Decay of singular values for infinite-dimensional systems with Gevrey regularity

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The final publication is available via
https://doi.org/10.1016/j.sysconle.2020.104644

Abstract

We consider the decay rate of the singular values of the input map, the output map and the Hankel operator for a class of infinite-dimensional systems. This class is characterized by the control operator (or the observation operator) having a smoothing effect. We capture this in the definition of a Gevrey operator (which generalizes the known concept of a Gevrey vector). In applications to PDEs, this abstract assumption on the control operator is typically satisfied when the input is multiplied by a function which is a compactly supported Gevrey function in the spatial variable. Using the theory of polynomial approximation (in particular: truncated Chebyshev expansions), we obtain that the singular values decay exponentially in a root of the approximation dimension. The power of the root depends on the order of the Gevrey operator and on whether the underlying semigroup is nilpotent, exponentially stable or polynomially stable.

1 Introduction

Ruth Curtain with Amol Sasane in [1, Theorem 4] showed that if A generates an exponentially stable strongly continuous semigroup on a Hilbert space $\mathcal{X}$ and $B \in \mathcal{L}(\mathbb{C}^m, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathbb{C}^p)$ then the controllability Gramian, the observability Gramian and the Hankel operator are nuclear (i.e. have summable singular values). The singular values of the Hankel operator appear in the error-bound for balanced truncation (established in the infinite-dimensioncal case by Ruth Curtain with Keith Glover and Jonathan Partington in [2]) and decay of singular values of the Gramians is important for numerical approximation of them. Decay of these singular values (for finite-dimensional systems) has therefore naturally received attention from the numerical linear algebra community as well [3, 4, 5, 6, 7].

In [8] we refined another result from [1], namely [1, Theorem 6] (where B and C are allowed to be unbounded, but A is assumed to generate an analytic semigroup); see [9] and [10] for related results. There we showed that the singular values of the Gramians and the Hankel operator decay exponentially in the square root which is much stronger than them just being summable. In an earlier article [11] we had already showed (through a very different approach) that the Hankel singular values in that case were in ℓ^p for all $p > 0$ (the case where $p < 1$ being the improvement on the Curtain–Sasane result)[1].

In this article we consider a refinement of the result from [1, Theorem 4] mentioned above. We make (for the controllability Gramian case) the stronger assumption that B maps into the domain of A^n for all $n \in \mathbb{N}$ and that furthermore certain estimates on $A^n B$ hold. These conditions are typically satisfied in PDE examples with interior control described by a smooth "shaping" function. We allow A to only generate a polynomially stable semigroup rather than an exponentially stable semigroup. In the exponentially stable case we obtain that the singular values of the controllability Gramian decay exponentially in $n^{1/(1+\delta)}$ where δ relates to the estimates on $A^n B$ (the typical case is $\delta = 2$ so that we obtain exponential decay in the cube root). With a similar "smoothness" condition on C, we obtain the analogues result for the observability Gramian. To obtain the result for the Hankel operator a "smoothness" assumption on either B or C suffices.

In Section 2 we elaborate on the above paragraph (but from the "observation" rather than the "control" point of view) as a guide to the technical content in the remainder of the article.

2 Discussion of the main result

We consider the observed system

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \qquad y(t) = Cx(t). \tag{1}$$

We assume that A generates the strongly continuous semigroup $T : [0, \infty) \to \mathcal{L}(\mathcal{X})$ on the Hilbert space $\mathcal{X}$ and that $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ for a finite-dimensional Hilbert space $\mathcal{Y}$. The general theory of such system is considered in e.g. [12]. The objective is to study the operator $\mathfrak{C} : \mathcal{X} \to L^2(0, \infty; \mathcal{Y})$ which maps the initial condition x_0 to the output y. Numerical approximation of (1) typically results in an approximation $\mathfrak{C}_n : \mathbb{R}^n \to L^2(0, \infty; \mathcal{Y})$ of $\mathfrak{C}$. Such an approximation $\mathfrak{C}_n$ has rank at most n. Therefore, how well we can approximate $\mathfrak{C}$ by such approximations is determined by the singular values of $\mathfrak{C}$. In this article we therefore are interested in those singular values $\sigma_n(\mathfrak{C})$, and in particular in their asymptotic behavior as $n \to \infty$. Since the observability Gramian of (1) is $\mathfrak{C}^*\mathfrak{C}$, results about the singular values of the observability Gramian can be deduced from this. By duality, results about the input map $\mathfrak{B}$ and the controllability

[1]Note that the proof regarding singular values of the Gramians in [11] is incorrect; [8] however gives a stronger result than that claimed with an erroneous proof in [11].

Gramian $\mathfrak{B}\mathfrak{B}^*$ can be obtained. Furthermore since the Hankel operator equals $\mathfrak{C}\mathfrak{B}$, results about the Hankel singular values follow.

In this article, we will make an additional assumption on C, namely that C^* is a Gevrey operator of order δ for A^* (for some given $\delta > 1$); see Definition 12. For the present discussion it suffices to keep in mind the following typical example of a transport equation (see Section 8 for more examples):

$$\begin{aligned} w_t(t,\xi) &= w_\xi(t,\xi), \quad t>0,\ \xi \in (-1,1), \\ w(0,\xi) &= w_0(\xi), \qquad \xi \in (-1,1), \\ w(t,1) &= 0, \qquad t>0, \\ y(t) &= \int_{-1}^{1} c(\xi) w(\xi)\, d\xi, \end{aligned}$$

where c is the standard bump function

$$c(\xi) = \exp\left(\frac{-1}{1-\xi^2}\right), \tag{2}$$

(see Figure 1), which satisfies this condition with $\delta = 2$. It then follows that for each initial condition x_0 the output y is a Gevrey function of order δ (see Corollary 17). The space of Gevrey functions sits in between the space of analytic functions and the space of infinitely differentiable functions; see Definition 1. A typical example of a Gevrey function (of order 2) is the bump function (2). In fact something stronger is true: not only is $t \mapsto CT(t)x_0$ a $\mathcal{Y}$-valued Gevrey function for each $x_0 \in \mathcal{X}$; but $t \mapsto CT(t)$ is a $\mathcal{L}(\mathcal{X},\mathcal{Y})$-valued Gevrey function. This stronger fact ensures that certain constants are independent of the initial condition x_0 which is crucial for the application to singular value analysis (since that deals with operator norms).

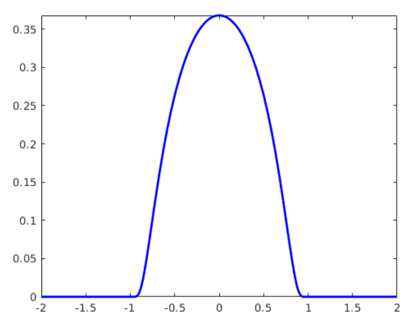


Figure 1: The function from (2)

A basic tenet of approximation theory is that since y is a smooth function, it can be well-approximated over a finite interval by a low degree polynomial. Instead of considering the best polynomial approximation we consider a Chebyshev projection; this is because the best polynomial approximation is non-linear whereas the Chebyshev projection is linear (for our application to singular value analysis this is important since singular values deal with *linear* approximations). Lemma 7 considers the error made when approximating a Gevrey function by its Chebyshev projection over a finite interval. In Lemmas 10 and 11 we consider *piecewise* polynomial approximation when the interval is unbounded. This requires an additional assumption on the decay of the to-be-approximated function at infinity; in the application to the observed system (1), these decay conditions will be satisfied by assuming that the semigroup is exponentially stable (for Lemma 10) or polynomially stable (for Lemma 11). To make this connection, in Section 6 we briefly study polynomially stable semigroups.

We end up with the following bound on the singular values (see Theorem 22): there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(\mathfrak{C}) \leq K \exp(-rn^{1/\varepsilon}),$$

where

$$\varepsilon := \begin{cases} \delta & T \text{ is nilpotent} \\ \delta + 1 & T \text{ is exponentially stable} \\ \delta\left(1 + \frac{1}{\beta}\right) + 1 & T \text{ is polynomially stable with rate } \beta. \end{cases}$$

As mentioned, in Section 8 we consider several examples to illustrate the theory. In Section 9 we illustrate the relevance of the assumptions by considering some examples which do not satisfy our assumptions and where the singular values do not decay or decay slowly.

