^{-\delta t}\big(qp(t)w(t)x(t) - c\big)u(t)\,\mathrm{d}t\,.$$

The resource-value balance method is applied to maximize (1.1), appealing to Case 1 as the variable z is absent. Setting $\alpha(t) := p(t)w(t)$, the to-be-maximized function (2.6) is given by

$$\begin{aligned} x \mapsto &-\delta\Big(\alpha(t)x + \frac{c}{q}\big(\ln(x(0)) - \ln(x)\big)\Big) + \dot{\alpha}(t)x + \Big(\alpha(t) - \frac{c}{qx}\Big)(-rx) \\ &= (\dot{\alpha} - r\alpha - \delta\alpha)\,x + \frac{c\delta}{q}\ln(x) + \frac{cr}{q} - \frac{c\delta}{q}\ln(x(0))\,. \end{aligned} \tag{3.21}$$

The first-order necessary condition (2.7) for a maximum of (3.21) is

$$\big(\dot{\alpha} - r\alpha - \delta\alpha\big) + \frac{c\delta}{qx} = 0\,,$$

which is easily solved to give

$$x_{\mathrm{s}}(t) := \frac{c\delta}{q\big((r+\delta)\alpha - \dot{\alpha}\big)}\,,$$

and coincides with [7, equation (6.27), p. 236] and, note, also determines $x_{\mathrm{s}}(0)$. The corresponding control u_{s} is determined by x_{s} and the differential equation in (3.20). Under the simplifying assumption that p is independent of t, the optimal biomass profile is then

$$x_{\mathrm{s}}(t)w(t) = \frac{c\delta}{pq(r + \delta - \dot{w}/w)}\,.$$

cf. [7, equation (6.29), p. 236]. Observe that both x_{s} and $x_{\mathrm{s}}w$ are only feasible if $r+\delta-\dot{w}/w > 0$ for all $t > 0$. The biomass per individual function w is typically assumed to be positive and bounded, with $\dot{w}/w$ a decreasing function with time. Therefore, under this assumption, the singular biomass may only be feasible beyond some $t_{\mathrm{f}} > 0$. In this case, the optimal solution of the performance criterion (1.5) subject to (3.20) and the control constraint $0 \leq u \leq u_{\max}$ is as in [6, pp. 237–238]. Indeed, let $x_0(t) = x(0)\mathrm{e}^{-rt}$ denote the unfished population, so that $x_0(t)w(t)$ equals the unfished biomass. Let t_{b} denote the point of intersection of $x_0(t)w(t)$ and the singular biomass $x_{\mathrm{s}}(t)w(t)$, and let t_{s} be such that

$$x_{\mathrm{s}}(t)w(t) = \frac{c}{qp}\,,$$

after which time harvesting is uneconomic. Then, assuming that input constraints are not violated, the optimal fishing strategy equals zero before t_{b} and after t_{s}, and equals the singular control u_{s} on $[t_{\mathrm{b}}, t_{\mathrm{s}}]$.

By way of examples of biomass functions, the von Bertalanffy growth function [29], see more recently [18],

$$w_1(t) = w_\infty\big(1 - \rho_1\mathrm{e}^{-a_1 t}\big)^3 \quad t \geq 0\,,$$

for positive constants w_∞, a_1 and $\rho_1 \in (0,1)$ is considered in [7, Section 6.2][2]. As another example, consider the Gompertz function

$$w_2(t) = w_\infty \mathrm{e}^{-\rho_2\mathrm{e}^{-a_2 t}} \quad t \geq 0\,,$$

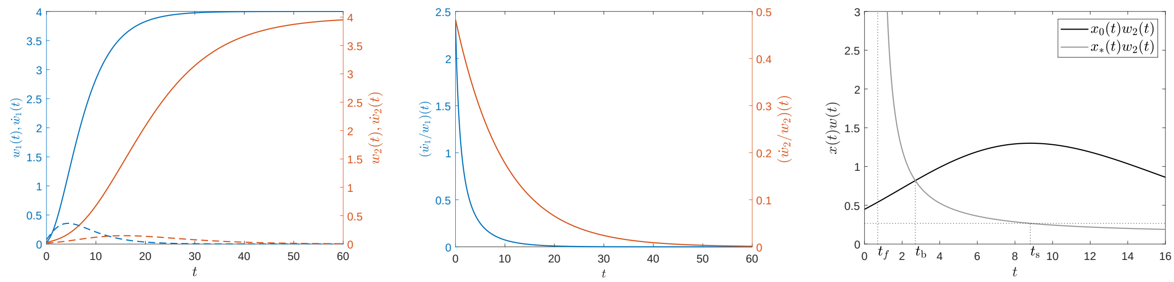
[2]The function in [7, equation (6.17), p. 234] is the more usual von Bertalanffy length function cubed, corresponding to volume rather than length.

