Writing Differential Algebraic Equations as Input/State/Output Systems: a State/Signal Perspective

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Abstract

We consider linear time-invariant differential–algebraic equations in both finite and infinite dimensions. Utilizing the theory of state/signals systems, we identify which variables are states, which variables are inputs and which variables are outputs and describe the dynamics though a standard input/state/output system.

1 Introduction

If one models a real-world dynamical system and puts all the relevant variables in a vector-valued function v, then in the linear finite-dimensional time-invariant case the equations obtained can be written as

$$\mathbf{E}_0\dot{v}(t) = \mathbf{A}_0 v(t), \tag{1}$$

for certain matrices $\mathbf{E}_0, \mathbf{A}_0 \in \mathbb{R}^{k\times n}$. Such a model is called a *differential–algebraic equation* (DAE). A DAE is difficult to analyze directly and a good first step to analyze a DAE is to restructure the equations. We show that we can write $\mathbb{R}^n = \mathcal{X}\dot{+}\mathcal{U}\dot{+}\mathcal{Y}$ (direct sum) and that accordingly decomposing $v(t) = x(t) + u(t) + y(t)$ we have

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \tag{2}$$

for certain (single-valued) bounded operators $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ which in the considered finite-dimensional case can be represented by matrices. Hence the DAE (1) can be re-written as an input/state/output system (2) with state space $\mathcal{X}$, input space $\mathcal{U}$ and output space $\mathcal{Y}$, which is much easier to analyze.

We also consider the infinite-dimensional situation where instead of (2) we obtain

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \tag{3}$$

where the (single-valued) operator S is in general unbounded and generalizes the block matrix $\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$. Generally, the infinite-dimensional system (3) may not have particularly nice properties (aside from S being single-valued), but under some mild additional assumptions we obtain that S is an operator node in the sense of [13].

In Section 2 we give sufficient detail about the above constructions to consider several examples in Section 3. Section 4 compares our results to results available in the literature. Finally in Section 5 we state precise theorems with proofs.

2 Informal statements of the main results

To the DAE (1) we can associate the subspace of $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$W := \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^n \\ \mathbb{R}^n \end{bmatrix} : \mathbf{E}_0 q = \mathbf{A}_0 v \right\}.$$

As a subspace of a product space, W can naturally be seen as the graph of a multi-valued operator $\mathcal{A}_0$ (also known as a linear relation):

$$\operatorname{gph}(\mathcal{A}_0) := W, \qquad \mathcal{A}_0 v = \left\{ q : \begin{bmatrix} q \\ v \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0) \right\}.$$

In Section 5 we will in fact formulate all results in terms of multi-valued operators.

Conversely, if $\mathcal{M}$ is a closed multi-valued operator on a Hilbert space $\mathcal{V}$ (closed meaning that $\operatorname{gph}(\mathcal{M})$ is closed), then by [1, Lemma 4.1.15] there exists a Hilbert space $\mathcal{Z}$ and bounded single-valued linear operators $\mathbf{E}, \mathbf{A} : \mathcal{V} \to \mathcal{Z}$ such that (this is called a kernel representation)

$$\operatorname{gph}(\mathcal{M}) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathcal{V} \\ \mathcal{V} \end{bmatrix} : \mathbf{E} q = \mathbf{A} v \right\}.$$

We have the following important subspaces associated to a multi-valued operator $\mathcal{M}$:

$$\begin{aligned} \operatorname{dom}(\mathcal{M}) &= \left\{ v : \exists q \text{ such that } \begin{bmatrix} q \\ v \end{bmatrix} \in \operatorname{gph}(\mathcal{M}) \right\}, \\ \operatorname{mul}(\mathcal{M}) &= \left\{ q : \begin{bmatrix} q \\ 0 \end{bmatrix} \in \operatorname{gph}(\mathcal{M}) \right\}, \\ \operatorname{im}(\mathcal{M}) &= \left\{ q : \exists v \text{ such that } \begin{bmatrix} q \\ v \end{bmatrix} \in \operatorname{gph}(\mathcal{M}) \right\}. \end{aligned}$$

We note that in terms of a kernel representation we have

$$\operatorname{dom}(\mathcal{M}) = \{ v : \exists q \text{ such that } \mathbf{E} q = \mathbf{A} v \}, \qquad \operatorname{mul}(\mathcal{M}) = N(\mathbf{E}),$$
$$\operatorname{im}(\mathcal{M}) = \{ q : \exists v \text{ such that } \mathbf{E} q = \mathbf{A} v \}.$$

2.1 The ultimate multi-valued operator

We specialize again to the finite-dimensional case and consider the multi-valued operator

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^n \\ \mathbb{R}^n \end{bmatrix} : \mathbf{E}_0 q = \mathbf{A}_0 v \right\},$$

associated to the DAE (1). As a first step, from this multi-valued operator we should obtain a multi-valued operator $\mathcal{A}$ with the same set of continuously differentiable trajectories but with the additional property that $\operatorname{im}(\mathcal{A}) \subset \operatorname{dom}(\mathcal{A})$. To this end, we consider the sequence of multi-valued operators $\mathcal{A}_k$ defined iteratively by

$$\operatorname{gph}(\mathcal{A}_{k+1}) = \operatorname{gph}(\mathcal{A}_k) \cap \begin{bmatrix} \operatorname{dom}(\mathcal{A}_k) \\ \operatorname{dom}(\mathcal{A}_k) \end{bmatrix}. \tag{4}$$

Because of finite-dimensionality, this sequence must terminate, i.e. there exists a $\mu \in \mathbb{N}_0$ such that $\mathcal{A}_{\mu+1} = \mathcal{A}_\mu$. We define $\mathcal{A} := \mathcal{A}_\mu$ and $\mathcal{A}$ then indeed has the property that $\operatorname{im}(\mathcal{A}) \subset \operatorname{dom}(\mathcal{A})$. Crucial is that all of these multi-valued operators have the same set of continuously differentiable trajectories: v is a continuously differentiable solution of (1) if and only if (for any $k \in \mathbb{N}$)

$$\begin{bmatrix} \dot{v}(t) \\ v(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0) \Longleftrightarrow \begin{bmatrix} \dot{v}(t) \\ v(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_k) \Longleftrightarrow \begin{bmatrix} \dot{v}(t) \\ v(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}).$$

Intuitively, in general the original multi-valued operator $\mathcal{A}_0$ has "too large" a graph and the graph of $\mathcal{A}$ is the subspace of $\operatorname{gph}(\mathcal{A}_0)$ which (minimally) suffices for the description of the continuously differentiable trajectories.

2.2 The input/state/output decomposition

We define $\mathcal{Y}_0$ as a direct complement to $\operatorname{dom}(\mathcal{A})$:

$$\mathbb{R}^n = \operatorname{dom}(\mathcal{A}) \dot{+} \mathcal{Y}_0,$$

(choosing $\mathcal{Y}_0$ as the orthogonal complement would make it unique, but there can be reasons to consider a non-orthogonal direct complement). We subsequently define the signal space

$$\mathcal{W} := \operatorname{mul}(\mathcal{A}) + \mathcal{Y}_0,$$

(this sum is actually direct since by construction $\operatorname{mul}(\mathcal{A}) \subset \operatorname{im}(\mathcal{A}) \subset \operatorname{dom}(\mathcal{A})$) and define the state space $\mathcal{X}$ as a direct complement (once again, choosing the orthogonal complement would make it unique, but there can be reasons to consider a non-orthogonal direct complement):

$$\mathbb{R}^n = \mathcal{X} \dot{+} \mathcal{W}.$$

We then decompose the signal space as a direct sum of the output space $\mathcal{Y}$ and the input space $\mathcal{U}$

$$\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U},$$

where the condition on the output space is that $\mathcal{Y}$ is a direct complement to $\operatorname{mul}(\mathcal{A})$. The choice $\mathcal{Y} := \mathcal{Y}_0$ and $\mathcal{U} := \operatorname{mul}(\mathcal{A})$ is always valid, but there can be reasons to consider other choices.

We then have

$$\mathbb{R}^n = \mathcal{X} \dot{+} \mathcal{U} \dot{+} \mathcal{Y},$$

and with respect to this decomposition we obtain that the dynamics is described by the input/state/output system (2) (where A, B, C and D can also be described, but in examples these are often obvious once we know the decomposition).

2.3 The infinite-dimensional case

In the infinite-dimensional case we consider initially a multi-valued operator $\mathcal{A}_0$ on the Hilbert space $\mathcal{V}$ (where the choice $\mathcal{V} = \mathbb{R}^n$ gives back the finite-dimensional case previously considered). In (4) we have to consider closures of domains, i.e. we instead have to consider

$$\operatorname{gph}(\mathcal{A}_{k+1}) = \operatorname{gph}(\mathcal{A}_k) \cap \begin{bmatrix} \overline{\operatorname{dom}(\mathcal{A}_k)} \\ \operatorname{dom}(\mathcal{A}_k) \end{bmatrix},$$

and the operator $\mathcal{A}$ is defined through

$$\mathcal{V}_\bullet = \bigcap_{k=1}^{\infty} \operatorname{dom}(\mathcal{A}_k), \qquad \operatorname{gph}(\mathcal{A}) = \operatorname{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \mathcal{V}_\bullet \\ \mathcal{V}_\bullet \end{bmatrix}.$$

It is no longer guaranteed that the sequence terminates, so that we instead have to assume this, i.e. we have to assume that there exists a $\mu \in \mathbb{N}_0$ such that $\mathcal{A}_{\mu+1} = \mathcal{A}_\mu$. We now obtain that $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ in case of termination (there are alternative assumptions under which this holds). The definitions of $\mathcal{Y}_0$, $\mathcal{X}$ and $\mathcal{W}$ remain essentially the same (the spaces $\mathcal{Y}_0$ and $\mathcal{X}$ are now explicitly assumed closed):

$$\mathcal{V} = \overline{\operatorname{dom}(\mathcal{A})} \dot{+} \mathcal{Y}_0, \qquad \mathcal{W} := \overline{\operatorname{mul}(\mathcal{A})} + \mathcal{Y}_0, \qquad \mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}.$$

The decomposition of the signal space $\mathcal{W}$ into an input space $\mathcal{U}$ and an output space $\mathcal{Y}$ becomes more complicated (unless $\operatorname{dom}(\mathcal{A})$ is closed in which case the situation is the same as described in the finite-dimensional case). Instead of the output space being taken as a direct complement in $\mathcal{W}$ to $\operatorname{mul}(\mathcal{A})$, it should be chosen as a direct complement in $\mathcal{W}$ to the characteristic signal bundle. Here we will restrict ourselves to the special case where $\mathcal{Y}_0 = \{0\}$ (Section 5 does deal with the general case). In that special case the characteristic signal bundle is given by (here $\lambda \in \mathbb{C}$)

$$\widehat{\mathfrak{F}}(\lambda) := \left\{ w \in \mathcal{W} : \exists x \in \mathcal{X} \text{ such that } \begin{bmatrix} \lambda x \\ x + w \end{bmatrix} \in \operatorname{gph}(\mathcal{A}) \right\}.$$

In the case that $\operatorname{dom}(\mathcal{A})$ is closed, $\widehat{\mathfrak{F}}(\infty)$ is well-defined and equals $\operatorname{mul}(\mathcal{A})$, but in general a finite value for λ needs to be used. The output space $\mathcal{Y}$ is chosen to satisfy $\mathcal{Y} \cap \operatorname{mul}(\mathcal{A}) \cap \operatorname{dom}(\mathcal{A}) = \{0\}$ and as a direct complement in $\mathcal{W}$ to $\widehat{\mathfrak{F}}(\lambda)$. Subsequently, the input space $\mathcal{U}$ is chosen as a direct complement in $\mathcal{W}$ to $\mathcal{Y}$:

$$\mathcal{W} = \mathcal{Y} \dot{+} \widehat{\mathfrak{F}}(\lambda) = \mathcal{Y} \dot{+} \mathcal{U}.$$

We then have that $\widehat{\mathfrak{F}}(\lambda)$ is the graph of a single-valued operator $\widehat{\mathfrak{G}}(\lambda) : \mathcal{U} \to \mathcal{Y}$, which is the transfer function of the input/state/output system. This knowledge can be used to distinguish more desirable choices of input and output spaces from less desirable ones (one good criterion would be to make a choice so that $\widehat{\mathfrak{G}}$ is well-posed, i.e. uniformly bounded on some right half-plane, if such a choice is possible).

3 Examples

3.1 Finite-dimensional examples

Example 3.1. We consider [7, Example 2.4] from our perspective. The DAE there models a simple RLC circuit with equations

$$L\dot{I} = V_L, \quad C\dot{V}_C = I, \quad 0 = -RI + V_R, \quad 0 = V_L + V_C + V_R - V_S.$$

We define

$$v_1 = I, \quad v_2 = V_L, \quad v_3 = V_C, \quad v_4 = V_R, \quad v_5 = V_S.$$

This corresponds to the multi-valued operator

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^5 \\ \mathbb{R}^5 \end{bmatrix} : \begin{array}{c} Lq_1 = v_2,\ Cq_3 = v_1,\ 0 = -Rv_1 + v_4, \\ 0 = v_2 + v_3 + v_4 - v_5 \end{array} \right\}.$$

The domain of this equals

$$\operatorname{dom}(\mathcal{A}_0) = \left\{ v \in \mathbb{R}^5 : 0 = -Rv_1 + v_4,\ 0 = v_2 + v_3 + v_4 - v_5 \right\},$$

so that

$$\operatorname{gph}(\mathcal{A}_1) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^5 \\ \mathbb{R}^5 \end{bmatrix} : \begin{array}{c} Lq_1 = v_2,\ Cq_3 = v_1,\ 0 = -Rv_1 + v_4, \\ 0 = v_2 + v_3 + v_4 - v_5, \\ 0 = -Rq_1 + q_4,\ 0 = q_2 + q_3 + q_4 - q_5 \end{array} \right\}.$$

We note that the added constraints in $\operatorname{gph}(\mathcal{A}_1)$ signify that for continuously differentiable trajectories we must have

$$0 = -R\dot{I} + \dot{V}_R, \quad 0 = \dot{V}_L + \dot{V}_C + \dot{V}_R - \dot{V}_S.$$

We have $\operatorname{dom}(\mathcal{A}_1) = \operatorname{dom}(\mathcal{A}_0)$ so that the sequence terminates and we have $\mathcal{A} = \mathcal{A}_1$. Therefore

$$\operatorname{gph}(\mathcal{A}) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^5 \\ \mathbb{R}^5 \end{bmatrix} : \begin{array}{c} Lq_1 = v_2,\ Cq_3 = v_1,\ 0 = -Rv_1 + v_4, \\ 0 = v_2 + v_3 + v_4 - v_5, \\ 0 = -Rq_1 + q_4,\ 0 = q_2 + q_3 + q_4 - q_5 \end{array} \right\},$$

and

$$\operatorname{dom}(\mathcal{A}) = \left\{ v \in \mathbb{R}^5 : 0 = -Rv_1 + v_4,\ 0 = v_2 + v_3 + v_4 - v_5 \right\}.$$