We have subdivided the article so that Section 4 uses ideas from approximation theory whereas Sections 5 and 6 uses ideas from the theory of strongly continuous semigroups.

We note that a crucial distinction between this article and earlier work (for example [11] and [8]) is that the semigroup is not assumed to be analytic. This allows us to deal with hyperbolic partial differential equations (rather than just parabolic partial differential equations or hyperbolic partial differential equations which are so strongly damped that they essentially behave like parabolic partial differential equations). In our earlier work [11] and [8], "smoothness" came from an assumption on the semigroup whereas in this article "smoothness" comes from an assumption on the observation operator C. In the numerical linear algebra literature on decay of singular values of Gramians, the hyperbolic–parabolic distinction is of course less clearly present; however, there are never restrictive assumptions on C but only spectral type assumptions on A; this effectively restricts applicability of those results to (discretizations of) parabolic-like partial differential equations.

We finally note the similarity between the approximation of $\mathfrak{C}$ used here, $\Pi_n\mathfrak{C}$, where Π_n is a projection onto a space of piecewise polynomials and the approximation computed by ADI which (in the interpretation from [13]) is given by a formula of the same form where instead Π_n is the orthogonal projection onto a space spanned by exponential functions.

3 Gevrey functions

The following recalls the notion of a Gevrey function which is the appropriate concept of smoothness for our purposes. We will use this notion both "in space" (for "shaping functions" such as (2)) and "in time" (for output functions such as $t \mapsto CT(t)x_0$). For functions "in time" we want to consider $\mathcal{L}(\mathcal{X}, \mathcal{Y})$-valued functions, which is why in the definition below we allow for functions with values in a Banach space.

Definition 1. Let $\mathcal{B}$ be a Banach space, let $\delta \geq 0$ and let $A_0, A_1 > 0$. By $G^{\delta}_{A_0,A_1}(a, b; \mathcal{B})$ we denote the set of functions $f : (a, b) \to \mathcal{B}$ which are infinitely differentiable on (a, b) and are such that for all $k \in \mathbb{N}_0$ and all $t \in (a, b)$

$$\|f^{(k)}(t)\|_{\mathcal{B}} \leq A_0 A_1^k (k!)^{\delta}.$$

We further define

$$G^{\delta}(a, b; \mathcal{B}) := \bigcup_{A_0,A_1>0} G^{\delta}_{A_0,A_1}(a, b; \mathcal{B}).$$

When $\mathcal{B} = \mathbb{R}$, then we simplify the notation to $G^{\delta}_{A_0,A_1}(a, b)$ and $G^{\delta}(a, b)$, respectively.

The following is the typical example that we have in mind for our "shaping" function (see also Figure 2).

Example 2. Let $a, \xi_1, \xi_2, b \in \mathbb{R}$ be such that $a < \xi_1 < \xi_2 < b$. Let $\gamma > 0$ and define $\delta := 1 + \frac{1}{\gamma}$. The function $f : (a, b) \to \mathbb{R}$ defined by

$$f(\xi) = \begin{cases} \exp\left(\dfrac{-1}{[(\xi - \xi_1)(\xi_2 - \xi)]^{\gamma}}\right) & \xi \in (\xi_1, \xi_2) \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $G^{\delta}(a, b)$.

The following defines exponentially decaying Gevrey functions which is important for the case of an exponentially stable semigroup.

Definition 3. Let $\mathcal{B}$ be a Banach space, let $\delta \geq 0$ and let $A_0, A_1, M, \omega > 0$. We denote by $G^{\delta}_{A_0,A_1,M,\omega}(0, \infty; \mathcal{B})$ the set of functions which satisfy $f \in G^{\delta}_{A_0,A_1}(0, \infty; \mathcal{B})$ and additionally for all $t \in (0, \infty)$

$$\|f(t)\|_{\mathcal{B}} \leq M e^{-\omega t}.$$

The following defines a class of Gevrey functions which in some sense decay polynomially. This is important for the case of a polynomially stable semigroup.

Definition 4. Let $\mathcal{B}$ be a Banach space, let $\delta \geq 0$ and let $C_0, C_1, \alpha > 0$. We denote by $G^{\delta}_{C_0,C_1,\alpha}(0, \infty; \mathcal{B})$ the set of functions which satisfy $f \in C^{\infty}(0, \infty; \mathcal{B})$ and additionally for all $k \in \mathbb{N}_0$ and all $t > 0$

$$\|f^{(k)}(t)\| \leq C_0 C_1^k (k!)^{\delta} \max\{t^{-\alpha-k}, 1\}.$$

Remark 5. We note that $G^{\delta}_{C_0,C_1,\alpha}(0, \infty; \mathcal{B}) \subset G^{\delta}_{C_0,C_1}(0, \infty; \mathcal{B})$.

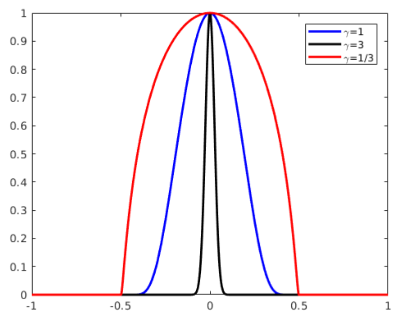


Figure 2: Functions from Example 2 with $\xi_1 = -1/2$, $\xi_2 = 1/2$ and various values of γ and with a multiplicative normalization so that $f(0) = 1$.

4 Approximation of Gevrey functions

We denote the orthogonal projection in $L^2(-1, 1; \frac{dx}{\sqrt{1-x^2}})$ onto the space of polynomials of degree less than n by Π_n. We note that $\Pi_n f$ then coincides with the truncated Chebyshev expansion of f. Since $L^\infty(-1, 1) \subset L^2(-1, 1; \frac{dx}{\sqrt{1-x^2}})$, the expression $\Pi_n f$ for $f \in L^\infty(-1, 1)$ is well-defined. The canonical affine change of variables between functions on (a, b) and functions on $(-1, 1)$ can be used to translate the above to a projection on $L^\infty(a, b)$ and we will use the notation $\Pi_n^{[a,b]}$ for that projection.

Figure 3 shows the standard bump function and a few of its approximations computed by Chebfun [14].

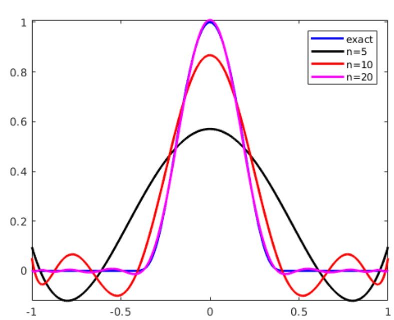


Figure 3: Function from Example 2 with $\xi_1 = -1/2$, $\xi_2 = 1/2$, $\gamma = 1$ (so that $\delta = 2$) and with a multiplicative normalization so that $f(0) = 1$ and its Chebyshev truncations for various degrees n.

The following is a result on approximation of C^k functions rather than Gevrey functions which is crucial in the proof of the Gevrey function case.

Lemma 6. *Let $n \in \mathbb{N}$ and $k \in \{2, \dots, n-1\}$. If $f \in C^k([a,b])$, then*

$$\|f - \Pi_n^{[a,b]} f\|_\infty \le 2 \left(\frac{b-a}{2}\right)^{k+1} \frac{1}{(k-1)(n-k)^{k-1}} \|f^{(k)}\|_\infty.$$

Proof. When $a = -1$ and $b = 1$, this follows from [15, Theorem 4.3] by noticing that for g a continuous function $\|x \mapsto \frac{g(x)}{\sqrt{1-x^2}}\|_1 \le \pi \|g\|_\infty$. The case of a general interval $[a,b]$ follows by using the standard affine change of variables. □

The following is the basic result on approximation of Gevrey functions over a finite interval.