for further positive constants a_2 and ρ_2; see, for example [19]. As is well known, w_∞ denotes the limiting/asymptotic value of w_i, the constants ρ_i determine $w_i(0) > 0$, and a_i capture rates of growth. Graphs of w_i and their derivatives, and the quotients $\dot{w}_i/w_i$ are contained in Figures 3.7a and 3.7b, respectively. We see graphically from the second figure that $\dot{w}_i/w_i$ is bounded and decreasing for $i = 1, 2$, and this is elementary to verify mathematically. Indeed, for instance, $\dot{w}_2/w_2 = \rho_2 a_2 \mathrm{e}^{-a_2 t}$. Graphs of the unfished and singular biomass are contained in Figure 3.7c, both assuming biomass function w_2. In both cases we, somewhat arbitrarily for the sake of illustration, take

$$w_\infty = 4, \quad \rho_1 = 0.8, \quad a_1 = 0.2, \quad \rho_2 = 4.8283, \quad a_2 = 0.1,$$
$$p = 1.5, \quad \delta = 0.25, \quad c = 0.4, \quad r = 0.2, \quad x_0(0) = 14\,.$$

The ρ_i are chosen so that $w_i(0) = 0.0320$ (viz. are equal). With these values, numerical calculations give

$$t_\mathrm{f} = 0.7042, \quad t_\mathrm{b} = 2.701, \quad t_\mathrm{s} = 8.8126\,.$$



(a) (b) (c)

Figure 3.7: Numerical simulations from the fisheries model of Example 3.3. (a) Graphs of biomass functions $w_i(t)$ against t (solid lines), and their derivatives (dashed lines) for $i = 1, 2$. (b) Graphs of quotients $\dot{w}_i/w_i$ against t for $i = 1, 2$. (c) Graphs of unfished biomass $x_0(t)w_2(t)$ and singular biomass $x_\mathrm{s}(t)w_2(t)$, both against t. The key times t_f, t_b and t_s are indicated.

Finally, we comment that the above analysis has crucially exploited that $\delta > 0$ and, indeed, breaks down when $\delta = 0$. In this case the optimal fishing policy is an infinite impulse at the maximum of the natural (unfished) biomass.

3.4 The resource-value function E

The resource-value function E in (3.2) in the basic single population harvesting problem, as well as those appearing in the inshore-offshore model and Beverton-Holt fisheries models, are unbounded as x approaches zero, presently corresponding to small population sizes. Here we consider this matter. More generally, consider

$$E(x) := px + \frac{c}{q}\big(C_0 - \ln(x)\big)$$

for a constant C_0, which is for now arbitrary. In bioeconomic contexts, the function E has the interpretation that $E(t, x(t))$ is the undiscounted net value of having a population of size x at time t, so that multiplying with $\mathrm{e}^{-\delta t}$ gives the discounted value. The term px and

$$C(x) := \frac{c}{q}\big(\ln(x) - C_0\big)\,,$$

have the respective interpretations of gross value (price times amount) and the cost, noting the sign change in C compared to E. In the latter case, it is easier to interpret this by considering

$$\frac{\partial C}{\partial x} = \frac{c}{qx},$$

which has the interpretation of marginal cost being a decreasing function of population size. Moreover, the marginal cost is unbounded as x decreases to zero. Both of these properties are economically realistic and meaningful. However, a consequence of the above expression for marginal cost is that it becomes impossible to choose the integration constant C_0 so that $C(x) > 0$ for all $x > 0$. This issue can be fixed as follows. We instead consider the marginal cost

$$\frac{\partial C^\sharp}{\partial x} = \begin{cases} \frac{-c}{qx_\mathrm{c}^2}(x - x_\mathrm{c}) + \frac{c}{qx_\mathrm{c}} & x < x_\mathrm{c} \\ \frac{c}{qx} & x \geq x_\mathrm{c}, \end{cases}$$

which precisely means that we make the marginal cost linear on $(0, x_\mathrm{c})$ and continuously differentiable at x_c (and, hence, for all $x > 0$). The cost then is

$$C^\sharp(x) := \begin{cases} \frac{-c}{2qx_\mathrm{c}^2}(x - x_\mathrm{c})^2 + \frac{c}{qx_\mathrm{c}}(x - x_\mathrm{c}) + C_0 & x < x_\mathrm{c} \\ \frac{c}{q}\big(\ln(x) - \ln(x_\mathrm{c})\big) + C_0 & x \geq x_\mathrm{c}, \end{cases}$$

where the integration constant C_0 now has the interpretation of $C_0 = C^\sharp(x_\mathrm{c})$. It makes sense to choose the integration constant so that $C^\sharp(0) = 0$ and this gives

$$C^\sharp(x) := \begin{cases} \frac{-c}{2qx_\mathrm{c}^2}(x - x_\mathrm{c})^2 + \frac{c}{qx_\mathrm{c}}(x - x_\mathrm{c}) + \frac{3c}{2q} & x < x_\mathrm{c} \\ \frac{c}{q}\big(\ln(x) - \ln(x_\mathrm{c})\big) + \frac{3c}{2q} & x \geq x_\mathrm{c}, \end{cases}$$

which, by construction, is positive for $x > 0$. Other adjustments to the marginal cost on $(0, x_c)$ would achieve the same outcome.