Note that $\operatorname{dom}(\mathcal{A})$ has dimension three. The multi-valued part of $\mathcal{A}$ equals

$$\operatorname{mul}(\mathcal{A}) = \left\{ z \in \mathbb{R}^5 : q_1 = 0,\ q_3 = 0,\ 0 = -Rq_1 + q_4,\ 0 = q_2 + q_3 + q_4 - q_5 \right\},$$

which has dimension one. Equivalently, we have

$$\operatorname{dom}(\mathcal{A}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ R \\ R \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \operatorname{mul}(\mathcal{A}) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The space $\mathcal{Y}_0$ should be taken as a direct complement of $\operatorname{dom}(\mathcal{A})$ and therefore must have dimension two. To align with the original variables we choose

$$\mathcal{Y}_0 = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This gives

$$\mathcal{W} = \mathcal{Y}_0 + \operatorname{mul}(\mathcal{A}) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The state space $\mathcal{X}$ should be chosen as a direct complement of $\mathcal{W}$. To align with the original variables we choose

$$\mathcal{X} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We choose the output space $\mathcal{Y} = \mathcal{Y}_0$. Finally, we must choose the input space $\mathcal{U}$ as a direct complement of $\mathcal{Y}$ in $\mathcal{W}$ and to align with the original variables we choose

$$\mathcal{U} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

These choices mean that v_1 and v_3 are states, v_5 is an input and v_2 and v_4 are outputs, i.e. I and V_C are states, V_S is an input and V_L and V_R are outputs. Re-writing the original equations gives

$$L\dot{I} = -RI - V_C + V_S, \qquad C\dot{V}_C = I, \qquad V_L = -RI - V_C + V_S, \qquad V_R = RI,$$

which is the input/state/output system

$$\begin{bmatrix} \dot{I} \\ \dot{V}_C \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} I \\ V_C \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_S, \qquad \begin{bmatrix} V_L \\ V_R \end{bmatrix} = \begin{bmatrix} -R & -1 \\ R & 0 \end{bmatrix} \begin{bmatrix} I \\ V_C \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_S.$$

Example 3.2. We consider [2, Example 6.1] from our perspective. To not prejudice the choice of input, we denote what is denoted u in [2, Example 6.1] by v_6. The equations then are

$$\dot{v}_1 = v_1 + v_2 + v_6, \qquad \dot{v}_2 = -v_1 + v_2 + v_6, \qquad \dot{v}_4 = v_3 - v_6, \qquad 0 = v_4,$$

(note that there are no constraints on v_5). This corresponds to the multi-valued operator

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^6 \\ \mathbb{R}^6 \end{bmatrix} : \begin{array}{c} q_1 = v_1 + v_2 + v_6, \ q_2 = -v_1 + v_2 + v_6, \\ q_4 = v_3 - v_6, \ 0 = v_4 \end{array} \right\}.$$

The domain of this equals

$$\operatorname{dom}(\mathcal{A}_0) = \left\{ v \in \mathbb{R}^6 : v_4 = 0 \right\},$$

so that

$$\operatorname{gph}(\mathcal{A}_1) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^6 \\ \mathbb{R}^6 \end{bmatrix} : \begin{array}{c} q_1 = v_1 + v_2 + v_6, \ q_2 = -v_1 + v_2 + v_6, \\ q_4 = v_3 - v_6, \ 0 = v_4, \ 0 = q_4 \end{array} \right\}.$$

The domain of this equals

$$\operatorname{dom}(\mathcal{A}_1) = \left\{ v \in \mathbb{R}^6 : v_4 = 0, \ v_3 = v_6 \right\},$$

so that

$$\operatorname{gph}(\mathcal{A}_2) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^6 \\ \mathbb{R}^6 \end{bmatrix} : \begin{array}{c} q_1 = v_1 + v_2 + v_6, \ q_2 = -v_1 + v_2 + v_6, \\ 0 = v_3 - v_6, \ 0 = v_4, \ 0 = q_4, \ 0 = q_3 - q_6 \end{array} \right\}.$$

We have $\mathrm{dom}(\mathcal{A}_2) = \mathrm{dom}(\mathcal{A}_1)$ so that the sequence terminates and we have $\mathcal{A} = \mathcal{A}_2$. Note that $\mathrm{dom}(\mathcal{A}) = \mathrm{dom}(\mathcal{A}_2) = \mathrm{dom}(\mathcal{A}_1)$ has dimension four. The multi-valued part of $\mathcal{A}$ equals

$$\mathrm{mul}(\mathcal{A}) = \left\{ z \in \mathbb{R}^6 : q_6 = q_3,\ q_1 = 0,\ q_2 = 0,\ q_4 = 0 \right\},$$

which has dimension two. Equivalently we have

$$\mathrm{dom}(\mathcal{A}) = \mathrm{span}\left\{ \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} \right\}, \qquad \mathrm{mul}(\mathcal{A}) = \mathrm{span}\left\{ \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} \right\}.$$

The space $\mathcal{Y}_0$ should be taken as a direct complement of $\mathrm{dom}(\mathcal{A})$ and therefore must have dimension two. To align with the original variables we choose

$$\mathcal{Y}_0 = \mathrm{span}\left\{ \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix} \right\}.$$

This gives

$$\mathcal{W} = \mathcal{Y}_0 + \mathrm{mul}(\mathcal{A}) = \mathrm{span}\left\{ \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \right\}.$$

The state space $\mathcal{X}$ should be chosen as a direct complement of $\mathcal{W}$. To align with the original variables we choose

$$\mathcal{X} = \mathrm{span}\left\{ \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \right\}.$$

We choose the output space $\mathcal{Y} = \mathcal{Y}_0$. Finally, we should choose the input space $\mathcal{U}$ as a direct complement of $\mathcal{Y}$ in $\mathcal{W}$ and to align with the original variables we choose

$$\mathcal{U} = \mathrm{span}\left\{ \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \right\}.$$

Therefore v_1 and v_2 are states, v_3 and v_4 are outputs and v_5 and v_6 are inputs. The input/state/output system is

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_5 \\ v_6 \end{bmatrix}, \qquad \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_5 \\ v_6 \end{bmatrix}.$$

In [2, Example 6.1] the following cost functional is considered

$$\int_0^\infty |v_6(t)|^2 + |v_1(t)|^2 + |v_2(t)|^2 \, dt.$$

If we ignore the input v_5 (which doesn't play a role in the dynamics or the cost), then this becomes a standard linear quadratic optimal control problem with input v_6, state $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ and control operator $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ (the variables v_3, v_4 and v_5 play no role and can at this stage be ignored). The corresponding algebraic Riccati equation can be easily numerically solved and results in the optimal feedback

$$v_6 = -5.1813 v_1 + 0.4142 v_2. \tag{5}$$

This part of the optimal control is unique, but since v_5 is an input which is unconstrained by the problem, it can be arbitrary in an optimal control (so that optimal controls are not unique, but the non-uniqueness is easily identified).

In [2, Example 6.1] the following initial conditions are specified:

$$v_1(0) = 1, \quad v_2(0) = -1, \quad v_3(0) = v_4(0) = v_5(0) = 0. \tag{6}$$

These initial conditions are consistent with the constraints in the sense that $v(0)$ belongs to $\mathrm{dom}(\mathcal{A})$ if (and only if) $v_6(0) = 0$ is chosen. However, they are not consistent with the optimal feedback (5) since that equation is then not satisfied at $t = 0$. This implies that with the initial conditions (6) the cost functional has an infimum which is not a minimum. This seems to have gone unnoticed in [2, Example 6.1]. The underlying issue is that (6) through the initial condition on v_3 imposes an initial condition on the control (since $v_6 = v_3$). By writing the DAE as an input/state/output system, it becomes clear which initial conditions can (and should) be imposed: only elements of the state space $\mathcal{X}$ (in this example v_1 and v_2) should have initial conditions and no initial conditions for the inputs or outputs should be imposed (although since in this example the input component v_5 plays no role, it doesn't hurt if an initial condition is imposed on v_5).

3.2 Infinite-dimensional examples

Example 3.3. We consider the example from [6]. There two coupled temperature equations on the spatial interval $(0, 1)$ are considered where in both equations diffusion is neglected (only convection is considered) and in one equation the time constant is set to zero. This leads to (here dot indicates time derivative and prime denotes spatial derivative)

$$\dot{T}_w = T_g - T_w, \qquad T_g' = T_w - T_g,$$

where for notational simplicity we set all the relevant constants in [6] equal to one. To minimize smoothness assumptions, we write the second equation in integral form as

$$T_g(\xi) = \mathrm{e}^{-\xi} T_g(0) + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} T_w(\eta) \, d\eta.$$

With $v_1 := T_w$, $v_2 := T_g$, $v_3 := T_g(0)$, a corresponding multi-valued operator is

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} L^2(0,1)^2 \times \mathbb{R} \\ L^2(0,1)^2 \times \mathbb{R} \end{bmatrix} : \begin{array}{c} q_1 = v_2 - v_1, \\ v_2(\xi) = \mathrm{e}^{-\xi} v_3 + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} v_1(\eta)\, d\eta \end{array} \right\}.$$

The domain of this equals

$$\operatorname{dom}(\mathcal{A}_0) = \left\{ v \in L^2(0,1)^2 \times \mathbb{R} : v_2(\xi) = \mathrm{e}^{-\xi} v_3 + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} v_1(\eta)\, d\eta \right\},$$

which leads to

$$\operatorname{gph}(\mathcal{A}_1) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} L^2(0,1)^2 \times \mathbb{R} \\ L^2(0,1)^2 \times \mathbb{R} \end{bmatrix} : \begin{array}{c} q_1 = v_2 - v_1, \\ v_2(\xi) = \mathrm{e}^{-\xi} v_3 + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} v_1(\eta)\, d\eta, \\ q_2(\xi) = \mathrm{e}^{-\xi} q_3 + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} q_1(\eta)\, d\eta \end{array} \right\}.$$

We then have $\operatorname{dom}(\mathcal{A}_1) = \operatorname{dom}(\mathcal{A}_0)$, so that $\mathcal{A} = \mathcal{A}_1$. We note that $\operatorname{dom}(\mathcal{A}) = \operatorname{dom}(\mathcal{A}_0)$ is closed. Because of this, the situation is actually like described in Section 2 for the finite-dimensional case and there is no need to consider the characteristic signal bundle. We should choose $\mathcal{Y}_0$ as a direct complement of $\operatorname{dom}(\mathcal{A})$. We can choose

$$\mathcal{Y}_0 = \begin{bmatrix} 0 \\ L^2(0,1) \\ 0 \end{bmatrix},$$

which is indeed a direct complement of $\operatorname{dom}(\mathcal{A})$ as we can uniquely write

$$\begin{bmatrix} L^2(0,1) \\ L^2(0,1) \\ \mathbb{R} \end{bmatrix} \ni \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ p_2 - \mathrm{e}^{-\xi} p_3 - \int_0^\xi \mathrm{e}^{-(\xi-\eta)} p_1(\eta)\, d\eta \\ 0 \end{bmatrix} + \begin{bmatrix} p_1 \\ \mathrm{e}^{-\xi} p_3 + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} p_1(\eta)\, d\eta \\ p_3 \end{bmatrix} \in \mathcal{Y}_0 + \operatorname{dom}(\mathcal{A}).$$

We have

$$\operatorname{mul}(\mathcal{A}) = \left\{ q \in L^2(0,1)^2 \times \mathbb{R} : q_1 = 0, \quad q_2(\xi) = \mathrm{e}^{-\xi} q_3 \right\},$$

so that

$$\mathcal{W} = \mathcal{Y}_0 + \operatorname{mul}(\mathcal{A}) = \begin{bmatrix} 0 \\ L^2(0,1) \\ \mathbb{R} \end{bmatrix}.$$

As a direct complement we choose

$$\mathcal{X} = \begin{bmatrix} L^2(0,1) \\ 0 \\ 0 \end{bmatrix}.$$

We choose

$$\mathcal{U} = \begin{bmatrix} 0 \\ 0 \\ \mathbb{R} \end{bmatrix}, \qquad \mathcal{Y} = \mathcal{Y}_0 = \begin{bmatrix} 0 \\ L^2(0,1) \\ 0 \end{bmatrix},$$

which is valid since $\mathcal{U}\dot{+}\mathcal{Y} = \mathcal{W}$. We therefore have that $x := v_1$ is a state, $y := v_2$ is an output and $u := v_3$ is an input. In terms of the original variables, this means that T_w is a state, T_g is an output and $T_g(0)$ is an input. The input/state/output equations are

$$\dot{x}(t,\xi) = \mathrm{e}^{-\xi}u(t) + \int_0^\xi \mathrm{e}^{-(\xi-\eta)}x(t,\eta)\,d\eta - x(t,\xi),$$
$$y(t,\xi) = \mathrm{e}^{-\xi}u(t) + \int_0^\xi \mathrm{e}^{-(\xi-\eta)}x(t,\eta)\,d\eta,$$

i.e.

$$(Ax)(\xi) = \int_0^\xi \mathrm{e}^{-(\xi-\eta)}x(\eta)\,d\eta - x(\xi), \quad (Bu)(\xi) = \mathrm{e}^{-\xi}u,$$
$$(Cx)(\xi) = \int_0^\xi \mathrm{e}^{-(\xi-\eta)}x(\eta)\,d\eta, \quad (Du)(\xi) = \mathrm{e}^{-\xi}u,$$

which we note are all bounded operators (this is because $\operatorname{dom}(\mathcal{A})$ is closed).