Lemma 7. *Let $\delta > 1$, let $\rho > 0$ and define $r_* := \delta \rho^{-1/\delta} \mathrm{e}^{1/2}$. Then for all $r \in (0, r_*)$ there exists a $C > 0$ such that for all $A_0, A_1 > 0$ and $a, b \in \mathbb{R}$ with $a < b$ and with $\rho = \frac{A_1(b-a)}{2}$ and all $f \in G^\delta_{A_0,A_1}(a,b)$ and all $n \in \mathbb{N}$ there holds*

$$\|f - \Pi_n^{[a,b]} f\|_\infty \le A_0 (b-a) C \mathrm{e}^{-rn^{1/\delta}}.$$

Proof. The proof is based on ideas from [16, Theorem 2.3]. Let $k \in \{2, \dots, n-1\}$. From Lemma 6 and the Stirling estimate $k! \le 1.1 (2\pi k)^{1/2} k^k \mathrm{e}^{-k}$ we obtain

$$\begin{aligned}
\|f - \Pi_n^{[a,b]} f\|_\infty &\le 2 \left(\frac{b-a}{2}\right)^{k+1} \frac{1}{(k-1)(n-k)^{k-1}} A_0 A_1^k (k!)^\delta \\
&\le \left[A_0 (b-a) 1.1^\delta (2\pi)^{\delta/2}\right] \left[A_1 \frac{b-a}{2} \mathrm{e}^{-\delta}\right]^k \frac{k^{\delta/2}(n-k)}{k-1} \frac{k^{\delta k}}{(n-k)^k}.
\end{aligned}$$

Defining

$$c_\delta := 1.1^\delta (2\pi)^{\delta/2}, \quad \rho_\delta := A_1 \frac{b-a}{2} \mathrm{e}^{-\delta}, \quad \phi(k) := \left(\frac{\rho_\delta k^\delta}{n-k}\right)^k,$$

this is exactly

$$\|f - \Pi_n^{[a,b]} f\|_\infty \le A_0 (b-a) c_\delta \frac{k^{\delta/2}(n-k)}{k-1} \phi(k). \tag{3}$$

Define

$$\kappa := \mathrm{e}^{-1/2} \left(\frac{n}{\rho_\delta}\right)^{1/\delta}, \qquad k = \lfloor \kappa \rfloor. \tag{4}$$

Note that there exists a $N_{\delta,\rho} \in \mathbb{N}$ depending only on δ and ρ such that for $n \ge N_{\delta,\rho}$ we have

$$\mathrm{e}^{-\delta/2} n \le n - \kappa \qquad \text{and} \qquad \kappa \ge 2.$$

For $n \ge N_{\delta,\rho}$ we then have

$$\phi(k) \le \left(\frac{\rho_\delta k^\delta}{\mathrm{e}^{-\delta/2} n}\right)^k \le \left(\frac{\rho_\delta \kappa^\delta}{\mathrm{e}^{-\delta/2} n}\right)^\kappa = \mathrm{e}^{-\delta\kappa}.$$

By choosing a $c_{\delta,\rho}$ large enough we have for $n < N_{\delta,\rho}$ that $\phi(k) \le c_{\delta,\rho}\mathrm{e}^{-\delta\kappa}$. Therefore we obtain that for all $n \in \mathbb{N}$ the choice (4) gives

$$\phi(k) \le c_{\delta,\rho}\mathrm{e}^{-\delta\kappa}.$$

Substituting this in (3) gives

$$\|f - \Pi_n^{[a,b]} f\|_\infty \le A_0(b-a)c_\delta \frac{k^{\delta/2}(n-k)}{k-1} c_{\delta,\rho}\mathrm{e}^{-\delta\kappa},$$

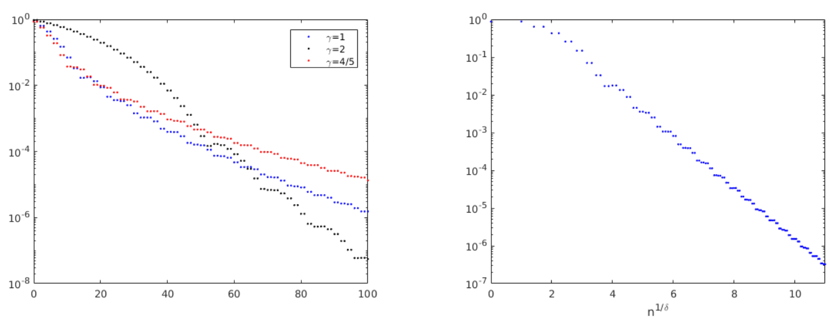
which can be re-written as

$$\|f - \Pi_n^{[a,b]} f\|_\infty \le A_0(b-a)c_\delta \frac{k^{\delta/2}(n-k)}{k-1} c_{\delta,\rho}\mathrm{e}^{-r_* n^{1/\delta}}.$$

Since $\frac{k^{\delta/2}(n-k)}{k-1}$ grows at most polynomially in n it can be absorbed by choosing $r < r_*$ and the result follows. $\square$

Remark 8. We emphasize that the constant C in Lemma 7 does not depend on the particular function f considered, but only on the parameters in the estimate from Definition 1. In fact: C is independent of A_0 and only depends on the length $b-a$ of the interval indirectly through its dependence on ρ.

Figure 4 illustrates the error bound from Lemma 7 when applied to bump functions for various values of γ (and therefore of δ).



(a) Various values of γ (and therefore of δ).

(b) Case of $\gamma = 1$ (and therefore $\delta = 2$) on a scale which makes the upper-bound a straight line.

Figure 4: Error in Chebyshev truncations for various degrees n for function from Example 2 with $\xi_1 = -1/2$, $\xi_2 = 1/2$ and with a multiplicative normalization so that $f(0) = 1$.

Corollary 9. *Under the same assumptions and with the same notation as in Lemma 7 we have for all* $p \in [1, \infty)$

$$\|f - \Pi_n^{[a,b]} f\|_p \le A_0(b-a)^{1+\frac{1}{p}} C\mathrm{e}^{-rn^{1/\delta}}.$$

Proof. This follows from Lemma 7 by utilizing the trivial estimate $\|g\|_{L^p(a,b)} \leq \|g\|_{L^\infty(a,b)}(b-a)^{1/p}$. $\square$

The following result considers approximation of exponentially decaying Gevrey functions. The approximation method using piecewise polynomials is illustrated in Figure 5.

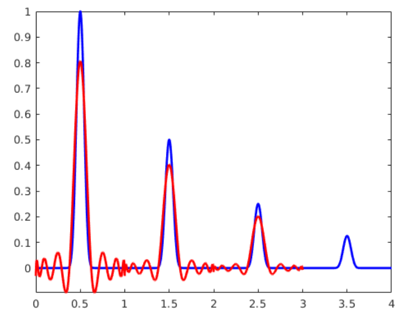


Figure 5: Approximation of an exponentially decaying Gevrey function using $m = 3$ intervals of equal length and $n = 15$ coefficients on each interval.