This revised cost $C^\sharp$ changes E and that in turn changes $\partial E/\partial x$ and, therefore, in particular changes the integrand in (1.1). We can choose $x_\mathrm{c} > 0$ to be arbitrarily small, so that any such change only happens for extremely small population sizes. Since these will not be involved in the optimal policy, the change is in fact irrelevant for optimal control purposes.

The issue of positivity of C is an example of model breakdown, along with the case $\delta = 0$ in Section 3.2. Here, the model (3.2) breaks down for small population sizes. However, in the optimal control problem small population sizes are never relevant, so this aspect of model breakdown is not an issue.

4 Conclusions

The resource-value balance method for simplifying the computation of optimal controls in optimal control problems commonly arising in natural resource management contexts has been presented. Summarising, the method rewrites the original optimal control problem, namely maximizing (1.1), via the so-called "resource-value balance" equation (2.1), the right-hand side of which is independent of the original control variable u. This result appears as Theorem 1. Since the original aim of maximizing the left-hand side of (2.1) is equivalent to maximizing the right-hand side of (2.1), where we now view the state variable x as a control, the resource-value balance method can obviate the need for classical optimal control theoretic techniques such as the Pontryagin Principle, as has been illustrated through a range of examples from bioeconomics. Indeed, when no dynamic variables (currently termed z) are present in the new optimization problem, then elementary calculus methods for optimization may be used instead, greatly simplifying the calculations involved. In particular, the concept of a shadow prices from economics is not required.

The relationship between solutions of the optimal control problem (P1): maximizing (1.1) subject to (1.2), and (the more common) problem (P2) of maximizing (1.5) — the integral term in (1.1) only — subject to the same dynamics (1.2) *and* the input constraint (2.18) was discussed in Section 2.1. Proposition 2 states that the critical controls of (P1) are precisely the (any) singular portions of optimal controls of (P2). The optimal solution to (P2) is bang-singular in general. In this sense, the inclusion of boundary conditions in (1.1) as compared to (1.5) may be seen as "correction terms" which ultimately simplifies the computation of the optimal control of (1.1), via the resource-value balance equation, and hence the (any) singular control of (1.5). The two main results, and the associated insights, comprise the main contribution of the present work, and arguably gather together and generalise a number of approaches of Clark deployed across [5]. It is our hope that this work helps to somewhat demystify singular controls in natural resource management and bioeconomic contexts.

In terms of future work, two avenues are proposed. The first is the optimal removal (control or management) of an invasive species, such as pest or weed control, considered in, for example [11, 21], with timely relevance to the societal challenge of ensuring food security. Optimal control problems naturally arise in this setting to balance the (economically harmful) presence of a pest with the cost (and potential environmental harm) of removal. The second is that the ideas underpinning the resource-value balance method have now been shown to apply to optimal control problems in bioeconomic/natural resource models (the present work), as well as in renewable energy conversion problems [16]. Whilst the resulting optimal control problems share some similarities, there are substantial differences. This motivates an exploration of the extent to which Theorem 1 and Proposition 2 generalise across optimal control problems.

Acknowledgement

Chris Guiver's contribution to this work was partially supported by a Personal Research Fellowship from the Royal Society of Edinburgh (RSE), award 2168. Chris Guiver expresses gratitude to the RSE for the financial support.

Statements and Declarations

There are no competing interests. No generative AI was used in the preparation of this work.

References

[1] A. Berman, M. Neumann, and R. J. Stern. *Nonnegative matrices in dynamic systems.* John Wiley & Sons Inc., New York, 1989.

[2] R. J. H. Beverton and S. J. Holt. On the dynamics of exploited fish populations. *Fish. Invest. Lond. Ser.*, 2:19, 1957.

[3] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge University Press, Cambridge, 2004.

[4] C. W. Clark. Bioeconomic modeling and resource management. In *Applied mathematical ecology (Trieste, 1986)*, volume 18 of *Biomathematics*, pages 11–57. Springer, Berlin, 1989.

[5] C. W. Clark. *Mathematical bioeconomics.* Pure and Applied Mathematics. John Wiley & Sons, Inc., Hoboken, NJ, second edition, 1990.