Example 3.4. We consider the following PDE on the spatial domain $(0,1)$ with boundary conditions

$$\dot{v}_1 = -v_1', \qquad v_1(0) = v_2, \qquad v_1(1) = v_3.$$

A corresponding multi-valued operator is

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} L^2(0,1)\times\mathbb{R}^2 \\ H^1(0,1)\times\mathbb{R}^2 \end{bmatrix} : q_1 = -v_1', \quad v_1(0) = v_2, \quad v_1(1) = v_3 \right\}.$$

The domain of this equals

$$\operatorname{dom}(\mathcal{A}_0) = \left\{ v \in H^1(0,1)\times\mathbb{R}^2 : v_1(0) = v_2, \quad v_1(1) = v_3 \right\},$$

which is dense in $L^2(0,1)\times\mathbb{R}^2$. Hence $\mathcal{A} = \mathcal{A}_0$ and $\overline{\operatorname{dom}(\mathcal{A})} = L^2(0,1)\times\mathbb{R}^2$, so that we must have $\mathcal{Y}_0 = \{0\}$ and therefore $\mathcal{W} = \overline{\operatorname{mul}(\mathcal{A})}$. We have

$$\operatorname{mul}(\mathcal{A}) = \left\{ q \in L^2(0,1)\times\mathbb{R}^2 : q_1 = 0 \right\},$$

which has dimension two (and is therefore closed). We note that $\operatorname{mul}(\mathcal{A}) \cap \operatorname{dom}(\mathcal{A}) = \{0\}$. We should choose $\mathcal{X}$ as a direct complement of $\operatorname{mul}(\mathcal{A})$ in $L^2(0,1)\times\mathbb{R}^2$. To align with the original variables we choose

$$\mathcal{X} = \left\{ v \in L^2(0,1)\times\mathbb{R}^2 : v_2 = v_3 = 0 \right\}.$$

Since $\operatorname{dom}(\mathcal{A})$ is not closed, we have to consider the characteristic signal bundle. We have

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix} \in \begin{bmatrix} L^2(0,1) \\ \mathbb{R} \\ \mathbb{R} \end{bmatrix} : \exists v_1 \in H^1(0,1) \text{ such that } \begin{array}{c} \lambda v_1 = -v_1', \\ v_1(0) = v_2, \ v_1(1) = v_3 \end{array} \right\}.$$

We have that $\lambda v_1 = -v_1'$ has the general solution $v_1(\xi) = \mathrm{e}^{-\lambda\xi}v_1(0)$, so that the boundary condition $v_1(0) = v_2$ gives $v_1(\xi) = \mathrm{e}^{-\lambda\xi}v_2$, and we must then have $v_3 = v_1(1) = \mathrm{e}^{-\lambda}v_2$, so that

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ \begin{bmatrix} 0 \\ 1 \\ \mathrm{e}^{-\lambda} \end{bmatrix} v_2 : v_2 \in \mathbb{R} \right\}.$$

We choose the output space $\mathcal{Y}$ as the following direct complement of the characteristic signal bundle (note that since $\mathrm{mul}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{A}) = \{0\}$, any direct complement would suffice)

$$\mathcal{Y} = \begin{bmatrix} 0 \\ 0 \\ \mathbb{R} \end{bmatrix},$$

and we choose the input space $\mathcal{U}$ as a direct complement (in fact orthogonal) to $\mathcal{Y}$ in $\mathcal{W} = \mathrm{mul}(\mathcal{A})$:

$$\mathcal{U} = \begin{bmatrix} 0 \\ \mathbb{R} \\ 0 \end{bmatrix}.$$

This means that v_2 is an input and v_3 is an output. This gives the transfer function $\mathrm{e}^{-\lambda}$ which is well-posed (the opposite choice of v_2 an output and v_3 an input would give the transfer function e^{λ} which is not well-posed and therefore this opposite choice, though allowed, is not sensible). The input/state/output system we obtain with input $u := v_2$, state $x := v_1$ and output $y := v_3$ is described by the operator

$$S \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -x' \\ x(1) \end{bmatrix}, \qquad \mathrm{dom}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} H^1(0,1) \\ \mathbb{R} \end{bmatrix} : x(0) = u \right\},$$

and is well-known to in fact be a well-posed linear system (here with some abuse of notation we identify the input space $\left[\begin{smallmatrix} 0 \\ \mathbb{R} \\ 0 \end{smallmatrix}\right]$ with $\mathbb{R}$).

Example 3.5. We consider the example from [11, Section 5] which is an electrical circuit with a transmission line. The equations are (where for notational simplicity we take $G_T = R_T = 0$ and $C_T = L_T = C = 1$ in the notation of [11] and we omit u_V which just equals u_C and therefore plays a trivial role, but complicates notation):

$$\dot{V} = -I', \qquad \dot{I} = -V',$$
$$\dot{u}_C = i_C, \qquad 0 = -i_V - i_C + I(0), \qquad 0 = -u_C + V(0), \qquad 0 = I(1).$$

We define

$$v_1 = V, \qquad v_2 = I, \qquad v_3 = u_C, \qquad v_4 = i_C, \qquad v_5 = i_V.$$

A corresponding multi-valued operator is

$$\mathrm{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} L^2(0,1)^2 \times \mathbb{R}^3 \\ H^1(0,1)^2 \times \mathbb{R}^3 \end{bmatrix} : \begin{array}{c} q_1 = -v_2', \quad q_2 = -v_1', \quad q_3 = v_4, \\ v_2(0) = v_4 + v_5, \quad v_1(0) = v_3, \quad v_2(1) = 0 \end{array} \right\}.$$

This gives

$$\mathrm{dom}(\mathcal{A}_0) = \left\{ v \in H^1(0,1)^2 \times \mathbb{R}^3 : v_2(0) = v_4 + v_5, \quad v_1(0) = v_3, \quad v_2(1) = 0 \right\},$$

whose closure is $\overline{\mathrm{dom}(\mathcal{A}_0)} = L^2(0,1)^2 \times \mathbb{R}^3$, so that $\mathcal{A} = \mathcal{A}_0$ and $\mathcal{Y}_0 = \{0\}$. We have

$$\mathrm{mul}(\mathcal{A}) = \{q : q_1 = q_2 = q_3 = 0\} = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

which is two-dimensional (and therefore closed). We should choose the state space $\mathcal{X}$ as a direct complement to $\mathcal{W}$ (which here equals $\mathrm{mul}(\mathcal{A})$) and to align with the original variables, we choose

$$\mathcal{X} = \left\{ v \in H^1(0,1)^2 \times \mathbb{R}^3 : v_4 = v_5 = 0 \right\}.$$

We note that

$$\mathrm{mul}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{A}) = \{x : v_1 = v_2 = v_3 = v_4 + v_5 = 0\} = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\},$$

which is one-dimensional.

Since $\mathrm{dom}(\mathcal{A})$ is not closed, we have to consider the characteristic signal bundle. We have

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ \begin{bmatrix} 0 \\ w \end{bmatrix} \in \begin{bmatrix} L^2(0,1) \times \mathbb{R} \\ \mathbb{R}^2 \end{bmatrix} : \exists v \in L^2(0,1) \times \mathbb{R} \text{ such that } \begin{array}{c} \lambda v_1 = -v_2', \\ \lambda v_2 = -v_1', \\ \lambda v_3 = w_1, \\ v_2(0) = w_1 + w_2, \\ v_1(0) = v_3, \\ v_2(1) = 0 \end{array} \right\}.$$

Solving the ODEs with the boundary conditions $v_1(0) = v_3 = \frac{w_1}{\lambda}$ and $v_2(1) = 0$ gives

$$\begin{aligned} v_1(\xi) &= \frac{w_1}{\lambda} \left(\cosh(\lambda\xi) - \tanh(\lambda) \sinh(\lambda\xi) \right), \\ v_2(\xi) &= \frac{w_1}{\lambda} \left(-\sinh(\lambda\xi) + \tanh(\lambda) \cosh(\lambda\xi) \right). \end{aligned}$$

From the final equation $v_2(0) = w_1 + w_2$ we then obtain

$$w_2 = \left(\frac{\tanh(\lambda)}{\lambda} + 1 \right) w_1.$$

Therefore

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ \begin{bmatrix} 0 \\ 1 \\ \frac{\tanh(\lambda)}{\lambda} + 1 \end{bmatrix} w_1 \right\}.$$

We choose

$$\mathcal{Y} = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

noting that this is a direct complement of the signal bundle and that $\mathcal{Y} \cap \mathrm{mul}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{A}) = \{0\}$. We choose the input space $\mathcal{U}$ as the orthogonal complement of $\mathcal{Y}$ in $\mathcal{W} = \mathrm{mul}(\mathcal{A})$:

$$\mathcal{U} = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This means that we consider v_4 as input and v_5 as output. We have that v_1, v_2 and v_3 are states. The input/state/output system is described by the operator

$$S\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -x_2' \\ -x_1' \\ u \end{bmatrix} \\ x_2(0) - u \end{bmatrix}, \qquad \operatorname{dom}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} H^1(0,1)^2 \times \mathbb{R} \\ \mathbb{R} \end{bmatrix} : \begin{array}{c} x_1(0) = x_3, \\ x_2(1) = 0 \end{array} \right\},$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} V \\ I \\ u_C \end{bmatrix}, \qquad u = v_4 = i_C, \qquad y = v_5 = i_V.$$

4 Literature comparison

By Proposition 5.11, in the finite-dimensional case we have $\operatorname{dom}(\mathcal{A}_{k-1}) = \operatorname{dom}(\mathcal{A}_0^k)$ for all $k \in \mathbb{N}$. The spaces $\operatorname{dom}(\mathcal{A}_0^k)$ and the space $\mathcal{V}_\bullet$ appear in the DAE literature in terms of kernel representations. In terms of a kernel representation of $\mathcal{A}_0$ we have (here $\mathbf{A}_0^{-1}\mathcal{W}$ for a set $\mathcal{W}$ refers to the inverse image under $\mathbf{A}_0$ of $\mathcal{W}$):

$$\operatorname{dom}(\mathcal{A}_0^{k+1}) = \mathbf{A}_0^{-1}\mathbf{E}_0 \operatorname{dom}(\mathcal{A}_0^k),$$

which is how it appears in the DAE literature (e.g. [3, 4]) as the first Wong sequence of the pencil $s\mathbf{E}_0 - \mathbf{A}_0$. The space $\mathcal{V}_\bullet$ then is the limit of this first Wong sequence.

The multi-valued operators $\mathcal{A}_k$ themselves (rather than just their domains) also appear in the DAE literature in terms of kernel representations. In terms of a kernel representation we have

$$\operatorname{gph}(\mathcal{A}_1) = \left\{ \begin{bmatrix} z \\ x \end{bmatrix} : \mathbf{E}_0 z = \mathbf{A}_0 x,\ \exists z_1 \text{ such that } \mathbf{E}_0 z_1 = \mathbf{A}_0 z \right\},$$

and more generally

$$\operatorname{gph}(\mathcal{A}_k) = \left\{ \begin{bmatrix} z \\ x \end{bmatrix} : \mathbf{E}_0 z = \mathbf{A}_0 x,\ \exists z_1, \dots z_k \text{ such that } \begin{array}{c} \mathbf{E}_0 z_1 = \mathbf{A}_0 z, \\ \mathbf{E}_0 z_j = \mathbf{A}_0 z_{j-1}, \\ j \in \{2, \dots, k\} \end{array} \right\}.$$

In terms of the *derivative array* [2]

$$M := \begin{bmatrix} \mathbf{E}_0 & & & \\ -\mathbf{A}_0 & \mathbf{E}_0 & & \\ & \ddots & \ddots & \\ & & -\mathbf{A}_0 & \mathbf{E}_0 \end{bmatrix}, \qquad N := \begin{bmatrix} \mathbf{A}_0 & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix},$$

this is (note that the x_j play a trivial role)

$$\operatorname{gph}(\mathcal{A}_k) = \left\{ \begin{bmatrix} z \\ x \end{bmatrix} : \exists z_1, \dots z_k, x_1, \dots, x_k \text{ such that } M \begin{bmatrix} z \\ z_1 \\ \vdots \\ z_k \end{bmatrix} = N \begin{bmatrix} x \\ x_1 \\ \vdots \\ x_k \end{bmatrix} \right\},$$

and the algorithm in [2, Theorem 3.1] can be seen as determining a kernel representation of $\mathrm{gph}(\mathcal{A}_k)$. Even though Wong sequences and derivative arrays both appear in the DAE literature, this connection between them (one giving the graph, the other the domain) does not seem to have been explicitly made before.

In [7] and [2] based on dimensions, it is realized that some variables should be interpreted as states and some as inputs, but no method is given for making this distinction. An algorithm for this is given in [5]. Also related is [8] where a DAE and is written as an input-state system with an output zeroing condition (the *explicitation* in [8, Proposition 6.7]). The above cited articles are all on finite-dimensional systems and take different approaches than we do. A particular difference is that we allow for quite some flexibility in the choice of the state, input and output spaces. We note that the Wong sequence appears in an infinite-dimensional context in [14] (although not under that name).

In behavioral theory, it is known that a linear differential system can always be written in standard input/state/output form [10], which is closely related to, but not quite indentical to the finite-dimensional version of our main result (as in the behavioral result a state is generally constructed rather than identified amongst the existing variables and it is therefore closer to classical realization theory than to our results).

5 Theorems and proofs

In this section we give precise statements and proofs for the results informally stated in Section 2. In Section 5.1 we detail the construction of $\mathcal{A}$ from $\mathcal{A}_0$ and we also give some results which allow us to connect this to the Wong sequence known in DAE theory. In Section 5.2 we consider the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}$ and how this gives a multi-valued operator $\mathcal{A}_\times$ obtained from $\mathcal{A}$. Section 5.3 reviews some results from state/signal theory [1] which are needed. Sections 5.4 and 5.5 construct a state/signal system from the multi-valued operator $\mathcal{A}$ (and conversely) and in Section 5.6 from this state/signal system an input/state/output system is obtained (this last step is a straightforward application of results from [1]). Finally, in Section 5.7 we state results, which easily follow from the earlier results, which more directly link the multi-valued operator $\mathcal{A}$ to the input/state/output system.

5.1 From a multi-valued operator to a multi-valued operator

Let $\mathcal{A}_0$ be a multi-valued operator on the Hilbert space $\mathcal{V}$. Define the sequence $(\mathcal{A}_k)_{k=1}^{\infty}$ of multi-valued operators on $\mathcal{V}$ iteratively by

$$\mathrm{gph}(\mathcal{A}_{k+1}) = \mathrm{gph}(\mathcal{A}_k) \cap \begin{bmatrix} \overline{\mathrm{dom}(\mathcal{A}_k)} \\ \overline{\mathrm{dom}(\mathcal{A}_k)} \end{bmatrix},$$

further define

$$\mathcal{V}_\bullet := \bigcap_{k=0}^{\infty} \mathrm{dom}(\mathcal{A}_k),$$

and define the multi-valued operator $\mathcal{A}$ on $\mathcal{V}$ by

$$\mathrm{gph}(\mathcal{A}) = \mathrm{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \mathcal{V}_\bullet \\ \mathcal{V}_\bullet \end{bmatrix}.$$

Lemma 5.1. *We have*

$$\mathrm{gph}(\mathcal{A}_{k+1}) = \mathrm{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \overline{\mathrm{dom}(\mathcal{A}_k)} \\ \mathrm{dom}(\mathcal{A}_k) \end{bmatrix}.$$

Proof. We show this by induction. It is trivially true for $k = 0$. We have (the first equality is by definition, the second by the induction hypothesis)

$$\begin{aligned} \mathrm{gph}(\mathcal{A}_{k+2}) = \mathrm{gph}(\mathcal{A}_{k+1}) \cap \begin{bmatrix} \overline{\mathrm{dom}(\mathcal{A}_{k+1})} \\ \mathrm{dom}(\mathcal{A}_{k+1}) \end{bmatrix} &= \mathrm{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \overline{\mathrm{dom}(\mathcal{A}_{k+1})} \\ \mathrm{dom}(\mathcal{A}_{k+1}) \end{bmatrix} \cap \begin{bmatrix} \overline{\mathrm{dom}(\mathcal{A}_k)} \\ \mathrm{dom}(\mathcal{A}_k) \end{bmatrix} \\ &= \mathrm{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \overline{\mathrm{dom}(\mathcal{A}_{k+1})} \\ \mathrm{dom}(\mathcal{A}_{k+1}) \end{bmatrix}, \end{aligned}$$

where in the last equality we used that $\mathrm{dom}(\mathcal{A}_{k+1}) \subset \mathrm{dom}(\mathcal{A}_k)$ as $\mathrm{gph}(\mathcal{A}_{k+1}) \subset \mathrm{gph}(\mathcal{A}_k)$. □

We say that the sequence $(\mathcal{A}_k)_{k=1}^{\infty}$ *terminates* if there exists a $\mu \in \mathbb{N}$ such that $\mathcal{A}_\mu = \mathcal{A}_{\mu+1}$ (in which case $\mathcal{V}_\bullet = \mathrm{dom}(\mathcal{A}_\mu)$).