Lemma 10. *Let $p \in [1, \infty)$, $\delta > 1$, $\omega, A_1 > 0$ and define $\rho := \frac{A_1}{2}$ and $r_* := \delta\rho^{-1/\delta}\mathrm{e}^{1/2}$. Then for all $r \in (0, r_*)$ there exists a $C > 0$ such that for all $A_0 > 0$ and $M > 0$ and all $f \in G^\delta_{A_0,A_1,M,\omega}(0,\infty)$ and all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ there holds*

$$\left\| f - \sum_{j=1}^{m} \Pi_n^{[j-1,j]} f \right\|_p \leq A_0 C m \mathrm{e}^{-rn^{1/\delta}} + \frac{M}{(\omega p)^{1/p}} \mathrm{e}^{-\omega m}. \tag{5}$$

Define $q_* := \min\{r_*, \omega\}$. Then for all $q \in (0, q_*)$ there exists a $D > 0$ such that for all $A_0 > 0$ and $M > 0$ and all $f \in G^\delta_{A_0,A_1,M,\omega}(0,\infty)$ and all $n \in \mathbb{N}$ there holds, with $m = \lceil n^{1/\delta} \rceil$ and $N := nm$,

$$\left\| f - \sum_{j=1}^{m} \Pi_n^{[j-1,j]} f \right\|_p \leq \max\{A_0, M\} D \mathrm{e}^{-qN^{1/(\delta+1)}}.$$

Proof. Applying Corollary 9 to the restriction of f to $[j-1, j]$ gives

$$\|f|_{[j-1,j]} - \Pi_n^{[j-1,j]} f\|_p \leq A_0 C \mathrm{e}^{-rn^{1/\delta}}.$$

Combining the inequalities for $j \in \{1, \ldots, m\}$ then gives

$$\left\| f_{[0,m]} - \sum_{j=1}^{m} \Pi_n^{[j-1,j]} f \right\|_p \leq \sum_{j=1}^{m} \|f|_{[j-1,j]} - \Pi_n^{[j-1,j]} f\|_p \leq A_0 C m \mathrm{e}^{-rn^{1/\delta}}.$$

We further have

$$\|f_{[m,\infty)}\|_p^p = \int_m^\infty |f(t)|^p \, dt \leq \int_m^\infty M^p \mathrm{e}^{-\omega p t} \, dt = \frac{M^p}{\omega p} \mathrm{e}^{-\omega p m}.$$

It follows that

$$\|f_{[m,\infty)}\|_p \leq \frac{M}{(\omega p)^{1/p}} \mathrm{e}^{-\omega m}.$$

Combining the estimates obtained on $[0,m]$ and $[m,\infty)$ gives the desired result.

Let $q \in (0, q_*)$ and choose $r \in (q, r_*)$ in the argument above. With the indicated choice of m we have

$$\frac{M}{(\omega p)^{1/p}} \mathrm{e}^{-\omega m} \leq \frac{M}{(\omega p)^{1/p}} \mathrm{e}^{-\omega n^{1/\delta}},$$

so that an upper-bound on the right-hand side of (5) is

$$\left(A_0 C m + \frac{M}{(\omega p)^{1/p}} \right) \mathrm{e}^{-\min\{r,\omega\} n^{1/\delta}}.$$

Let $\tilde{q} \in (q, \min\{r,\omega\})$. Since m grows sub-linearly in n, we can absorb it in the exponential using that $\tilde{q} < \min\{r,\omega\}$ to obtain the upper-bound

$$\max\{A_0, M\} D_0 \mathrm{e}^{-\tilde{q} n^{1/\delta}},$$

for a constant $D_0 > 0$ depending only on p, q, ρ, δ and ω. Since we have $N^{1/(\delta+1)} \leq n^{1/\delta} + n^{1/(\delta+1)} \leq \frac{\tilde{q}}{q} n^{1/\delta}$ for $n \geq C_{\delta,q}$ for some $C_{\delta,q}$ depending only on δ and q, we have for those n

$$\mathrm{e}^{-\tilde{q} n^{1/\delta}} \leq \mathrm{e}^{-q N^{1/(\delta+1)}}.$$

To deal with $n < C_{\delta,q}$, we increase the constant D_0 to D. This gives the desired result. □

The following result considers approximation of "polynomially decaying" Gevrey functions.

Lemma 11. *Let $p \in [1,\infty)$, $\delta > 1$, $\alpha > 1 + \frac{1}{p}$, $C_1 > 0$, $\theta > 1$ and define $r_* = \delta \left(\frac{C_1(\theta-1)}{2} \right)^{-1/\delta} \mathrm{e}^{1/2}$. Then for all $r \in (0, r_*)$ there exists a $C > 0$ such that for all $C_0 > 0$ and all $f \in G^\delta_{C_0,C_1,\alpha}(0,\infty)$ and all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ there holds*

$$\left\| f - \sum_{j=1}^m \Pi_n^{[\theta^{j-1},\theta^j]} f - \Pi_n^{[0,1]} f \right\|_p \leq C_0 \left(\frac{\theta - 1}{1 - \theta^{1-\alpha+\frac{1}{p}}} + 1 \right) C \mathrm{e}^{-r n^{1/\delta}} + C_0 \frac{\theta^{m(1-\alpha p)/p}}{(1-\alpha p)^{1/p}}. \tag{6}$$

Define $q_* := \min\{r_*, \frac{\alpha p-1}{p}\ln(\theta)\}$. *Then for all* $q \in (0, q_*)$ *there exists a* $D > 0$ *such that for all* $C_0 > 0$ *and all* $f \in G^{\delta}_{C_0,C_1,\alpha}(0,\infty)$ *and all* $n \in \mathbb{N}$ *there holds, with* $m = \lceil n^{1/\delta} \rceil$ *and* $N := n(m+1)$,

$$\left\| f - \sum_{j=1}^{m} \Pi_n^{[\theta^{j-1},\theta^j]} f - \Pi_n^{[0,1]} f \right\|_p \le C_0 D \mathrm{e}^{-qN^{1/(\delta+1)}}.$$

Proof. By Corollary 9 applied to the interval (θ^{j-1}, θ^j) and with $A_0 = C_0\theta^{-\alpha(j-1)}$, $A_1 = C_1\theta^{-(j-1)}$ we obtain the existence of $r > 0$ and $C > 0$ depending only on r, C_1, θ and δ (we note that $\rho = \frac{C_1(\theta-1)}{2}$) such that

$$\|f|_{[\theta^{j-1},\theta^j]} - \Pi_n^{[\theta^{j-1},\theta^j]} f\|_p \le C_0\theta^{(1-\alpha+\frac{1}{p})(j-1)}(\theta-1)^{1+\frac{1}{p}} C\mathrm{e}^{-rn^{1/\delta}}.$$

By Corollary 9 applied to the interval $(0, 1)$ with $A_0 = C_0$ and $A_1 = C_1$ we have that there exists a $\tilde{C}$ depending only on r, C_1 and δ such that

$$\|f|_{[0,1]} - \Pi_n^{[0,1]}\|_p \le C_0\tilde{C}\mathrm{e}^{-rn^{1/\delta}}.$$

We further have

$$\|f|_{[\theta^m,\infty)}\|_p^p = \int_{\theta^m}^{\infty} |f(t)|^p \, dt \le \int_{\theta^m}^{\infty} C_0^p t^{-\alpha p} \, dt = C_0^p \frac{\theta^{m(1-\alpha p)}}{1-\alpha p}.$$

Combining these three inequalities and using that

$$\sum_{j=1}^{m} \theta^{(1-\alpha+\frac{1}{p})(j-1)} \le \frac{\theta-1}{1-\theta^{1-\alpha+\frac{1}{p}}},$$

we obtain (6).

Let $q \in (0, q_*)$ and apply the above with $r := q$. With the indicated choice of m we have that

$$\theta^{m(1-\alpha p)/p} \le \mathrm{e}^{\frac{1-\alpha p}{p}\ln(\theta)n^{1/\delta}},$$

so that from (6) we obtain the upper-bound

$$C_0 \max\left\{ \frac{\theta-1}{1-\theta^{1-\alpha+\frac{1}{p}}} C, \frac{1}{(1-\alpha p)^{1/p}} \right\} \mathrm{e}^{-qn^{1/\delta}}.$$

Utilizing the definition of N gives the desired result. □

5 Gevrey operators

For an operator A on a Banach space $\mathcal{X}$ define

$$D(A^\infty) := \bigcap_{n=0}^{\infty} D(A^n).$$

Definition 12. Let A be a closed densely defined operator with non-empty resolvent set on a Banach space $\mathcal{X}$ and let $\delta \geq 0$. Let $\mathcal{U}$ be a Banach space and let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$. Then B is called a *Gevrey operator* for A of order δ, denoted $B \in G^\delta(A)$, if $B\mathcal{U} \subset D(A^\infty)$ and there exist $C_0, C_1 > 0$ such that for all $n \in \mathbb{N}_0$ there holds

$$\|A^n B\|_{\mathcal{L}(\mathcal{U},\mathcal{X})} \leq C_0 C_1^n (n!)^\delta.$$

Remark 13. If A is a bounded operator, then any bounded operator B is a Gevrey operator for B with $\delta = 0$ (take $C_0 = \|B\|_{\mathcal{L}(\mathcal{U},\mathcal{X})}$ and $C_1 = \|A\|_{\mathcal{L}(\mathcal{X})}$). However, the interesting case in applications to partial differential equations is where A is unbounded.