[6] C. W. Clark. *Mathematical bioeconomics.* Pure and Applied Mathematics. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2010.

[7] F. Clarke. *Functional analysis, calculus of variations and optimal control*, volume 264 of *Graduate Texts in Mathematics.* Springer, London, 2013.

[8] M. De Lara and L. Doyen. *Sustainable management of natural resources: mathematical models and methods.* Environmental Science and Engineering. Springer, Berlin, 2008.

[9] G. Di Letizia, C. De Lucia, P. Pazienza, and G. M. Cappelletti. Forest bioeconomy at regional scale: A systematic literature review and future policy perspectives. *For. Policy Econ.*, 155:103052, 2023.

[10] R. Dorfman, P. A. Samuelson, and R. M. Solow. *Linear programming and economic analysis*. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. A Rand Corporation Research Study.

[11] C. J. Edholm, B. Tenhumberg, C. Guiver, Y. Jin, S. Townley, and R. Rebarber. Management of invasive insect species using optimal control theory. *Ecol. Mod.*, 381:36–45, 2018.

[12] T. Faulwasser and D. Bonvin. On the design of economic NMPC based on an exact turnpike property. *IFAC-PapersOnLine*, 48(8):525–530, 2015.

[13] T. Faulwasser, K. Flaß kamp, S. Ober-Blöbaum, M. Schaller, and K. Worthmann. Manifold turnpikes, trims, and symmetries. *Math. Control Signals Systems*, 34(4):759–788, 2022.

[14] T. Faulwasser and L. Grüne. Turnpike properties in optimal control: An overview of discrete-time and continuous-time results. *Handb. Numer. Anal.*, 23:367–400, 2022.

[15] T. Faulwasser, M. Korda, C. N. Jones, and D. Bonvin. On turnpike and dissipativity properties of continuous-time optimal control problems. *Automatica*, 81:297–304, 2017.

[16] C. Guiver and M. R. Opmeer. The energy-balance method for optimal control in renewable energy applications. *Renew. Energy Focus*, 50:100582, 2024.

[17] W. M. Haddad, V. Chellaboina, and Q. Hui. *Nonnegative and compartmental dynamical systems*. Princeton University Press, Princeton, 2010.

[18] J. Hamre, E. Johnsen, and K. Hamre. A new model for simulating growth in fish. *PeerJ*, 2:e244, 2014.

[19] T. B. Kirkwood. Deciphering death: a commentary on Gompertz (1825) 'On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies'. *Philos. Trans. R. Soc. Lond. B. Biol. Sci.*, 370(1666):20140379, 2015.

[20] U. Krause. *Positive dynamical systems in discrete time*, volume 62 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2015. Theory, models, and applications.

[21] A. Lampert, A. Hastings, E. D. Grosholz, S. L. Jardine, and J. N. Sanchirico. Optimal approaches for balancing invasive species eradication and endangered species management. *Science*, 344(6187):1028–1031, 2014.

[22] D. Liberzon. *Calculus of variations and optimal control theory*. Princeton University Press, Princeton, 2011.

[23] G. R. Munro. The optimal management of transboundary renewable resources. *Can. J. Econ.*, pages 355–376, 1979.

[24] J. D. Murray. *Mathematical biology. I*, volume 17 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, third edition, 2002. An introduction.

[25] A. Seierstad and K. Sydsaeter. *Optimal Control Theory with Economic Applications*, volume 24 of *Advanced Textbooks in Economics*. North Holland, Amsterdam, 1987.

[26] S. N. Seo. *The Economics of Optimal Growth Pathways*. Palgrave Macmillan, Cham, Switzerland, 2023.

[27] Team Commands, Inria Saclay. Bocop: an open source toolbox for optimal control. `http://bocop.org`, 2019.

[28] A. C. Tsikliras and R. Froese. Maximum sustainable yield. In B. Fath, editor, *Encyclopedia of Ecology*, volume 1, pages 108–115. Elsevier, 2019.

[29] L. Von Bertalanffy. A quantitative theory of organic growth (inquiries on growth laws. ii). *Hum. Biol.*, 10(2):181–213, 1938.

[30] M. I. Zelikin. Pontryagin's Principle of Optimality. In M. Vernengo, E. Perez Caldentey, and J. Ghosh, editors, *The New Palgrave Dictionary of Economics*, pages 1–4. Palgrave Macmillan UK, London, 2017.

[31] Y. Zhang. Phase Portrait Plotter on 2D phase plane, version 1.0.3. `https://uk.mathworks.com/matlabcentral/fileexchange/110785-phase-portrait-plotter-on-2d-phase-plane`, 2022. Accessed: September 2024.