Lemma 5.2. *If the sequence $(\mathcal{A}_k)_{k=1}^{\infty}$ terminates, then $\mathrm{im}(\mathcal{A}) \subset \overline{\mathrm{dom}(\mathcal{A})}$.*

Proof. Using termination for the first and last equalities we have

$$\mathrm{im}(\mathcal{A}) = \mathrm{im}(\mathcal{A}_{\mu+1}) \subset \overline{\mathrm{dom}(\mathcal{A}_\mu)} = \overline{\mathrm{dom}(\mathcal{A})},$$

as desired. □

The following definition gives a boundedness notion for multi-valued operators generalizing that for single-valued operators. The terminology quasi-bounded is not standard, but will be useful for us to distinguish two related but different notions which often appear in the literature with the same name (in many instances, bounded requires the domain to be the whole space).

Definition 5.3. A multi-valued operator $\mathcal{M} : \mathrm{dom}(\mathcal{M}) \subset \mathcal{V} \to \mathcal{V}$ is called *quasi-bounded* if there exists a $M > 0$ such that for all $x \in \mathrm{dom}(\mathcal{M})$ there exists a $z \in \mathcal{M}x$ such that $\|z\| \leq M\|x\|$.

The closed-graph theorem for multi-valued operators (see e.g. [12]) states that the following are equivalent for a closed multi-valued operator $\mathcal{M}$:

- $\mathrm{dom}(\mathcal{M})$ is closed;
- $\mathcal{M}$ is quasi-bounded.

Proposition 5.4. *If $\mathcal{A}_0$ is closed and $\mathrm{dom}(\mathcal{A}_k)$ is closed for all $k \in \mathbb{N}_0$, then $\mathcal{V}_\bullet = \mathcal{V} = \mathrm{dom}(\mathcal{A})$.*

Proof. As the intersection of the closed spaces $\mathrm{dom}(\mathcal{A}_k)$, we have that $\mathcal{V}_\bullet$ is closed, so that in particular $\mathcal{V}_\bullet = \overline{\mathcal{V}_\bullet}$. Let $x \in \mathcal{V}_\bullet$ and let $k \in \mathbb{N}_0$. Then $x \in \mathrm{dom}(\mathcal{A}_{k+1})$ so that there exists a $z_k \in \mathrm{dom}(\mathcal{A}_k)$ such that $\left[\begin{smallmatrix} z_k \\ x \end{smallmatrix}\right] \in \mathrm{gph}(\mathcal{A}_0)$. By the closed graph theorem for multi-valued operators, we have that $\mathcal{A}_0$ is quasi-bounded (as its graph and domain are closed). It follows that we can choose the sequence $(z_k)_{k\in\mathbb{N}_0}$ to be bounded. Because we are in a Hilbert space, it follows that $(z_k)_{k\in\mathbb{N}_0}$ has a weakly convergent subsequence. Let z denote the limit of this subsequence. Then $\left[\begin{smallmatrix} z_k \\ x \end{smallmatrix}\right] \in \mathrm{gph}(\mathcal{A}_0)$ converges weakly along a subsequence to $\left[\begin{smallmatrix} z \\ x \end{smallmatrix}\right]$. Since $\mathrm{gph}(\mathcal{A}_0)$ is a closed subspace of a Hilbert space, it is weakly closed [9, Problem 19] and therefore the limit $\left[\begin{smallmatrix} z \\ x \end{smallmatrix}\right]$ is in $\mathrm{gph}(\mathcal{A}_0)$. Similarly, since the elements of $(z_j)_{j\geq k}$ are all in $\mathrm{dom}(\mathcal{A}_k)$ and $\mathrm{dom}(\mathcal{A}_k)$ is weakly closed, we have $z \in \mathrm{dom}(\mathcal{A}_k)$ and since this is true for all $k \in \mathbb{N}_0$ we have $z \in \mathcal{V}_\bullet$. It follows that for all $x \in \mathcal{V}_\bullet$ there exists a $z \in \mathcal{V}_\bullet$ such that $\left[\begin{smallmatrix} z \\ x \end{smallmatrix}\right] \in \mathrm{gph}(\mathcal{A}_0)$. This precisely means that $\mathrm{dom}(\mathcal{A}) = \mathcal{V}_\bullet$. □

Corollary 5.5. *If $\mathcal{A}_0$ is closed and* $\operatorname{dom}(\mathcal{A}_k)$ *is closed for all* $k \in \mathbb{N}_0$*, then* $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})} = \operatorname{dom}(\mathcal{A})$*.*

Proof. By definition we have $\operatorname{im}(\mathcal{A}) \subset \mathcal{V}_\bullet$, so that Proposition 5.4 gives $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})} = \operatorname{dom}(\mathcal{A})$. □

Example 5.6. We consider an example where the sequence $(\mathcal{A}_k)_{k=1}^{\infty}$ does not terminate. Define

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} z \\ x \end{bmatrix} \in \begin{bmatrix} \ell^2(\mathbb{N}) \\ \ell^2(\mathbb{N}) \end{bmatrix} : x_1 = 0, \quad z_k = x_{k+1} \right\}.$$

We then have

$$\operatorname{dom}(\mathcal{A}_0) = \left\{ x \in \ell^2(\mathbb{N}) : x_1 = 0 \right\},$$

so that

$$\operatorname{gph}(\mathcal{A}_1) = \left\{ \begin{bmatrix} z \\ x \end{bmatrix} \in \begin{bmatrix} \ell^2(\mathbb{N}) \\ \ell^2(\mathbb{N}) \end{bmatrix} : x_1 = z_1 = 0, \quad z_k = x_{k+1} \right\},$$

and therefore

$$\operatorname{dom}(\mathcal{A}_1) = \left\{ x \in \ell^2(\mathbb{N}) : x_1 = x_2 = 0 \right\}.$$

Generally we have

$$\operatorname{dom}(\mathcal{A}_k) = \left\{ x \in \ell^2(\mathbb{N}) : x_1 = x_2 = \ldots = x_{k+1} = 0 \right\}.$$

We therefore see that there does not exist a $\mu \in \mathbb{N}$ such that $\operatorname{dom}(\mathcal{A}_\mu) = \operatorname{dom}(\mathcal{A}_{\mu+1})$.

We do have that $\operatorname{dom}(\mathcal{A}_k)$ is closed for all $k \in \mathbb{N}_0$ (and we have that $\mathcal{A}_0$ is closed) and we have $\mathcal{V}_\bullet = \{0\}$. In accordance with Proposition 5.4 we have $\operatorname{dom}(\mathcal{A}) = \{0\}$.

Theorem 5.7. *If at least one of the following conditions hold*

- *The sequence* $(\mathcal{A}_k)_{k=1}^{\infty}$ *terminates;*
- $\mathcal{A}_0$ *is closed and* $\operatorname{dom}(\mathcal{A}_k)$ *is closed for all* $k \in \mathbb{N}_0$*;*

then $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$*.*

Proof. Under the first condition this follows from Lemma 5.2. Under the second condition it follows from Corollary 5.5. □

Definition 5.8. Let $\mathcal{M}$ be a multi-valued operator on $\mathcal{V}$. Then $v : [0, \infty) \to \mathcal{V}$ is a *continuously differentiable trajectory* if v is continously differentiable (from the right at $t = 0$) and

$$\begin{bmatrix} \dot{v}(t) \\ v(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{M}), \qquad \forall t \geq 0.$$

Theorem 5.9. *Continuously differentiable trajectories of $\mathcal{A}_0$ and $\mathcal{A}$ coincide.*

Proof. Since $\operatorname{gph}(\mathcal{A}) \subset \operatorname{gph}(\mathcal{A}_0)$, it is obvious that a continuously differentiable trajectory of $\mathcal{A}$ is a continuously differentiable trajectory of $\mathcal{A}_0$. Let v be a continuously differentiable trajectory of $\mathcal{A}_0$. Then $v(t) \in \operatorname{dom}(\mathcal{A}_0)$ and the difference quotient $\frac{1}{h}(v(t+h) - v(t))$ is in $\operatorname{dom}(\mathcal{A}_0)$. It follows that $\dot{v}(t)$ as the limit of the difference quotient belongs to $\overline{\operatorname{dom}(\mathcal{A}_0)}$. Hence v is a continuously differentiable trajectory of $\mathcal{A}_1$. Arguing similarly, we obtain that v is a continuously differentiable trajectory of $\mathcal{A}_k$ for all $k \in \mathbb{N}$. In particular $v(t) \in \operatorname{dom}(\mathcal{A}_k)$ for all $k \in \mathbb{N}$ so that $v(t) \in \mathcal{V}_\bullet$. As before, using difference quotients, it follows from this that $\dot{v}(t) \in \overline{\mathcal{V}_\bullet}$. We conclude that $\left[\begin{smallmatrix} \dot{v}(t) \\ v(t) \end{smallmatrix}\right] \in \operatorname{gph}(\mathcal{A})$, i.e. that v is a continuously differentiable trajectory of $\mathcal{A}$. □

The following two results are to make the connection with the Wong sequence in Section 4 and are also used when discussing discrete-time trajectories in Appendix B.

Lemma 5.10. *Let $\mathcal{M}$ be a multi-valued operator. Then*

$$\operatorname{dom}(\mathcal{M}^{k+1}) = \left\{ x \in \operatorname{dom}(\mathcal{M}^k) : \exists z \in \operatorname{dom}(\mathcal{M}^k) \text{ such that } \begin{bmatrix} z \\ x \end{bmatrix} \in \operatorname{gph}(\mathcal{M}) \right\}.$$

Proof. Since $\mathcal{M}^{k+1} = \mathcal{M}^k \mathcal{M}$ we have

$$\operatorname{gph}(\mathcal{M}^{k+1}) = \left\{ \begin{bmatrix} y \\ x \end{bmatrix} : \exists z \text{ such that } \begin{bmatrix} z \\ x \end{bmatrix} \in \operatorname{gph}(\mathcal{M}), \begin{bmatrix} y \\ z \end{bmatrix} \in \operatorname{gph}(\mathcal{M}^k) \right\},$$

so that (using that $\exists y$ such that $\left[\begin{smallmatrix} y \\ z \end{smallmatrix}\right] \in \operatorname{gph}(\mathcal{M}^k)$ is equivalent to $z \in \operatorname{dom}(\mathcal{M}^k)$)

$$\operatorname{dom}(\mathcal{M}^{k+1}) = \left\{ x \in \operatorname{dom}(\mathcal{M}) : \exists z \in \operatorname{dom}(\mathcal{M}^k) \text{ such that } \begin{bmatrix} z \\ x \end{bmatrix} \in \operatorname{gph}(\mathcal{M}) \right\}.$$

Since $\operatorname{dom}(\mathcal{M}^k) \subset \operatorname{dom}(\mathcal{M}^{k+1})$ we can replace $x \in \operatorname{dom}(\mathcal{M})$ with $x \in \operatorname{dom}(\mathcal{M}^k)$ without changing the set. This gives the desired result. □

Proposition 5.11. *Assume that $\operatorname{dom}(\mathcal{A}_k)$ is closed for all $k \in \mathbb{N}_0$. Then we have $\operatorname{dom}(\mathcal{A}_k) = \operatorname{dom}(\mathcal{A}_0^{k+1})$.*

Proof. We use induction. The statement is trivially true for $k = 0$, so assume that $\operatorname{dom}(\mathcal{A}_k) = \operatorname{dom}(\mathcal{A}_0^{k+1})$. We have (since the domains are assumed closed, we can omit closures)

$$\begin{aligned} \operatorname{dom}(\mathcal{A}_{k+1}) &= \left\{ x \in \operatorname{dom}(\mathcal{A}_k) : \exists z \in \operatorname{dom}(\mathcal{A}_k) \text{ such that } \begin{bmatrix} z \\ x \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0) \right\} \\ &= \left\{ x \in \operatorname{dom}(\mathcal{A}_0^{k+1}) : \exists z \in \operatorname{dom}(\mathcal{A}_0^{k+1}) \text{ such that } \begin{bmatrix} z \\ x \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0) \right\} \\ &= \operatorname{dom}(\mathcal{A}_0^{k+2}), \end{aligned}$$

where in the last equality we have used Lemma 5.10. □

5.2 To a multi-valued operator on a product space

Let $\mathcal{A}$ be a multi-valued operator on $\mathcal{V}$ and let $\mathcal{X}$ and $\mathcal{W}$ be closed subspaces such that

$$\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}.$$

Then we can define the multi-valued operator $\mathcal{A}_\times$ from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ to itself by

$$\operatorname{gph}(\mathcal{A}_\times) = \left\{ \begin{bmatrix} z \\ p \\ x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} : \begin{bmatrix} z + p \\ x + w \end{bmatrix} \in \operatorname{gph}(\mathcal{A}) \right\}.$$

Conversely, given two Hilbert spaces $\mathcal{X}$ and $\mathcal{W}$ we can consider the product space $\widetilde{\mathcal{V}} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ (with the product inner-product) and consider $\mathcal{X}$ and $\mathcal{W}$ canonically as subspaces through identifying them with $\begin{bmatrix} \mathcal{X} \\ \{0\} \end{bmatrix}$ and $\begin{bmatrix} \{0\} \\ \mathcal{W} \end{bmatrix}$ respectively. We can then define the multi-valued operator $\widetilde{\mathcal{A}}$ on $\widetilde{\mathcal{V}}$ though

$$\operatorname{gph}(\widetilde{\mathcal{A}}) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \widetilde{\mathcal{V}} \\ \widetilde{\mathcal{V}} \end{bmatrix} : \begin{bmatrix} \Pi_{\mathcal{X}|\mathcal{W}} q \\ \Pi_{\mathcal{W}|\mathcal{X}} q \\ \Pi_{\mathcal{X}|\mathcal{W}} v \\ \Pi_{\mathcal{W}|\mathcal{X}} v \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times) \right\}.$$

We have that $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are algebraically and topologically equivalent; however the inner-products in $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ are different unless the direct sum $\mathcal{V} = \mathcal{X} \dotplus \mathcal{W}$ is orthogonal. Since inner-products play no role in our theory (only the induced topology does), this difference is immaterial.