The following result is used in the proof of Proposition 15 and is crucial in ensuring that, with reference to the discussion in Section 2, not only is $t \mapsto CT(t)x_0$ a $\mathcal{Y}$-valued Gevrey function for all initial conditions $x_0 \in \mathcal{X}$, but in fact $t \mapsto CT(t)$ is a $\mathcal{L}(\mathcal{X}, \mathcal{Y})$-valued Gevrey function.

Lemma 14. *Let A be the generator of the strongly continuous semigroup T on the Banach space $\mathcal{X}$. Let $\mathcal{U}$ be a Banach space and let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$. Let $n \in \mathbb{N}_0$ and assume that $B\mathcal{U} \subset D(A^n)$. Then the function $f : (0, \infty) \to \mathcal{L}(\mathcal{U}, \mathcal{X})$ defined by $f(t) := T(t)B$ is $n-1$ times continously differentiable in the uniform operator topology.*

Proof. The method of proof follows [17, Lemma 2.4.2, Corollary 2.4.3].

Let $k \in \{0, \dots, n\}$ and $t \geq 0$. Since A^k is closed and $T(t)B$ is bounded, $A^k T(t)B$ is closed and since it is everywhere defined it follows from the Closed Graph Theorem that it is bounded.

Let $u \in \mathcal{U}$ and define $x_0 = Bu$. Then $x_0 \in D(A^n)$ and it follows by standard semigroup theory that $g : (0, \infty) \to \mathcal{X}$ defined by $g(t) = f(t)u$, i.e. $g(t) = T(t)x_0$, is n times continously differentiable with $g^{(k)}(t) = A^k T(t)x_0$, i.e. $g^{(k)}(t) = A^k f(t)u$, for $k \in \{0, \dots, n\}$. This precisely means that f is n times continously differentiable in the *strong* operator topology.

We now proceed to show that $t \mapsto A^k f(t)$ is continuous in the uniform operator topology for $k \in \{0, \dots, n-1\}$. Let t_1 and t_2 be such that $0 \leq t_1 \leq t_2 \leq t_1 + 1$ and let $M > 0$ be such that $\|T(t)\| \leq M$ for all $t \in [0, t_1 + 1]$. Then

$$A^k f(t_2)u - A^k f(t_1)u = g^{(k)}(t_2) - g^{(k)}(t_1) = \int_{t_1}^{t_2} g^{(k+1)}(t)\,dt = \int_{t_1}^{t_2} A^{k+1} T(t)Bu\,dt. \tag{7}$$

From this we obtain

$$\|A^k f(t_2)u - A^k f(t_1)u\| \leq |t_2 - t_1| M \|A^{k+1}B\| \|u\|,$$

which show that $t \mapsto A^k f(t)$ is indeed continuous in the uniform operator topology.

From (7) we have, with $f^{[k]}$ the k-th derivative of f in the strong operator topology, for $k \in \{0, \dots, n-2\}$, $t \geq 0$ and $h > 0$

$$f^{[k]}(t+h) - f^{[k]}(t) = \int_t^{t+h} A^{k+1} f(s)\,ds,$$

so that

$$\frac{f^{[k]}(t+h) - f^{[k]}(t)}{h} - A^{k+1}f(t) = \int_t^{t+h} A^{k+1}f(s) - A^{k+1}f(t)\,ds,$$

from which we obtain

$$\left\| \frac{f^{[k]}(t+h) - f^{[k]}(t)}{h} - A^{k+1}f(t) \right\| \leq h \sup_{s\in[t,t+h]} \|A^{k+1}f(s) - A^{k+1}f(t)\|,$$

which by continuity of $A^{k+1}f$ in the uniform operator topology implies that $f^{[k]}$ is differentiable at t in the uniform operator topology with derivative $A^{k+1}f(t)$. We conclude that f is $n-1$ times continuously differentiable in the uniform operator topology. □

The following result makes the connection between B being a Gevrey operator and $t \mapsto T(t)B$ being a Gevrey function.

Proposition 15. *Let $\delta \geq 0$ and let A be the generator of the strongly continuous semigroup T on the Banach space $\mathcal{X}$. Let $\mathcal{U}$ be a Banach space and let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$. If $B \in G^\delta(A)$, then for all $\tau > 0$ the function defined by $t \mapsto T(t)B$ is in $G^\delta(0, \tau; \mathcal{L}(\mathcal{U}, \mathcal{X}))$.*

If T is uniformly bounded, then the function is in $G^\delta(0, \infty; \mathcal{L}(\mathcal{U}, \mathcal{X}))$.

Proof. From Lemma 14 we know that $t \mapsto T(t)B$ is infinitely differentiable in the uniform operator topology.

Let C_0 and C_1 be as in Definition 12. By the theory of strongly continuous semigroups, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$. Define $A_1 := C_1$ and $A_0 = C_0 M \max\{1, \mathrm{e}^{\omega\tau}\}$.

Let $n \in \mathbb{N}$ and let $u \in \mathcal{U}$. Define $x(t) := T(t)B$. Since $x^{(n)}(t) = A^n T(t)B = T(t)A^n B$, we have that

$$\|x^{(n)}(t)\| \leq M\mathrm{e}^{\omega t}\|A^n B\|.$$

From the estimate in Definition 12 we then obtain for $t \in [0, \tau]$

$$\|x^{(n)}(t)\| \leq M \max\{1, \mathrm{e}^{\omega\tau}\} C_0 C_1^n (n!)^\delta = A_0 A_1^n (n!)^\delta.$$

Since $n \in \mathbb{N}_0$ was arbitrary, this shows that $t \mapsto T(t)B$ belongs to $G^\delta(0, \tau; \mathcal{L}(\mathcal{U}, \mathcal{X}))$.

If T is uniformly bounded then we may choose $\omega = 0$ in the above and we see (since A_0 and A_1 then do not depend on τ) that $t \mapsto T(t)B$ belongs to $G^\delta(0, \infty; \mathcal{L}(\mathcal{U}, \mathcal{X}))$. □

The following result uses the sun-dual semigroup (see e.g. [18, Section 3.5] and [19]). When $\mathcal{X}$ is reflexive (e.g. when it is a Hilbert space), then the sun-dual semigroup is simply equal to the dual semigroup (the reason for considering the sun-dual semigroup is that the dual semigroup need not be strongly continuous if $\mathcal{X}$ is non-reflexive). We define $B^\odot$ as the restriction of B^* to $\mathcal{X}^\odot$.

Proposition 16. *Let $\delta \geq 0$ and let A be the generator of the strongly continuous semigroup T on the Banach space $\mathcal{X}$ and let $\mathcal{U}$ be a Hilbert space. If B is a Gevrey operator of order δ for A, then for all $\tau > 0$ the function defined by $t \mapsto B^{\odot} T^{\odot}(t)$ is in $G^{\delta}(0, \tau; \mathcal{L}(\mathcal{X}^{\odot}, \mathcal{U}))$.*

If T is uniformly bounded, then the function is in $G^{\delta}(0, \infty; \mathcal{L}(\mathcal{X}^{\odot}, \mathcal{U}))$.