5.3 A brief review of state/signal systems

We recall several notions from state/signal theory [1]. A state/signal system with state space $\mathcal{X}$ and signal space $\mathcal{W}$ is a subspace V of $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$. The state/signal system is called *closed* if V is a closed subspace. The *multi-valued part* of the state/signal system is

$$\left\{ z \in \mathcal{X} : \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \right\}.$$

The *canonical input space* of the state/signal system is

$$\mathcal{W}_0 := \left\{ w \in \mathcal{W} : \exists z \text{ such that } \begin{bmatrix} z \\ 0 \\ w \end{bmatrix} \in V \right\}.$$

The *classical state space* of the state/signal system is

$$\mathcal{X}_{\text{cls}} := \left\{ x \in \mathcal{X} : \exists z, w \text{ such that } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}.$$

The *observation subspace* of the state/signal system is

$$\mathcal{H}_0 := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} : \exists z \text{ such that } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}.$$

Definition 5.12. A state/signal system V is *bounded* if the following conditions all hold

(i) V is closed;

(ii) the multi-valued part of V equals $\{0\}$;

(iii) for the classical state space we have $\mathcal{X}_{\text{cls}} = \mathcal{X}$;

(iv) the canonical input space $\mathcal{W}_0$ is closed.

By [1, Theorem 4.2.31] this is equivalent to the definition of bounded in [1, Definition 2.1.37].

Definition 5.13. We say that $x : [0,\infty) \to \mathcal{X}$ and $w : [0,\infty) \to \mathcal{W}$ form a *continuously differentiable trajectory* of V if x and w are continuously differentiable (from the right at $t = 0$) and

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad \forall t \geq 0.$$

Remark 5.14. The notion of continuously differentiable trajectory is similar to but distinct from the notion of classical trajectory from [1]. For a classical trajectory it is only assumed that w is continuous. By density, this difference is immaterial. We consider the notion of continuously differentiable trajectory since this connects more easily to trajectories for multi-valued operators.

5.4 From a multi-valued operator to a state/signal system

Let $\mathcal{A}$ be a multi-valued operator on $\mathcal{V}$. Let $\mathcal{Y}_0$ be a closed subspace such that

$$\mathcal{V} = \overline{\operatorname{dom}(\mathcal{A})} \dot{+} \mathcal{Y}_0,$$

define

$$\mathcal{W} := \overline{\operatorname{mul}(\mathcal{A})} + \mathcal{Y}_0,$$

and let $\mathcal{X}$ be a closed subspace such that

$$\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}.$$

Assume throughout that $\operatorname{mul}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$. We denote the multi-valued operator on $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ induced by $\mathcal{A}$ and the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}$ by $\mathcal{A}_\times$ as in Section 5.2.

Lemma 5.15. *We have* $\mathcal{W} = \overline{\operatorname{mul}(\mathcal{A})} \dot{+} \mathcal{Y}_0$.

Proof. We only have to show directness. Let $w \in \overline{\operatorname{mul}(\mathcal{A})} \cap \mathcal{Y}_0$. Since $\overline{\operatorname{mul}(\mathcal{A})} \subset \overline{\operatorname{dom}(\mathcal{A})}$ we have $w \in \overline{\operatorname{dom}(\mathcal{A})} \cap \mathcal{Y}_0$ and therefore $w = 0$. □

Lemma 5.16. *We have* $\mathcal{W} \cap \overline{\operatorname{dom}(\mathcal{A})} = \overline{\operatorname{mul}(\mathcal{A})}$.

Proof. Since $\overline{\operatorname{mul}(\mathcal{A})} \subset \mathcal{W}$ by definition of $\mathcal{W}$ and $\overline{\operatorname{mul}(\mathcal{A})} \subset \overline{\operatorname{dom}(\mathcal{A})}$ by assumption, one inclusion is obvious.

Let $v \in \mathcal{W} \cap \overline{\operatorname{dom}(\mathcal{A})}$. Then $v \in \mathcal{W}$, so $v = u_0 + y \in \overline{\operatorname{mul}(\mathcal{A})} + \mathcal{Y}_0$, so that $y = v - u_0 \in \overline{\operatorname{dom}(\mathcal{A})}$ using that $v \in \overline{\operatorname{dom}(\mathcal{A})}$ and $u_0 \in \overline{\operatorname{mul}(\mathcal{A})} \subset \overline{\operatorname{dom}(\mathcal{A})}$. Therefore $y \in \overline{\operatorname{dom}(\mathcal{A})} \cap \mathcal{Y}_0$ so that $y = 0$. It follows that $v = u_0 \in \overline{\operatorname{mul}(\mathcal{A})}$. □

Lemma 5.17. *The space* $\mathcal{W}$ *is closed.*

Proof. Let $\mathcal{W} \ni w_n = u_n + y_n \in \overline{\operatorname{mul}(\mathcal{A})} + \mathcal{Y}_0$ be such that $w_n \to w$. Denote by Π to oblique projection onto $\overline{\operatorname{dom}(\mathcal{A})}$ along $\mathcal{Y}_0$. Define $u := \Pi w$ and $y := (I - \Pi)w$. We have (using that $\overline{\operatorname{mul}(\mathcal{A})} \subset \overline{\operatorname{dom}(\mathcal{A})}$)

$$u_n = \Pi u_n = \Pi(u_n + y_n) \to \Pi w = u,$$

which since $u_n \in \overline{\operatorname{mul}(\mathcal{A})}$ gives that $u \in \overline{\operatorname{mul}(\mathcal{A})}$. Similarly we have

$$y_n = (I - \Pi)y_n = (I - \Pi)(u_n + y_n) \to (I - \Pi)w = y,$$

which since $\mathcal{Y}_0$ is closed gives $y \in \mathcal{Y}_0$. Therefore $w = u + y \in \overline{\operatorname{mul}(\mathcal{A})} + \mathcal{Y}_0$. Hence $\mathcal{W}$ is closed. □

We define the state/signal system with state space $\mathcal{X}$ and signal space $\mathcal{W}$ (as defined above) by

$$V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} : \exists p \in \mathcal{W} \text{ such that } \begin{bmatrix} z \\ p \\ x \\ w \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times) \right\}. \tag{7}$$

Lemma 5.18. *With the assumptions and definitions as above, we have*

(i) The multi-valued part of V equals $\{0\}$;

(ii) For the canonical input space of V we have $\mathcal{W}_0 = \overline{\operatorname{mul}(\mathcal{A})} \cap \operatorname{dom}(\mathcal{A})$.

(iii) For the classical state space of V we have

$$\mathcal{X}_{\text{cls}} = \{x \in \mathcal{X} : \exists w \in \mathcal{W} \textit{ such that } x + w \in \operatorname{dom}(\mathcal{A})\},$$

and $\mathcal{X}_{\text{cls}}$ is dense in $\mathcal{X}$.

(iv) If $\mathcal{A}$ is closed and quasi-bounded, then V is bounded.

(v) For the observation subspace of V we have $\mathcal{H}_0 = \operatorname{dom}(\mathcal{A}_\times)$.

If additionally we have $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$, then additionally the following holds:

(vi) We have

$$\left\{ z \in \mathcal{X} : \exists x, w \textit{ such that } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\} \subset \left\{ z \in \mathcal{X} : \exists p \textit{ such that } \begin{bmatrix} z \\ p \end{bmatrix} \in \overline{\mathcal{H}_0} \right\}. \tag{8}$$

Proof. (i) By definition the multi-valued part of V is

$$\left\{ z \in \mathcal{X} : \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \right\} = \left\{ z \in \mathcal{X} : \exists p \in \mathcal{W} \text{ such that } \begin{bmatrix} z + p \\ 0 \end{bmatrix} \in \operatorname{gph}(\mathcal{A}) \right\}.$$

We then have $z + p \in \operatorname{mul}(\mathcal{A}) \subset \mathcal{W}$, which since $p \in \mathcal{W}$ implies $z \in \mathcal{W}$, but then $z \in \mathcal{X} \cap \mathcal{W} = \{0\}$. Therefore the multi-valued part of V indeed equals $\{0\}$.

(ii) The canonical input space by definition is

$$\begin{aligned} \mathcal{W}_0 &= \left\{ w \in \mathcal{W} : \exists z \text{ such that } \begin{bmatrix} z \\ 0 \\ w \end{bmatrix} \in V \right\} \\ &= \left\{ w \in \mathcal{W} : \exists z \in \mathcal{X}, p \in \mathcal{W} \text{ such that } \begin{bmatrix} z + p \\ w \end{bmatrix} \in \operatorname{gph}(\mathcal{A}) \right\} \\ &= \left\{ w \in \mathcal{W} : \exists q \in \mathcal{V} \text{ such that } \begin{bmatrix} q \\ w \end{bmatrix} \in \operatorname{gph}(\mathcal{A}) \right\} \\ &= \operatorname{dom}(\mathcal{A}) \cap \mathcal{W}. \end{aligned}$$

Using Lemma 5.16, it follows that $\mathcal{W}_0 \subset \overline{\mathrm{dom}(\mathcal{A})} \cap \mathcal{W} = \overline{\mathrm{mul}(\mathcal{A})}$. Combined with the above established $\mathcal{W}_0 \subset \mathrm{dom}(\mathcal{A})$, this gives $\mathcal{W}_0 \subset \overline{\mathrm{mul}(\mathcal{A})} \cap \mathrm{dom}(\mathcal{A})$. Conversely, assume that $w \in \overline{\mathrm{mul}(\mathcal{A})} \cap \mathrm{dom}(\mathcal{A})$. Then since $\overline{\mathrm{mul}(\mathcal{A})} \subset \mathcal{W}$ we have $w \in \mathcal{W} \cap \mathrm{dom}(\mathcal{A}) = \mathcal{W}_0$.

(iii) By definition the classical state space of V is

$$\begin{aligned}\mathcal{X}_{\mathrm{cls}} &= \left\{ x : \exists z, w \text{ such that } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\} \\ &= \left\{ x \in \mathcal{X} : \exists z \in \mathcal{X}, w, p \in \mathcal{W} \text{ such that } \begin{bmatrix} z+p \\ x+w \end{bmatrix} \in \mathrm{gph}(\mathcal{A}) \right\} \\ &= \left\{ x \in \mathcal{X} : \exists w \in \mathcal{W} \text{ such that } x + w \in \mathrm{dom}(\mathcal{A}) \right\}.\end{aligned}$$

We now show that $\mathcal{X}_{\mathrm{cls}}$ is dense in $\mathcal{X}$. Let $x \in \mathcal{X}$. Then $\mathcal{V} \ni x = a + w \in \overline{\mathrm{dom}(\mathcal{A})} \dot{+} \mathcal{Y}_0$. Since $a \in \overline{\mathrm{dom}(\mathcal{A})}$, there exists a sequence $\mathrm{dom}(\mathcal{A}) \ni a_n \to a$. Let Π be the oblique projection onto $\mathcal{X}$ along $\mathcal{W}$. Define $x_n := \Pi a_n \in \mathcal{X}$ and $w_n := (I - \Pi) a_n \in \mathcal{W}$. Then $x_n + w_n = a_n \in \mathrm{dom}(\mathcal{A})$, so that x_n is in $\mathcal{X}_{\mathrm{cls}}$. We have $x_n = \Pi a_n \to \Pi a$. We further have $x = \Pi x = \Pi(a + w) = \Pi a$ since $w \in \mathcal{Y}_0 \subset \mathcal{W}$ and the projection is along $\mathcal{W}$. Therefore $x_n \to x$, showing that $\mathcal{X}_{\mathrm{cls}}$ is dense in $\mathcal{X}$.

(iv) Assume that $\mathcal{A}$ is closed and quasi-bounded. By the closed graph theorem for multi-valued operators we have that $\mathrm{dom}(\mathcal{A})$ is closed. By (i) we always have that the multi-valued part of V equals $\{0\}$. Since $\mathrm{dom}(\mathcal{A})$ and $\overline{\mathrm{mul}(\mathcal{A})}$ are closed we have using (ii) that the canonical input space as the intersection of these is closed. Let $x \in \mathcal{X}$. We can write $\mathcal{V} \ni x = a + y \in \overline{\mathrm{dom}(\mathcal{A})} \dot{+} \mathcal{Y}_0$, so that $x - y = a \in \overline{\mathrm{dom}(\mathcal{A})}$. Since $\mathrm{dom}(\mathcal{A})$ is closed, with $w := -y \in \mathcal{W}$ we have $x + w \in \mathrm{dom}(\mathcal{A})$, so that $x \in \mathcal{X}_{\mathrm{cls}}$ by (iii). It follows that $\mathcal{X} = \mathcal{X}_{\mathrm{cls}}$. It remains to show that V is closed.

Let $V \ni \left[\begin{smallmatrix} z_n \\ x_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right]$. Then there exist $p_n \in \mathcal{W}$ such that

$$\mathrm{gph}(\mathcal{A}) \ni \begin{bmatrix} z_n + p_n \\ x_n + w_n \end{bmatrix}.$$

By quasi-boundedness, p_n can be chosen as a bounded sequence. It follows that (p_n) has a weakly convergent subsequence. Since $\mathcal{W}$ as a closed subspace of a Hilbert space is weakly closed [9, Problem 19], we have that the weak limit p belongs to $\mathcal{W}$. It follows that along a subsequence

$$\mathrm{gph}(\mathcal{A}) \ni \begin{bmatrix} z_n + p_n \\ x_n + w_n \end{bmatrix} \to \begin{bmatrix} z + p \\ x + w \end{bmatrix},$$

which since $\mathrm{gph}(\mathcal{A})$ is closed implies $\left[\begin{smallmatrix} z+p \\ x+w \end{smallmatrix}\right] \in \mathrm{gph}(\mathcal{A})$. It follows that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ and therefore that V is closed. We conclude that V is bounded.

(v) The claim about the observation subspace is immediate from the definitions.

(vi) The condition $\mathrm{im}(\mathcal{A}) \subset \overline{\mathrm{dom}(\mathcal{A})}$ translates to $\mathrm{im}(\mathcal{A}_\times) \subset \overline{\mathrm{dom}(\mathcal{A}_\times)}$.

Let z be such that there exist x and w such that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$. By definition of V this implies that there exists a $p \in \mathcal{W}$ such that $\left[\begin{smallmatrix} z \\ p \\ x \\ w \end{smallmatrix}\right] \in \mathrm{gph}(\mathcal{A}_\times)$. Then $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \mathrm{im}(\mathcal{A}_\times)$ which by the additional assumption in this part of the result gives $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \overline{\mathrm{dom}(\mathcal{A}_\times)}$. Using that by (v) we have $\mathcal{H}_0 = \mathrm{dom}(\mathcal{A}_\times)$ then gives $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \overline{\mathcal{H}_0}$ which gives the desired result. □

5.5 From a state/signal system to a multi-valued operator

Let V be a state/signal system whose multi-valued part equals $\{0\}$ and which satisfies (8). Define the multi-valued operator $\mathcal{M}$ on $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ by

$$\operatorname{gph}(\mathcal{M}) := \left\{ \begin{bmatrix} z \\ p \\ x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} : \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, \quad \begin{bmatrix} z \\ p \end{bmatrix} \in \overline{\mathcal{H}_0} \right\}. \tag{9}$$

Lemma 5.19. *With the assumptions and definitions as above, we have*

(i) $\operatorname{dom}(\mathcal{M}) = \mathcal{H}_0$*;*

(ii) $\operatorname{mul}(\mathcal{M}) = \left[\begin{smallmatrix} 0 \\ \mathcal{W}_0 \end{smallmatrix}\right]$ *where* $\mathcal{W}_0$ *is the canonical input space;*

(iii) If V is closed, then $\mathcal{M}$ is closed.