Proof. From Proposition 15 that there exist A_1 and A_2 such that

$$\|T(t)A^n B\|_{\mathcal{L}(\mathcal{X},\mathcal{U})} \leq A_0 A_1^n (n!)^{\delta}.$$

Using that with $x(t) := B^{\odot} T^{\odot}(t)$

$$\|x^{(m)}(t)\|_{\mathcal{L}(\mathcal{X}^{\odot},\mathcal{U})} = \|B^{\odot} A^{\odot m} T^{\odot}(t)\|_{\mathcal{L}(\mathcal{X}^{\odot},\mathcal{U})} = \|T(t)A^m B\|_{\mathcal{L}(\mathcal{U},\mathcal{X})},$$

we then have the desired

$$\|x^{(m)}(t)\|_{\mathcal{L}(\mathcal{X}^{\odot},\mathcal{U})} \leq A_0 A_1^n (n!)^{\delta},$$

so that indeed $x \in G^{\delta}(0, \tau; \mathcal{L}(\mathcal{X}^{\odot}, \mathcal{U}))$. The case where T is uniformly bounded follows as well. $\square$

The following result gives the "smoothing" property of the output map $\mathfrak{C}$ which is crucial for us.

Corollary 17. *Let $\delta \geq 0$ and let A be the generator of the uniformly bounded strongly continuous semigroup T on the reflexive Banach space $\mathcal{X}$ and let $\mathcal{Y}$ be a Hilbert space. If C^* is a Gevrey operator of order δ for A^*, then $\mathfrak{C}\mathcal{X} \subset G^{\delta}(0, \infty; \mathcal{Y})$ and we have that there exist $\tilde{A}_0, A_1 > 0$ such that for all $m \in \mathbb{N}_0$, all $t \in (0, \infty)$ and all $x_0 \in \mathcal{X}$*

$$\|(\mathfrak{C}x_0)^{(m)}(t)\|_{\mathcal{Y}} \leq A_0 A_1^m (m!)^{\delta},$$

where $A_0 := \tilde{A}_0 \|x_0\|$.

Proof. This follows from Proposition 16 with $B = C^*$ and (so that $B^{\odot} = C$ since both $\mathcal{X}$ and $\mathcal{Y}$ are reflexive) noting that $A^{\odot} = A^*$ since $\mathcal{X}$ is reflexive. $\square$

Note that a crucial part of Corollary 17 is that the constants $\tilde{A}_0$ and A_1 do not depend on the "initial condition" x_0.

6 Polynomial stability

In this section we very briefly discuss the notion of a polynomially stable semigroup (see e.g. [20] and [21] for more details on this notion).

We note that for A a closed operator in $\mathcal{X}$, $\mathcal{X}_1$ denotes the space $D(A)$ equipped with the norm $\|x\|_{\mathcal{X}_1} := \|(\alpha I - A)x\|_{\mathcal{X}}$ where $\alpha \in \rho(A)$ (different α's give rise to equivalent norms).

Definition 18. Let A be the generator of the strongly continuous semigroup T on the Banach space $\mathcal{X}$. This strongly continuous semigroup is called *polynomially stable* (with rate β) if there exist $M > 0$ and $\beta > 0$ such that for all $x_0 \in D(A)$ and all $t \geq 1$ there holds

$$\|T(t)x_0\|_{\mathcal{X}} \leq Mt^{-\beta}\|x_0\|_{\mathcal{X}_1}.$$

Remark 19. We note that when $0 \in \rho(A)$, the estimate in the definition of polynomial stability is equivalent to:

$$\|T(t)x_0\|_{\mathcal{X}} \leq Mt^{-\beta}\|Ax_0\|_{\mathcal{X}},$$

which by defining $z_0 := Ax_0$ is equivalent to

$$\|T(t)A^{-1}z_0\|_{\mathcal{X}} \leq Mt^{-\beta}\|z_0\|_{\mathcal{X}},$$

which by taking the supremum over all z_0 with $\|z_0\|_{\mathcal{X}} = 1$ gives

$$\|T(t)A^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq Mt^{-\beta}.$$

Proposition 20. *Let A be the generator of the strongly continuous semigroup T on the Banach space $\mathcal{X}$ with $0 \in \rho(A)$and let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ where $\mathcal{U}$ is a Banach space. Assume that T is polynomially stable (with rate β) and that B is a Gevrey operator of order δ for A. Then the function $x : (0, \infty) \to \mathcal{L}(\mathcal{U}, \mathcal{X})$ defined by $x(t) := T(t)B$ satisfies: for given $\gamma > 0$ there exist $A_0, A_1 > 0$ such that for all $m \in \mathbb{N}_0$ and all $t > 0$*

$$\|x^{(m)}(t)\|_{\mathcal{L}(\mathcal{U},\mathcal{X})} \leq A_0 A_1^m (m!)^{\widehat{\delta}} \min\{t^{-\gamma-m}, 1\},$$

with

$$\widehat{\delta} := \delta\left(1 + \frac{1}{\beta}\right) + 1.$$

Proof. Let C_0 and C_1 be as in the definition of Gevrey operator and let $u \in \mathcal{U}$. We have for $n \in \mathbb{N}_0$ (using Remark 19)

$$\begin{aligned}
&\|x^{(m)}(t)\|_{\mathcal{X}} = \|T(t)A^m Bu\|_{\mathcal{X}} \leq \|T(t)A^{-n}\|_{\mathcal{L}(\mathcal{X})}\|A^{n+m}Bu\|_{\mathcal{X}} \\
&= \left\|\left(T\left(\frac{t}{n}\right)A^{-1}\right)^n\right\|_{\mathcal{L}(\mathcal{X})} \|A^{n+m}Bu\|_{\mathcal{X}} \leq \left\|T\left(\frac{t}{n}\right)A^{-1}\right\|_{\mathcal{L}(\mathcal{X})}^n \|A^{n+m}B\|_{\mathcal{L}(\mathcal{U},\mathcal{X})}\|u\|_{\mathcal{U}} \\
&\leq \left(M\left(\frac{t}{n}\right)^{-\beta}\right)^n C_0 C_1^{n+m}(n+m)!^{\delta}\|u\|_{\mathcal{U}} = t^{-\beta n}M^n n^{\beta n} C_0 C_1^{n+m}(n+m)!^{\delta}\|u\|_{\mathcal{U}}.
\end{aligned}$$

Choosing $n := \left\lceil \frac{\gamma}{\beta} + \frac{m}{\beta} \right\rceil$ gives for $t \geq 1$ and certain A_0 and A_1

$$\|x^{(m)}(t)\|_{\mathcal{X}} \leq t^{-\gamma-m} A_0 A_1^m (m!)^{\widehat{\delta}} \|u\|_{\mathcal{U}}.$$

For $t \leq 1$ we can use Proposition 15 to obtain the desired estimate. $\square$

7 Main results

Theorem 21. *Let A be the generator of the strongly continuous semigroup T on the Hilbert space $\mathcal{X}$. Let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ where $\mathcal{U}$ is a finite-dimensional Hilbert space. Let $\delta > 1$ and assume that $B \in G^\delta(A)$. Additionally assume one of the following:*

(i) T is nilpotent, in which case define $\varepsilon := \delta$;

(ii) T is exponentially stable, in which case define $\varepsilon := \delta + 1$;

(iii) T is polynomially stable with rate β, in which case define $\varepsilon := \delta\left(1 + \frac{1}{\beta}\right) + 1$.