(iv) If V is bounded, then $\mathcal{M}$ is closed and quasi-bounded.

Proof. (i) We have

$$\operatorname{dom}(\mathcal{M}) = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} : \exists z, p \text{ such that } \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, \quad \begin{bmatrix} z \\ p \end{bmatrix} \in \overline{\mathcal{H}_0} \right\}.$$

The condition $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ precisely means $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{H}_0$ and by the assumption (8), the condition $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \overline{\mathcal{H}_0}$ doesn't impose any further restrictions. Therefore $\operatorname{dom}(\mathcal{M}) = \mathcal{H}_0$.

(ii) We have (using that the multi-valued part of V is trivial)

$$\operatorname{mul}(\mathcal{M}) = \left\{ \begin{bmatrix} z \\ p \end{bmatrix} : \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V, \quad \begin{bmatrix} z \\ p \end{bmatrix} \in \overline{\mathcal{H}_0} \right\} = \left\{ \begin{bmatrix} 0 \\ p \end{bmatrix} : \begin{bmatrix} 0 \\ p \end{bmatrix} \in \overline{\mathcal{H}_0} \right\} = \begin{bmatrix} 0 \\ \overline{\mathcal{W}_0} \end{bmatrix},$$

using that the multi-valued part of the closure is the closure of the multi-valued part.

(iii) Let

$$\operatorname{gph}(\mathcal{M}) \ni \begin{bmatrix} z_n \\ p_n \\ x_n \\ w_n \end{bmatrix} \to \begin{bmatrix} z \\ p \\ x \\ w \end{bmatrix}.$$

Then

$$V \ni \begin{bmatrix} z_n \\ x_n \\ w_n \end{bmatrix} \to \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \qquad \overline{\mathcal{H}_0} \ni \begin{bmatrix} z_n \\ p_n \end{bmatrix} \to \begin{bmatrix} z \\ p \end{bmatrix},$$

which since V and $\overline{\mathcal{H}_0}$ are closed gives $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$ and $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \overline{\mathcal{H}_0}$, so that $\left[\begin{smallmatrix} z \\ p \\ x \\ w \end{smallmatrix}\right] \in \operatorname{gph}(\mathcal{M})$.

(iv) Since V is bounded, it is closed so that by (iii) we have that $\mathcal{M}$ is closed. By [1, Definition 2.1.37] we have that the observation subspace $\mathcal{H}_0$ is closed, which by (i) gives that $\operatorname{dom}(\mathcal{M})$ is closed. It follows from the closed graph theorem for multi-valued operators that $\mathcal{M}$ is quasi-bounded. □

Lemma 5.20. *If we start with a multi-valued operator* $\mathcal{A}$ *that satisfies* $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$*, then form* V *using* (7)*, and then form* $\mathcal{M}$ *using* (9)*, we have* $\operatorname{gph}(\mathcal{A}_\times) \subset \operatorname{gph}(\mathcal{M})$*, where we have equality if moreover* $\operatorname{mul}(\mathcal{A})$ *is closed.*

Proof. The condition $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ translates to $\operatorname{im}(\mathcal{A}_\times) \subset \overline{\operatorname{dom}(\mathcal{A}_\times)}$.

Let $\begin{bmatrix} z \\ p \\ x \\ w \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times)$. By definition of V we then have $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$. Moreover, $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \operatorname{im}(\mathcal{A}_\times)$, so that by assumption $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \overline{\operatorname{dom}(\mathcal{A}_\times)}$, which by Lemma 5.18 gives $\left[\begin{smallmatrix} z \\ p \end{smallmatrix}\right] \in \overline{\mathcal{H}_0}$. Therefore $\operatorname{gph}(\mathcal{A}_\times) \subset \operatorname{gph}(\mathcal{M})$.

For the converse, let $\begin{bmatrix} z \\ p \\ x \\ w \end{bmatrix} \in \operatorname{gph}(\mathcal{M})$. Since $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$, we have that there exists a p_0 such that $\begin{bmatrix} z \\ p_0 \\ x \\ w \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times)$. From the first part of the proof it follows that we must have $\left[\begin{smallmatrix} z \\ p_0 \end{smallmatrix}\right] \in \overline{\mathcal{H}_0}$. We then have

$$\begin{bmatrix} 0 \\ p - p_0 \end{bmatrix} = \begin{bmatrix} z \\ p \end{bmatrix} - \begin{bmatrix} z \\ p_0 \end{bmatrix} \in \overline{\mathcal{H}_0}.$$

Using Lemma 5.18 it follows that

$$\begin{bmatrix} 0 \\ p - p_0 \end{bmatrix} \in \overline{\operatorname{dom}(\mathcal{A}_\times)}.$$

It follows that $p - p_0 \in \overline{\operatorname{dom}(\mathcal{A})} \cap \mathcal{W}$ which by Lemma 5.16 equals $\overline{\operatorname{mul}(\mathcal{A})}$. Using that $\operatorname{mul}(\mathcal{A})$ is closed, we obtain

$$\begin{bmatrix} 0 \\ p - p_0 \\ 0 \\ 0 \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times),$$

so that

$$\begin{bmatrix} z \\ p \\ x \\ w \end{bmatrix} = \begin{bmatrix} z \\ p_0 \\ x \\ w \end{bmatrix} - \begin{bmatrix} 0 \\ p - p_0 \\ 0 \\ 0 \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times),$$

as desired. □

Lemma 5.21. *If* $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ *and* $\operatorname{mul}(\mathcal{A})$ *is closed, then we have the following. Continuously differentiable trajectories of* $\mathcal{A}$ *and* V *coincide in the sense that if* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a continuously differentiable trajectory of* V*, then* $v := x + w$ *is a continuously differentiable trajectory of* $\mathcal{A}$ *and if* v *is a continuously differentiable trajectory of* $\mathcal{A}$*, and we (uniquely) write* $v = x + w$ *with respect to the decomposition* $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}$*, then* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *is a continuously differentiable trajectory of* V*.*

Proof. Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be a continuously differentiable trajectory of V. By definition we have that both x and w are continuously differentiable and that for all $t \geq 0$

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V.$$

We then have $\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right] \in \mathcal{H}_0$ so that (by considering difference quotients) $\left[\begin{smallmatrix} \dot{x}(t) \\ \dot{w}(t) \end{smallmatrix}\right] \in \overline{\mathcal{H}_0}$. From Lemma 5.20 it follows that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \\ x(t) \\ w(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times),$$

which precisely means

$$\begin{bmatrix} \dot{x}(t) + \dot{w}(t) \\ x(t) + w(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}),$$

so that indeed $v = x + w$ is a continuously differentiable trajectory of $\mathcal{A}$.

Let v be a continuously differentiable trajectory of $\mathcal{A}$. Then for all $t \geq 0$

$$\begin{bmatrix} \dot{v}(t) \\ v(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}).$$

Write $v(t) = x(t) + w(t)$ with respect to the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}$. Then x and w are continuously differentiable and $\dot{v}(t) = \dot{x}(t) + \dot{w}(t)$. It follows that

$$\begin{bmatrix} \dot{x}(t) + \dot{w}(t) \\ x(t) + w(t) \end{bmatrix} = \begin{bmatrix} \dot{v}(t) \\ v(t) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}).$$

From this we conclude that (with $p(t) := \dot{w}(t) \in \mathcal{W}$)

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V,$$

so that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a continuously differentiable trajectory of V. □

Example 5.22. We give an example to illustrate the imporance of the condition $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$.

Let

$$\operatorname{gph}(\mathcal{A}) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^3 \\ \mathbb{R}^3 \end{bmatrix} : q_1 = v_2, \quad q_2 = v_3, \quad v_1 = 0 \right\}.$$

Then

$$\operatorname{dom}(\mathcal{A}) = \left\{ v \in \mathbb{R}^3 : v_1 = 0 \right\}, \quad \operatorname{im}(\mathcal{A}) = \mathbb{R}^3, \quad \operatorname{mul}(\mathcal{A}) = \left\{ q \in \mathbb{R}^3 : q_1 = q_2 = 0 \right\},$$

so that the condition $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ is not satisfied but the standing assumption that $\operatorname{mul}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ is satisfied. We consider

$$\mathcal{Y}_0 = \left\{ v \in \mathbb{R}^3 : v_2 = v_3 = 0 \right\}, \quad \mathcal{W} = \mathcal{Y}_0 + \operatorname{mul}(\mathcal{A}) = \left\{ v \in \mathbb{R}^3 : v_2 = 0 \right\},$$
$$\mathcal{X} = \left\{ v \in \mathbb{R}^3 : v_1 = v_3 = 0 \right\}.$$

We then have

$$V = \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} : z_2 = w_3, \quad w_1 = 0 \right\}.$$

For f twice continuously differentiable we have that

$$x(t) = \begin{bmatrix} 0 \\ f(t) \\ 0 \end{bmatrix}, \qquad w(t) = \begin{bmatrix} 0 \\ 0 \\ f'(t) \end{bmatrix},$$

is a continuously differentiable trajectory of V. However, the equations describing trajectories of $\mathcal{A}$ are

$$\dot{v}_1 = v_2, \qquad \dot{v}_2 = v_3, \qquad v_1 = 0.$$

and it easily follows that $v = 0$ is the only continuously differentiable trajectory of $\mathcal{A}$. Hence the conclusion of Lemma 5.21 is false (and as mentioned above, the assumptions in Lemma 5.21 are not satisfied). Of course, we can use the iteration from Section 5.1 with $\mathcal{A}_0$ the above multi-valued operator to arrive at a multi-valued operator which does satisfy $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$, and produce the corresponding state/signal system (and this is what one should do for this example).

5.6 From a state/signal system to an input/state/output system

Let V be a state/signal system with state space $\mathcal{X}$ and signal space $\mathcal{W}$. Let $\mathcal{U}$ and $\mathcal{Y}$ be closed subsspaces such that (this is called an input/output representation in [1, Definition 4.2.5])

$$\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y},$$

and let S be the multi-valued operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ given by (this is called an input/state/output representation in [1, Definition 4.2.10])

$$\operatorname{gph}(S) = \left\{ \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} : \begin{bmatrix} z \\ x \\ u + y \end{bmatrix} \in V \right\}. \tag{10}$$

Definition 5.23. We say that $x : [0,\infty) \to \mathcal{X}$, $u : [0,\infty) \to \mathcal{U}$ and $y : [0,\infty) \to \mathcal{Y}$ form a *continuously differentiable trajectory* if x, u and y are continuously differentiable (from the right at $t = 0$) and

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \\ x(t) \\ u(t) \end{bmatrix} \in \operatorname{gph}(S), \qquad \forall t \geq 0.$$

Lemma 5.24. *With the assumptions and definitions as at the start of this section we have the following. Continuously differentiable trajectories of V and S coincide in the sense that if $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a continuously differentiable trajectory of S, then $\left[\begin{smallmatrix} x \\ u+y \end{smallmatrix}\right]$ is a continuously differentiable trajectory of V and if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a continuously differentiable trajectory of V and we (uniquely) write $w = u + y$ with respect to the decomposition $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$, then $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a continuously differentiable trajectory of S.*

Proof. This is immediate from the definitions. □

The *classical state space* of S is

$$\left\{ x \in \mathcal{X} : \exists z, u, y \text{ such that } \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \operatorname{gph}(S) \right\},$$

and it is trivial to see that the classical state spaces of V and S coincide. It is also easy to see that V is closed if and only if S is closed (see [1, Lemma 4.2.11 (i)]).

Lemma 5.25. *We have that S is single-valued if and only if both the multi-valued part of V equals $\{0\}$ and $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$ where $\mathcal{W}_0$ is the canonical input space of V.*

Proof. This is essentially contained in [1, Theorem 4.2.15].

Assume that S is single-valued. Let $y \in \mathcal{Y} \cap \mathcal{W}_0$. Then there exists a $z \in \mathcal{X}$ such that $\left[\begin{smallmatrix} z \\ 0 \\ y \end{smallmatrix}\right] \in V$, which implies $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] \in S \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. Since S is single-valued, this implies $y = 0$ (and $z = 0$). Therefore $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$. Let z be in the multi-valued part of V. Then $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$ so that $\left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right] \in S \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ which since S is single-valued implies $z = 0$. Hence the multi-valued part of V equals $\{0\}$.

Assume that the multi-valued part of V equals $\{0\}$ and that $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$. Let $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] \in S \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} z \\ 0 \\ y \end{smallmatrix}\right] \in V$, so that $y \in \mathcal{W}_0$. It follows that $y \in \mathcal{Y} \cap \mathcal{W}_0$, so that $y = 0$. Then $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$, so that z is in the multi-valued part of V, so that by assumption $z = 0$. It follows that S is single-valued. □

Proposition 5.26. *We have that S is a bounded single-valued everywhere-defined operator if and only if the following conditions both hold:*

(i) V is bounded;

(ii) $\mathcal{Y}$ is a direct complement to $\mathcal{W}_0$.

Proof. Assume the two stated conditions. By [1, Theorem 4.2.33] the second condition is equivalent to $(\mathcal{U}, \mathcal{Y})$ being an input/state/output bounded input/output representation of $\mathcal{W}$ in the sense of [1, Definition 4.2.16], which precisely means that S is bounded.

Assume that S is bounded. It follows from [1, Lemma 4.2.11 (iv)] that V is bounded. Using [1, Theorem 4.2.33] we obtain that $\mathcal{Y}$ is a direct complement to $\mathcal{W}_0$. □

The formal resolvent of the state/signal system V is the family of multi-valued operators $\widehat{\mathfrak{E}}(\lambda)$ indexed by $\lambda \in \mathbb{C}$ defined by

$$\widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} 0 & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} V,$$

the *resolvent set* $\rho(V)$ consists of those $\lambda \in \mathbb{C}$ for which $\widehat{\mathfrak{E}}(\lambda)$ is a bounded everywhere-defined single-valued operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ to $\mathcal{X}$, and V is called *resolvable* if its resolvent set is non-empty [1, Definition 3.4.33]. We note that a resolvable state/signal system is closed [1, Lemma 3.4.34].

The *characteristic signal bundle* of V is [1, Definition 3.4.15]

$$\widehat{\mathfrak{F}}(\lambda) := \left\{ w \in \mathcal{W} : \exists x \text{ such that } \begin{bmatrix} \lambda x \\ x \\ w \end{bmatrix} \in V \right\}.$$

The formal resolvent of the input/state/output system S is the family of multi-valued operators $\widehat{\mathfrak{G}}(\lambda)$ indexed by $\lambda \in \mathbb{C}$ defined by [1, (10.2.1(a)]

$$\mathrm{gph}(\widehat{\mathfrak{G}}(\lambda)) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{gph}(S),$$

the *resolvent set* $\rho_{\mathrm{iso}}(S)$ is the set of those$\lambda \in \mathbb{C}$ for which $\widehat{\mathfrak{G}}(\lambda)$ is a bounded everywhere-defined single-valued operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$, and S is called *resolvable* if its resolvent set is non-empty.

Lemma 5.27. *Let V be closed. The following are equivalent:*

(a) $\lambda \in \rho_{\mathrm{iso}}(S)$*;*

(b) $\lambda \in \rho(V)$ *and* $\mathcal{Y}$ *is a direct complement of the characteristic signal bundle* $\widehat{\mathfrak{F}}(\lambda)$ *in* $\mathcal{W}$*.*

Proof. This is [1, Theorem 10.3.6] (in particular (vii) implies (ii) in that result). □

The *main operator* A of an input/state/output system S is the multi-valued operator from $\mathcal{X}$ to $\mathcal{X}$ defined by

$$\mathrm{gph}(A) = \left\{ \begin{bmatrix} z \\ x \end{bmatrix} : \exists y \in \mathcal{Y} \text{ such that } \begin{bmatrix} z \\ y \\ x \\ 0 \end{bmatrix} \in \mathrm{gph}(S) \right\}.$$

We note that by [1, Theorem 10.2.4], resolvability of S implies that A has a non-empty resolvent set and that $\rho_{\mathrm{iso}}(S) = \rho(A)$ (where $\rho(A)$ is the usual resolvent set of the multi-valued operator A).

The following definition is adapted from [13] (by [13, Lemma 4.7.7] this definition is equivalent to [13, Definition 4.7.2]).

Definition 5.28. An *operator node* is a single-valued linear operator $S : \mathrm{dom}(S) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with the following properties. We decompose $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ where $A\&B : \mathrm{dom}(S) \to \mathcal{X}$ and $C\&D : \mathrm{dom}(S) \to \mathcal{Y}$. We denote $\mathrm{dom}(A) := \{x \in \mathcal{X} : \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathrm{dom}(S)\}$, define $A : \mathrm{dom}(A) \subset \mathcal{X} \to \mathcal{X}$ by $Ax = A\&B \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ and require the following to hold:

(i) S is closed;

(ii) $A\&B$ is closed;

(iii) A has a nonempty resolvent set and $\mathrm{dom}(A)$ is dense in $\mathcal{X}$;

(iv) For all $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}(S)$.

Note that A in Definition 5.28 is the main operator of S.

Lemma 5.29. *We have that S is an operator node if and only if the following conditions both hold:*

(i) S is single-valued;

(ii) S is resolvable.

Proof. This is [1, Theorem 10.2.26] noting that S resolvable implies S closed. □

Theorem 5.30. *We have that S is an operator node if and only if the following conditions all hold:*

(i) the multi-valued part of V equals $\{0\}$;

(ii) V is resolvable;

(iii) $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$;

(iv) $\mathcal{Y}$ is a direct complement of the characteristic signal bundle $\widehat{\mathfrak{F}}(\lambda)$ in $\mathcal{W}$ for some $\lambda \in \rho(V)$.

Proof. Assume that S is an operator node. Then by Lemma 5.29, S is single-valued which by Lemma 5.25 implies (i) and (iii). By Lemma 5.29, S is resolvable, which implies that S is closed which implies that V is closed. From Lemma 5.27 we then obtain (ii) and (iv).

Assume the above condititions. By Lemma 5.25 we obtain that S is single-valued. Since V resolvable implies V closed, by Lemma 5.27 we obtain that S is resolvable. From Lemma 5.29 we obtain that S is an operator node. □

Remark 5.31. With respect to $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$, the characteristic signal bundle $\widehat{\mathfrak{F}}(\lambda)$ becomes the graph of the input/output resolvent $\widehat{\mathfrak{G}}(\lambda)$ (this follows from [1, 10.3.7(a)]). For S to be well-posed (for the definition of a well-posed input/state/output system (as in [13]), we refer to [1, Definition 14.1.1]) it is necessary that $\widehat{\mathfrak{G}}$ is uniformly bounded on some right half-plane. This can further inform the choice of the decomposition $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$ in Theorem 5.30.

5.7 From a multi-valued operator to an input/state/output system

We assume that the multi-valued operator $\mathcal{A}$ is as in Section 5.4 (in particular, it is assumed that $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$), that V is defined in terms of $\mathcal{A}$ as in (7) and that S is defined in terms of V as in (10).

We note that if the conditions on the multi-valued operator are not satisfied then the construction in Section 5.1 can often be used to obtain a multi-valued operator with the same set of continuously differentiable trajectories which does satisfy the assumptions and can therefore be used as $\mathcal{A}$ in this section.

Theorem 5.32. *With the assumptions and definitions as at the start of this section and the additional assumption that* $\operatorname{mul}(\mathcal{A})$ *is closed we have the following. Continuously differentiable trajectories of $\mathcal{A}$ and S coincide in the sense that if $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ is a continuously differentiable trajectory of S, then $v := x + u + y$ is a continuously differentiable trajectory of $\mathcal{A}$ and if v is a continuously differentiable trajectory of $\mathcal{A}$ and we (uniquely) write $v = x+u+y$ with respect to the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{U} \dot{+} \mathcal{Y}$, then $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ is a continuously differentiable trajectory of S.*

Proof. This follows from combining Lemma 5.21 and Lemma 5.24. □

Theorem 5.33. *We have that S is single-valued if and only if $\mathcal{Y} \cap \overline{\operatorname{mul}(\mathcal{A})} \cap \operatorname{dom}(\mathcal{A}) = \{0\}$.*

Proof. From Lemma 5.25, we have that S is single-valued if and only if both the multi-valued part of V equals $\{0\}$ and $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$ where $\mathcal{W}_0$ is the canonical input space of V. By Lemma 5.18 the first of these conditions is always true and by that same lemma $\mathcal{W}_0 = \overline{\text{mul}(\mathcal{A})} \cap \text{dom}(\mathcal{A})$, giving the desired result. □

Theorem 5.34. *We have that S is a bounded single-valued everywhere-defined operator if the following conditions all hold:*

(i) $\mathcal{A}$ *is closed;*

(ii) $\text{dom}(\mathcal{A})$ *is closed;*

(iii) $\mathcal{Y}$ *is a direct complement to* $\overline{\text{mul}(\mathcal{A})} \cap \text{dom}(\mathcal{A})$.

If $\text{mul}(\mathcal{A})$ *is closed, then the converse is also true.*

Proof. We verify the conditions in Proposition 5.26.

Assume the listed conditions. By Lemma 5.18 and the closed graph theorem for multi-valued operators we have that V is bounded. By Lemma 5.18 we have that the canonical input space $\mathcal{W}_0$ equals $\overline{\text{mul}(\mathcal{A})} \cap \text{dom}(\mathcal{A})$ so that $\mathcal{Y}$ is a direct complement to $\mathcal{W}_0$. Therefore the conditions in Proposition 5.26 are satisfied, so that S is bounded.

Assume that S is bounded and $\text{mul}(\mathcal{A})$ is closed. Then by Proposition 5.26 we have that V is bounded. From Lemma 5.19 we have that $\mathcal{M}$ as given there is closed and quasi-bounded. From Lemma 5.20 it follows that $\mathcal{A}_\times$ and therefore $\mathcal{A}$ is closed and quasi-bounded. From the closed graph theorem for multi-valued operators we then obtain that $\mathcal{A}$ is closed and $\text{dom}(\mathcal{A})$ is closed. The last condition follows from Proposition 5.26 using that $\mathcal{W}_0$ equals $\overline{\text{mul}(\mathcal{A})} \cap \text{dom}(\mathcal{A})$ by Lemma 5.18. □

Theorem 5.35. *We have that S is an operator node if and only if the following conditions all hold:*

(i) $\mathcal{Y} \cap \overline{\text{mul}(\mathcal{A})} \cap \text{dom}(\mathcal{A}) = \{0\}$*;*

(ii) V *is resolvable;*

(iii) $\mathcal{Y}$ *is a direct complement of the characteristic signal bundle* $\widehat{\mathfrak{F}}(\lambda)$ *in* $\mathcal{W}$ *for some* $\lambda \in \rho(V)$.

Proof. We verify the conditions in Theorem 5.30. By Lemma 5.18 we have that the condition that the multi-valued part of V equals $\{0\}$ is always satisfied. By Lemma 5.18 we have that canonical input space $\mathcal{W}_0$ equals $\overline{\text{mul}(\mathcal{A})} \cap \text{dom}(\mathcal{A})$, so that the two conditions $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$ and $\mathcal{Y} \cap \overline{\text{mul}(\mathcal{A})} \cap \text{dom}(\mathcal{A}) = \{0\}$ are the same. The remaining condition (resolvability of V) is literally identical. □

Remark 5.36. Whereas Theorem 5.34 does not mention V, Theorem 5.35 does. In principle the two conditions involving V in Theorem 5.35 could be phrased in terms of $\mathcal{A}$, but its seems more natural to phrase them in terms of V.

A Frequency domain trajectories

In addition to continuously differentiable trajectories, as we have discussed up to now, there is also the notion of frequency domain trajectories (which is for example important in optimal control problems).

Definition A.1. Let Ω be a non-empty open subset of $\mathbb{C}$. A frequency domain Ω trajectory is defined as follows.

- For a multivalued operator $\mathcal{A}$ on $\mathcal{V}$ it is a pair $(\hat{v}, v^0)$ where $\hat{v}$ is a holomorphic function defined on Ω with values in $\mathcal{V}$ and $v^0 \in \mathcal{V}$ such that for all $\lambda \in \Omega$

$$\begin{bmatrix} -v^0 + \lambda\hat{v}(\lambda) \\ \hat{v}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}).$$

- For a state/signal system V it is a triple $(\hat{x}, \hat{w}, x^0)$ where $\hat{x}$ and $\hat{w}$ are holomorphic functions defined on Ω with values in $\mathcal{X}$ and $\mathcal{W}$ respectively and $x^0 \in \mathcal{X}$ such that for all $\lambda \in \Omega$

$$\begin{bmatrix} -x^0 + \lambda\hat{x}(\lambda) \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V. \tag{11}$$

- For a multi-valued operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ it is a quadruple $(\hat{x}, \hat{y}, x^0, \hat{u})$ where $\hat{x}$, $\hat{y}$ and $\hat{u}$ are holomorphic functions defined on Ω with values in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$ respectively and $x^0 \in \mathcal{X}$ such that for all $\lambda \in \Omega$

$$\begin{bmatrix} -x^0 + \lambda\hat{x}(\lambda) \\ \hat{y}(\lambda) \\ \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{gph}(S). \tag{12}$$

The state/signal definition of frequency domain Ω trajectory is [1, Definition 12.1.1] and the input/state/output one is [1, Definition 11.1.1].

We note that the equations (11) and (12) are respectively equivalent to

$$\begin{bmatrix} \hat{x}(\lambda) \\ x^0 \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda), \qquad \begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \\ x^0 \\ \hat{u}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{G}}(\lambda),$$

where $\widehat{\mathfrak{E}}$ and $\widehat{\mathfrak{G}}$ are the respective formal resolvents.

In the remainder of this section, we assume that that V is defined in terms of $\mathcal{A}$ as in (7) and that S is defined in terms of V as in (10).

Proposition A.2. *If* $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ *and* $\operatorname{mul}(\mathcal{A})$ *is closed, then we have the following. Frequency domain Ω trajectories of $\mathcal{A}$ and V coincide in the sense that if $(\hat{x}, \hat{w}, x^0)$ is a frequency domain Ω trajectory of V, then with $\hat{v} := \hat{x} + \hat{w}$ there exists a $w^0 \in \mathcal{W}$ such that with $v^0 := x^0 + w^0$ we have that $(\hat{v}, v^0)$ is a frequency domain Ω trajectory of $\mathcal{A}$ and if $(\hat{v}, v^0)$ is a frequency domain Ω trajectory of $\mathcal{A}$, and we (uniquely) write $\hat{v} = \hat{x} + \hat{w}$ and $v^0 = x^0 + w^0$ with respect to the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}$, then $(\hat{x}, \hat{w}, x^0)$ is a frequency domain Ω trajectory of V.*

Proof. Let $(\hat{x}, \hat{w}, x^0)$ be a frequency domain Ω trajectory of V, i.e. for all $\lambda \in \Omega$

$$\begin{bmatrix} -x^0 + \lambda \hat{x}(\lambda) \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V. \tag{13}$$

Using (8) we have that there exists a $p(\lambda) \in \mathcal{W}$ such that

$$\begin{bmatrix} -x^0 + \lambda \hat{x}(\lambda) \\ p(\lambda) \end{bmatrix} \in \overline{\mathcal{H}_0}. \tag{14}$$

For fixed $\alpha \in \Omega$, define $w^0 := \alpha \hat{w}(\alpha) - p(\alpha)$. Then (14) with $\lambda = \alpha$ precisely is

$$\begin{bmatrix} -x^0 + \alpha \hat{x}(\alpha) \\ -w^0 + \alpha \hat{w}(\alpha) \end{bmatrix} \in \overline{\mathcal{H}_0}.$$

For $\lambda \in \Omega$ we have

$$\begin{bmatrix} -x^0 + \lambda \hat{x}(\lambda) \\ -w^0 + \lambda \hat{w}(\lambda) \end{bmatrix} = \begin{bmatrix} -x^0 + \alpha \hat{x}(\alpha) \\ -w^0 + \alpha \hat{w}(\alpha) \end{bmatrix} + \lambda \begin{bmatrix} \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} - \alpha \begin{bmatrix} \hat{x}(\alpha) \\ \hat{w}(\alpha) \end{bmatrix} \in \overline{\mathcal{H}_0},$$

where we have used that by (13) (applied with λ and with $\lambda = \alpha$) we have that $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \hat{x}(\alpha) \\ \hat{w}(\alpha) \end{smallmatrix}\right]$ are both in $\mathcal{H}_0$. It follows from Lemma 5.20 that

$$\begin{bmatrix} -x^0 + \lambda \hat{x}(\lambda) \\ -w^0 + \lambda \hat{w}(\lambda) \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times),$$

which precisely means that

$$\begin{bmatrix} -x^0 - w^0 + \lambda \hat{x}(\lambda) + \lambda \hat{w}(\lambda) \\ \hat{x}(\lambda) + \hat{w}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}).$$

This gives that $(\hat{v}, v^0)$ is a frequency domain Ω trajectory of $\mathcal{A}$.