Then the input map $\mathfrak{B} : L^2(0, \infty; \mathcal{U}) \to \mathcal{X}$ of the pair (A, B) satisfies: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(\mathfrak{B}) \leq K \exp(-rn^{1/\varepsilon}).$$

The controllability Gramian $Q := \mathfrak{B}\mathfrak{B}^* : \mathcal{X} \to \mathcal{X}$ of the pair (A, B) satisfies: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(Q) \leq K \exp(-rn^{1/\varepsilon}).$$

If additionally $C : D(A) \to \mathcal{Y}$ is an infinite-time admissible observation operator for A with $\mathcal{Y}$ a Hilbert space (see e.g. [22, Section 4.6] or [18, Chapter 10]), then the Hankel operator $\mathfrak{H} := \mathfrak{C}\mathfrak{B} : L^2(0, \infty; \mathcal{U}) \to L^2(0, \infty; \mathcal{Y})$ of the triple (A, B, C) satisfies: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(\mathfrak{H}) \leq K \exp(-rn^{1/\varepsilon}).$$

Proof. By Corollary 17 we have that $\mathfrak{C} := \mathfrak{B}^* : \mathcal{X} \to L^2(0, \infty; \mathcal{U})$ has the properties listed there. Let $\{e_i\}_{i=1}^{\dim \mathcal{Y}}$ be an orthonormal basis for $\mathcal{Y}$ and define $\mathfrak{C}_i : \mathcal{X} \to L^2(0, \infty)$ by $(\mathfrak{C}_i z)(t) := \langle (\mathfrak{C}z)(t), e_i \rangle_{\mathcal{Y}}$. It follows that

$$\mathfrak{C}z = \sum_{i=1}^{\dim \mathcal{Y}} (\mathfrak{C}_i z) e_i.$$

We first consider the case where T is nilpotent. We can then apply Lemma 7 to $f := \mathfrak{C}_i z$ where by Corollary 17 the constant A_1 does not depend on z so that by Lemma 7 the constant C is independent of z. We therefore obtain (where $\tau > 0$ is such that $T(t) = 0$ for all $t \geq \tau$)

$$\|\mathfrak{C}_i z - \Pi_n \mathfrak{C}_i z\|_{L^2(0,\infty)} \leq \tilde{A}_0 \|z\| \tau^{3/2} C \mathrm{e}^{-rn^{1/\delta}}.$$

It follows that

$$\|\mathfrak{C}_i - \Pi_n \mathfrak{C}_i\|_{\mathcal{L}(\mathcal{X}, L^2(0,\infty))} \leq \tilde{A}_0 \tau^{3/2} C \mathrm{e}^{-rn^{1/\delta}}. \tag{8}$$

Define $\Pi_{n,\mathcal{Y}} : \mathcal{X} \to L^2(0,\infty;\mathcal{Y})$ by

$$\Pi_{n,\mathcal{Y}} z = \sum_{i=1}^{\dim \mathcal{Y}} (\Pi_n z) e_i.$$

Since Π_n has rank at most n, it follows that $\Pi_{n,\mathcal{Y}}$ has rank at most $n \dim \mathcal{Y}$ and from (8) we obtain

$$\|\mathfrak{C} - \Pi_{n,\mathcal{Y}} \mathfrak{C}\|_{\mathcal{L}(\mathcal{X}, L^2(0,\infty;\mathcal{Y}))} \leq \tilde{A}_0 \tau^{3/2} C \mathrm{e}^{-rn^{1/\delta}} \dim \mathcal{Y}.$$

It follows that

$$\sigma_{n \dim \mathcal{Y}+1}(\mathfrak{B}) \leq \tilde{A}_0 \tau^{3/2} C \mathrm{e}^{-rn^{1/\delta}} \dim \mathcal{Y}.$$

From this the indicated bound follows.

The exponentially stable case follows similarly by utilizing Lemma 10 instead of Lemma 7. The polynomially stable case follows by utilizing Lemma 11 (where we obtain the required estimate from Proposition 20) instead of Lemma 7.

Since we have

$$\sigma_n(\mathfrak{B}\mathfrak{B}^*) \leq \sigma_n(\mathfrak{B}) \|\mathfrak{B}^*\|,$$

and

$$\sigma_n(\mathfrak{C}\mathfrak{B}) \leq \|\mathfrak{C}\| \sigma_n(\mathfrak{B}),$$

the statements about the controllability Gramian and Hankel operator readily follow. □

The dual of Theorem 21 is the following:

Theorem 22. *Let A be the generator of the strongly continuous semigroup T on the Hilbert space $\mathcal{X}$. Let $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ where $\mathcal{Y}$ is a finite-dimensional Hilbert space. Let $\delta > 1$ and assume that $C^* \in G^\delta(A^*)$. Additionally assume one of the following:*

(i) T is nilpotent, in which case define $\varepsilon := \delta$;

(ii) T is exponentially stable, in which case define $\varepsilon := \delta + 1$;

(iii) T is polynomially stable with rate β, in which case define $\varepsilon := \delta\left(1 + \frac{1}{\beta}\right) + 1$.

Then the output map $\mathfrak{C}$ of the pair (A, C) satisfies: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(\mathfrak{C}) \leq K \exp(-rn^{1/\varepsilon}).$$

The observability Gramian $Q := \mathfrak{C}^*\mathfrak{C}$ of the pair (A, C) satisfies: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(Q) \leq K \exp(-rn^{1/\varepsilon}).$$

If additionally $B : \mathcal{U} \to \mathcal{X}_{-1}$ is an infinite-time admissible control operator for A with $\mathcal{U}$ a Hilbert space, then the Hankel operator $\mathfrak{H} := \mathfrak{C}\mathfrak{B}$ of the triple (A, B, C) satisfies: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(\mathfrak{H}) \leq K \exp(-rn^{1/\varepsilon}).$$

Proof. This follows by duality from Theorem 21. □

8 Examples

Examples 23, 24 and 25 consider a transport equation which illustrates the nilpotent case; examples 26 (a transport equation) and Example 27 (a wave equation) illustrate the exponentially stable case whereas Example 28 (a Rayleigh beam) and Example 29 (a coupled wave-heat system) illustrate the polynomially stable case.

Example 23. Consider the partial differential equation

$$\begin{aligned} w_t(t,\xi) &= w_\xi(t,\xi) + b(\xi)u(t), \quad t>0,\ \xi\in(0,1), \\ w(t,1) &= 0, \qquad t>0, \end{aligned}$$

where

$$b(\xi) = \begin{cases} \exp\left(\dfrac{-1}{(\xi-\frac{1}{4})(\frac{3}{4}-\xi)}\right) & \xi\in\left(\frac{1}{4},\frac{3}{4}\right), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Abstractly, this is given by $\dot{x} = Ax + Bu$ where $x(t) := \xi \mapsto w(t,\xi)$, $\mathcal{X} := L^2(0,1)$, $\mathcal{U} = \mathbb{R}$,

$$Af = f', \qquad D(A) = \{f \in H^1(0,1) : f(1) = 0\}, \qquad (Bu)(\xi) = b(\xi)u.$$

We have for the set of Gevrey vectors of this particular A:

$$\begin{aligned} G^\delta(A) = \{&f\in L^2(0,1) : f\in H^n(0,1) \text{ and } f^{(n)}(1)=0, \text{ for all } n\in\mathbb{N}_0, \\ &\exists A_0, A_1>0 \text{ such that } \forall n\in\mathbb{N}_0 \text{ there holds } \|f^{(n)}\|_{L^2(0,1)} \le A_0A_1^n(n!)^\delta\}. \end{aligned}$$

We note that a Gevrey function f of order δ as defined in Definition 1 which additionally satisfies $f^{(n)}(1) = 0$ for all $n \in \mathbb{N}_0$ belongs to $G^\delta(A)$. In particular, since b is a Gevrey function of order 2 and satisfies $b^{(n)}(1) = 0$ for all $n \in \mathbb{N}_0$ we have $B \in G^2(A)$.

Since $T(t) = 0$ for $t > 1$, we have that the semigroup is nilpotent. We conclude from Theorem 21 that for the controllability Gramian Q of this system we have that there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(Q) \le K\exp(-r\sqrt[2]{n}).$$

Example 24. Consider the same PDE as in example 23, but now with the observation $y(t) = w(t,0)$. The observation operator then is

$$Cf = f(0),$$

which is unbounded. We however have that the initial state to output map $\mathfrak{C}$ is bounded $\mathcal{X} \to L^2(0,\infty)$ as it is given by

$$(\mathfrak{C}x_0)(t) = \begin{cases} x_0(t) & t\in[0,1], \\ 0 & t>1, \end{cases}$$

where $x_0 \in \mathcal{X}$ is the initial state (i.e., C is an admissible observation operator for A). For the Hankel operator $\mathfrak{H} = \mathfrak{C}\mathfrak{B}$ we have from Theorem 21: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(\mathfrak{H}) \leq K \exp(-r \sqrt[2]{n}).$$