Let $(\hat{v}, v^0)$ be a frequency domain Ω trajectory of $\mathcal{A}$, i.e. for all $\lambda \in \Omega$

$$\begin{bmatrix} -v^0 + \lambda \hat{v}(\lambda) \\ \hat{v}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}).$$

Then

$$\begin{bmatrix} -x^0 - w^0 + \lambda \hat{x}(\lambda) + \lambda \hat{w}(\lambda) \\ \hat{x}(\lambda) + \hat{w}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}),$$

so that

$$\begin{bmatrix} -x^0 + \lambda \hat{x}(\lambda) \\ -w^0 + \lambda \hat{w}(\lambda) \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times),$$

so that

$$\begin{bmatrix} -x^0 + \lambda \hat{x}(\lambda) \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V,$$

which precisely means that $(\hat{x}, \hat{w}, x^0)$ is a frequency domain Ω trajectory of V. $\square$

Proposition A.3. *Frequency domain Ω trajectories of V and S coincide in the sense that if $(\hat{x}, \hat{y}, x^0, \hat{u})$ is a frequency domain Ω trajectory of S, then $(\hat{x}, \hat{u} + \hat{y}, x^0)$ is a frequency domain Ω trajectory of V and if $(\hat{x}, \hat{w}, x^0)$ is a frequency domain Ω trajectory of V and we (uniquely) write $\hat{w} = \hat{u} + \hat{y}$ with respect to the decomposition $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$, then $(\hat{x}, \hat{y}, x^0, \hat{u})$ is a frequency domain Ω trajectory of S.*

Proof. This is [1, Lemma 12.1.4] □

Corollary A.4. *If $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ and $\operatorname{mul}(\mathcal{A})$ is closed, then we have the following. Frequency domain Ω trajectories of $\mathcal{A}$ and S coincide in the sense that if $(\hat{x}, \hat{y}, x^0, \hat{u})$ is a frequency domain Ω trajectory of S, then with $\hat{v} := \hat{x} + \hat{u} + \hat{y}$ there exist $u^0 \in \mathcal{U}$ and $y^0 \in \mathcal{Y}$ such that with $v^0 := x^0 + u^0 + y^0$ we have that $(\hat{v}, v^0)$ is a frequency domain Ω trajectory of $\mathcal{A}$ and if $(\hat{v}, v^0)$ is a frequency domain Ω trajectory of $\mathcal{A}$ and we (uniquely) write $\hat{v} = \hat{x} + \hat{u} + \hat{y}$ and $v^0 = x^0 + u^0 + y^0$ with respect to the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{U} \dot{+} \mathcal{Y}$, then $(\hat{x}, \hat{y}, x^0, \hat{u})$ is a frequency domain Ω trajectory of S.*

Proof. This follows from combining Proposition A.2 and Proposition A.3. □

Example A.5. It is not true that frequency domain trajectories are preserved by the iteration from Section 5.1 (unlike continuously differentiable trajectories as shown in Theorem 5.9). Let

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^2 \end{bmatrix} : q_1 = v_2, \quad v_1 = 0 \right\}.$$

Applying the iteration from Section 5.1 gives

$$\operatorname{gph}(\mathcal{A}) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^2 \end{bmatrix} : q_1 = q_2 = v_1 = v_2 = 0 \right\},$$

so that the only frequency domain trajectory of $\mathcal{A}$ is $(0, 0)$ (the choice of Ω is immaterial). On the other hand, $\mathcal{A}_0$ has the nontrivial frequency domain trajectory $\hat{v} = \begin{bmatrix} 0 \\ -v_1^0 \end{bmatrix}$, $v^0 = \begin{bmatrix} v_1^0 \\ v_2^0 \end{bmatrix}$ where $v_1^0, v_2^0 \in \mathbb{R}$ are arbitrary.

We now give some positive results on the preservation of frequency domain trajectories under the iteration from Section 5.1.

Proposition A.6. *Let $(\hat{v}, v^0)$ be an Ω trajectory of $\mathcal{A}_0$. Then it is an Ω trajectory of $\mathcal{A}_1$ if and only if $v^0 \in \overline{\operatorname{dom}(\mathcal{A}_0)}$. Moreover, it is an Ω trajectory of $\mathcal{A}$ if and only if $v^0 \in \overline{\mathcal{V}_\bullet}$.*

Proof. Let $v^0 \in \overline{\operatorname{dom}(\mathcal{A}_0)}$. Since $(\hat{v}, v^0)$ is an Ω trajectory of $\mathcal{A}_0$ we have $\hat{v}(\lambda) \in \operatorname{dom}(\mathcal{A}_0)$, so that $-v^0 + \lambda \hat{v}(\lambda) \in \overline{\operatorname{dom}(\mathcal{A}_0)}$. It follows that

$$\begin{bmatrix} -v^0 + \lambda \hat{v}(\lambda) \\ \hat{v}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \overline{\operatorname{dom}(\mathcal{A}_0)} \\ \overline{\operatorname{dom}(\mathcal{A}_0)} \end{bmatrix} = \operatorname{gph}(\mathcal{A}_1), \tag{15}$$

which shows that $(\hat{v}, v^0)$ is an Ω trajectory of $\mathcal{A}_1$.

Let $(\hat{v}, v^0)$ be an Ω trajectory of $\mathcal{A}_1$. Then (15) holds, so that $\hat{v}(\lambda) \in \operatorname{dom}(\mathcal{A}_0)$ and $-v^0 + \lambda \hat{v}(\lambda) \in \overline{\operatorname{dom}(\mathcal{A}_0)}$, which implies that $v^0 \in \overline{\operatorname{dom}(\mathcal{A}_0)}$.

Let $v^0 \in \overline{\mathcal{V}_\bullet}$. Then $v^0 \in \overline{\operatorname{dom}(\mathcal{A}_0)}$, so that $(\hat{v}, v^0)$ is an Ω trajectory of $\mathcal{A}_1$ by the above. Moreover $v^0 \in \overline{\operatorname{dom}(\mathcal{A}_1)}$, which implies similarly that $(\hat{v}, v^0)$ is an Ω trajectory of $\mathcal{A}_2$. Proceeding like this, we obtain that $(\hat{v}, v^0)$ is an Ω trajectory of $\mathcal{A}_k$ for all $k \in \mathbb{N}$. In particular, $\hat{v}(\lambda) \in \cap_{k=1}^{\infty} \operatorname{dom}(\mathcal{A}_k) = \mathcal{V}_\bullet$. It follows that

$$\begin{bmatrix} -v^0 + \lambda\hat{v}(\lambda) \\ \hat{v}(\lambda) \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \overline{\mathcal{V}_\bullet} \\ \overline{\mathcal{V}_\bullet} \end{bmatrix} = \operatorname{gph}(\mathcal{A}), \tag{16}$$

so that $(\hat{v}, v^0)$ is an Ω trajectory of $\mathcal{A}$.

Conversely, assume that $(\hat{v}, v^0)$ is an Ω trajectory of $\mathcal{A}$. Then (16) holds, which implies that $\hat{v}(\lambda) \in \overline{\mathcal{V}_\bullet}$ and $-v^0 + \lambda\hat{v}(\lambda) \in \overline{\mathcal{V}_\bullet}$, the combination of which implies $v^0 \in \overline{\mathcal{V}_\bullet}$. □

B Discrete-time trajectories

In addition to continuous-time trajectories, it is possible to consider discrete-time trajectories.

Definition B.1. Infinite-time discrete-time trajectories are defined as follows.

- For a multi-valued operator $\mathcal{A}$ it is a sequence $v : \mathbb{N}_0 \to \mathcal{V}$ such that for all $n \in \mathbb{N}_0$

$$\begin{bmatrix} v_{n+1} \\ v_n \end{bmatrix} \in \operatorname{gph}(\mathcal{A}). \tag{17}$$

- For a state/signal system V it is a pair of sequences $x : \mathbb{N}_0 \to \mathcal{X}$ and $w : \mathbb{N}_0 \to \mathcal{W}$ such that for all $n \in \mathbb{N}_0$

$$\begin{bmatrix} x_{n+1} \\ x_n \\ w_n \end{bmatrix} \in V.$$

- For a multi-valued operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ it is a triple of sequences $x : \mathbb{N}_0 \to \mathcal{X}$, $y : \mathbb{N}_0 \to \mathcal{Y}$ and $u : \mathbb{N}_0 \to \mathcal{U}$ such that for all $n \in \mathbb{N}_0$

$$\begin{bmatrix} x_{n+1} \\ y_n \\ x_n \\ u_n \end{bmatrix} \in \operatorname{gph}(S).$$

The state/signal definition of discrete-time trajectory is [1, Definition 7.4.1] and the input/state/output one is adapted from [1, Definition 6.5.2].

In the remainder of this section, we assume that that V is defined in terms of $\mathcal{A}$ as in (7) and that S is defined in terms of V as in (10).

Proposition B.2. *If* $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ *and* $\operatorname{mul}(\mathcal{A})$ *is closed, then we have the following. Infinite-time discrete-time trajectories of $\mathcal{A}$ and V coincide in the sense that if (x, w) is an infinite-time discrete-time trajectory of V, then v defined by $v_n := x_n + w_n$ is an infinite-time discrete-time trajectory of $\mathcal{A}$ and if v is an infinite-time discrete-time trajectory of $\mathcal{A}$, and we (uniquely) write $v_n = x_n + w_n$ with respect to the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{W}$, then (x, w) is an infinite-time discrete-time trajectory of V.*

Proof. Let (x, w) be an infinite-time discrete-time trajectory of V. Then for all $n \in \mathbb{N}_0$

$$\begin{bmatrix} x_{n+1} \\ x_n \\ w_n \end{bmatrix} \in V.$$

It follows that also $\left[\begin{smallmatrix} x_{n+2} \\ x_{n+1} \\ w_{n+1} \end{smallmatrix}\right] \in V$, so that $\left[\begin{smallmatrix} x_{n+1} \\ w_{n+1} \end{smallmatrix}\right] \in \mathcal{H}_0$. It follows from Lemma 5.20 that

$$\begin{bmatrix} x_{n+1} \\ w_{n+1} \\ x_n \\ w_n \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times),$$

which precisely means that

$$\begin{bmatrix} x_{n+1} + w_{n+1} \\ x_n + w_n \end{bmatrix} \in \operatorname{gph}(\mathcal{A}),$$

i.e. that v is an infinite-time discrete-time trajectory of $\mathcal{A}$.

Let v be an infinite-time discrete-time trajectory of $\mathcal{A}$. Then $\left[\begin{smallmatrix} v_{n+1} \\ v_n \end{smallmatrix}\right] \in \operatorname{gph}(\mathcal{A})$, i.e.

$$\begin{bmatrix} x_{n+1} + w_{n+1} \\ x_n + w_n \end{bmatrix} \in \operatorname{gph}(\mathcal{A}),$$

so that

$$\begin{bmatrix} x_{n+1} \\ w_{n+1} \\ x_n \\ w_n \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_\times),$$

which gives that

$$\begin{bmatrix} x_{n+1} \\ x_n \\ w_n \end{bmatrix} \in V.$$

Hence (x, w) is an infinite-time discrete-time trajectory of V. □

Proposition B.3. *Infinite-time discrete-time trajectories of V and S coincide in the sense that if (x, y, u) is an infinite-time discrete-time trajectory of S, then $(x, u + y)$ is an infinite-time discrete-time trajectory of V and if (x, w) is an infinite-time discrete-time trajectory of V and we (uniquely) write $w_n = u_n + y_n$ with respect to the decomposition $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$, then (x, y, u) is an infinite-time discrete-time trajectory of S.*

Proof. This is essentially [1, Lemma 7.4.3]. □

Corollary B.4. *If $\operatorname{im}(\mathcal{A}) \subset \overline{\operatorname{dom}(\mathcal{A})}$ and $\operatorname{mul}(\mathcal{A})$ is closed, then we have the following. Infinite-time discrete-time trajectories of $\mathcal{A}$ and S coincide in the sense that if (x, y, u) is an infinite-time discrete-time trajectory of S, then v defined by $v_n := x_n + u_n + y_n$ is an infinite-time discrete-time trajectory of $\mathcal{A}$ and if v is an infinite-time discrete-time trajectory of $\mathcal{A}$, and we (uniquely) write $v_n = x_n + u_n + y_n$ with respect to the decomposition $\mathcal{V} = \mathcal{X} \dot{+} \mathcal{U} \dot{+} \mathcal{Y}$, then (x, y, u) is an infinite-time discrete-time trajectory of S.*

Proof. This follows from combining Proposition B.2 and Proposition B.3. □

The following result considers the iteration from Section 5.1 in the context of discrete-time trajectories.

Proposition B.5. *Assume that* $\operatorname{dom}(\mathcal{A}_k)$ *is closed for all* $k \in \mathbb{N}_0$*. Then infinite-time discrete-time trajectories of* $\mathcal{A}$ *and* $\mathcal{A}_0$ *coincide.*

Proof. As $\operatorname{gph}(\mathcal{A}) \subset \operatorname{gph}(\mathcal{A}_0)$, we always have that an infinite-time discrete-time trajectory of $\mathcal{A}$ is an infinite-time discrete-time trajectory of $\mathcal{A}_0$.

From Proposition 5.11 we have that $\operatorname{dom}(\mathcal{A}_k) = \operatorname{dom}(\mathcal{A}_0^{k+1})$, so that by Proposition 5.4 we have

$$\operatorname{dom}(\mathcal{A}) = \bigcap_{k=0}^{\infty} \operatorname{dom}(\mathcal{A}_0^{k+1}), \tag{18}$$

and

$$\operatorname{gph}(\mathcal{A}) = \operatorname{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \operatorname{dom}(\mathcal{A}) \\ \operatorname{dom}(\mathcal{A}) \end{bmatrix}. \tag{19}$$

Let v be an infinite-time discrete-time trajectory of $\mathcal{A}_0$. Then as for all $k \in \mathbb{N}_0$

$$\begin{bmatrix} v_1 \\ v_0 \end{bmatrix}, \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}, \ldots, \begin{bmatrix} v_{k+1} \\ v_k \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0),$$

we have $\left[\begin{smallmatrix} v_{k+1} \\ v_0 \end{smallmatrix}\right] \in \operatorname{gph}(\mathcal{A}_0^{k+1})$, so that $v_0 \in \operatorname{dom}(\mathcal{A}_0^{k+1})$. From (18) we obtain $v_0 \in \operatorname{dom}(\mathcal{A})$. Similarly, we obtain that $v_n \in \operatorname{dom}(\mathcal{A})$ for all $n \in \mathbb{N}$. We then have for all $n \in \mathbb{N}_0$

$$\begin{bmatrix} v_{n+1} \\ v_n \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0) \cap \begin{bmatrix} \operatorname{dom}(\mathcal{A}) \\ \operatorname{dom}(\mathcal{A}) \end{bmatrix},$$

which by (19) gives that v is an infinite-time discrete-time trajectory of $\mathcal{A}$. □

Example B.6. We show that finite-time discrete-time trajectories (defined in the obvious way by requiring (17) to hold for $n \in \{0, \ldots, N-1\}$ for some $N \geq 1$ where N is called the length) are not necessarily preserved by the iteration from Section 5.1. Let

$$\operatorname{gph}(\mathcal{A}_0) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^2 \end{bmatrix} : q_1 = v_2, \quad v_1 = 0 \right\}.$$

Applying the iteration from Section 5.1 gives

$$\operatorname{gph}(\mathcal{A}) = \left\{ \begin{bmatrix} q \\ v \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^2 \end{bmatrix} : q_1 = q_2 = v_1 = v_2 = 0 \right\}.$$

We have that for any $a \in \mathbb{R}$ and $b \in \mathbb{R}$

$$\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix} \in \operatorname{gph}(\mathcal{A}_0),$$

so that the finite sequence $v : \{0\} \to \mathbb{R}^2$ defined by $v_0 := \left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right]$ is a discrete-time trajectory of length one of $\mathcal{A}_0$. However, for $a \neq 0$ this is not a discrete-time trajectory of length one of $\mathcal{A}$.

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