Example 25. Consider the observed transport equation

$$\begin{aligned} w_t(t,\xi) &= w_\xi(t,\xi), \quad t>0,\ \xi \in (0,1), \\ w(t,1) &= 0, \qquad t > 0, \\ y(t) &= \int_0^1 b(\xi) w(\xi)\, d\xi, \end{aligned}$$

where b is given again by (9). We now have

$$A^* f = -f', \qquad D(A^*) = \{f \in H^1(0,1) : f(0) = 0\},$$

and

$$(C^* u)(\xi) = b(\xi) u.$$

Therefore we have $C^* \in G^2(A^*)$. We conclude from Theorem 22 that for the observability Gramian Q of this system we have that there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(Q) \leq K \exp(-r \sqrt[2]{n}).$$

Example 26. Consider the same PDE as in example 23, but now with damped periodic boundary conditions:

$$\begin{aligned} w_t(t,\xi) &= w_\xi(t,\xi) + b(\xi) u(t), \quad t>0,\ \xi \in (0,1), \\ w(t,1) &= \frac{1}{2} w(t,0), \qquad t > 0, \end{aligned}$$

where b is given again by (9). We now have

$$Af = f', \qquad D(A) = \{f \in H^1(0,1) : f(1) = \frac{1}{2} f(0)\},$$

and we again have $B \in G^2(A)$. The semigroup generated by A is now no longer nilpotent, but it is exponentially stable. From Theorem 21 we therefore have: there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(Q) \leq K \exp(-r \sqrt[3]{n}).$$

Example 27. Consider the damped wave equation

$$\begin{aligned} y_{tt}(t,\xi) + y_t(t,\xi) &= y_{\xi\xi}(t,\xi) + b(\xi) u(t), \qquad t>0,\ \xi \in (0,1), \\ y(t,0) = 0, \quad y(t,1) &= 0, \qquad t > 0, \end{aligned}$$

where b is as in (9). Abstractly, this is given by $\dot{x} = Ax + Bu$ where

$$x(t) := \xi \mapsto \begin{bmatrix} y(t,\xi) \\ y_t(t,\xi) \end{bmatrix},$$

$\mathcal{X} := H_0^1(0,1) \times L^2(0,1)$, $\mathcal{U} = \mathbb{R}$,

$$A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_1'' - f_2 \end{bmatrix}, \quad D(A) = \left\{ f \in \begin{bmatrix} H^2(0,1) \\ H_0^1(0,1) \end{bmatrix} : f_1(0) = f_1(1) = 0 \right\},$$

and

$$(Bu)(\xi) = \begin{bmatrix} 0 \\ b(\xi)u \end{bmatrix}.$$

Similarly as in Example 23, since b is a Gevrey function of order 2 which has a zero of infinite order at the boundary points, it follows that $B \in G^2(A)$. The semigroup generated by A is exponentially stable (this follows e.g. from [23, Theorem 8.3]). We therefore obtain from Theorem 21 that there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(Q) \leq K \exp(-r \sqrt[3]{n}).$$

Example 28. Consider a weakly damped clamped Rayleigh beam :

$$\begin{gathered} y_{tt}(t,\xi) - y_{tt\xi\xi}(t,\xi) + y_t(t,\xi) + y_{\xi\xi\xi\xi}(t,\xi) = b(\xi)u(t), \qquad t > 0,\ \xi \in (0,1), \\ y(t,0) = y_\xi(t,0) = y(t,1) = y_\xi(t,1) = 0, \qquad t > 0, \end{gathered}$$

where b is given by (9). Abstractly, this is given by $\dot{x} = Ax + Bu$ where

$$x(t) := \xi \mapsto \begin{bmatrix} y(t,\xi) \\ y_t(t,\xi) \end{bmatrix},$$

$\mathcal{X} := H_0^2(0,1) \times H_0^1(0,1)$, $\mathcal{U} = \mathbb{R}$,

$$A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_2 \\ -M^{-1}K_0 f_1 - M^{-1} f_2 \end{bmatrix}, \quad D(A) = \begin{bmatrix} H^3(0,1) \cap H_0^2(0,1) \\ H_0^2(0,1) \end{bmatrix},$$

where

$$\begin{gathered} M : H_0^1(0,1) \to H^{-1}(0,1), \qquad Mg = g - g'', \\ K_0 : H^3(0,1) \cap H_0^2(0,1) \to H^{-1}(0,1), \qquad K_0 g = g^{(4)}, \end{gathered}$$

and

$$(Bu)(\xi) = \begin{bmatrix} 0 \\ b(\xi)u \end{bmatrix}.$$

Similarly as in Example 23, since b is a Gevrey function of order 2 which has a zero of infinite order at the boundary points, it follows that $B \in G^2(A)$. The semigroup generated by A is polynomially stable with rate $\beta = 1/2$ (this follows from [24]). We therefore obtain from Theorem 21 that, with Q the controllability Gramian, there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$\sigma_n(Q) \leq K \exp(-r \sqrt[7]{n}).$$

Example 29. Consider the system of partial differential equations (a coupled wave-heat system) from [25] (considered there without the control).

$$
\begin{gathered}
y_{tt}(t,\xi) = y_{\xi\xi}(t,\xi) + b(\xi)u(t), \qquad t>0,\ \xi \in (-1,0), \\
w_t(t,\xi) = w_{\xi\xi}(t,\xi), \qquad t>0,\ \xi \in (0,1), \\
y_\xi(t,-1) = 0, \quad w(t,1) = 0, \qquad t>0 \\
y_t(t,0) = w(t,0), \quad y_\xi(t,0) = w_\xi(t,0), \qquad t>0.
\end{gathered}
$$

where

$$
b(\xi) = \begin{cases} \exp\left(\dfrac{-1}{(\xi - \frac{-3}{4})(\frac{-1}{4} - \xi)}\right) & \xi \in \left(\frac{-3}{4}, \frac{-1}{4}\right), \\ 0 & \text{otherwise.} \end{cases}
$$

Abstractly, this is given by $\dot{x} = Ax + Bu$ where

$$
x(t) := \xi \mapsto \begin{bmatrix} y(t,\xi) \\ y_t(t,\xi) \\ w(t,\xi) \end{bmatrix},
$$

$\mathcal{X} := H^1(-1,0) \times L^2(-1,0) \times L^2(0,1)$, $\mathcal{U} = \mathbb{R}$,

$$
A \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_1'' \\ f_3'' \end{bmatrix}, \quad D(A) = \left\{ f \in \begin{bmatrix} H^2(-1,0) \\ H^1(-1,0) \\ H^2(0,1) \end{bmatrix} : f_1'(-1) = f_3(1) = 0, f_2(0) = f_3(0), f_1'(0) = f_3'(0) \right\},
$$

and

$$
(Bu)(\xi) = \begin{bmatrix} 0 \\ b(\xi)u \\ 0 \end{bmatrix}.
$$

Similarly as in Example 23, since b is a Gevrey function of order 2 which has a zero of infinite order at -1, 0 and 1, it follows that $B \in G^2(A)$. By [25, Theorem 4.2], the semigroup is polynomially stable with rate $\beta = 2$. We conclude from Theorem 21 that for the controllability Gramian Q of this system we have that there exist $K, r > 0$ such that for all $n \in \mathbb{N}$

$$
\sigma_n(Q) \le K \exp(-r \sqrt[5]{n}).
$$

9 Importance of the assumptions

In this section we consider some examples which illustrate the relevance of the assumptions in our results.

Example 30 considers a transport equation with an unbounded observation operator where the singular values of the observability Gramian don't decay at all, whereas the more complicated Example 31 has a bounded observation operator which is not Gevrey where the singular values decay slowly. In Example 32 the infinite-dimensionality of the output space is the cause of slow decay of the singular values.

Example 30. Consider on $\mathcal{X} = L^2(0,1)$

$$Af = f', \qquad D(A) = \{f \in H^1(0,1) : f(1) = 0\}, \qquad Cf = f(0).$$

Then the observability Gramian Q equals the identity. In particular $\sigma_n(Q) = 1$ so that there is no decay of its singular values. Note that C is unbounded and that therefore Theorem 22 does not apply.

Example 31. Consider on $\mathcal{X} = L^2(0,1)$

$$Af = -f', \qquad D(A) = \{f \in H^1(0,1) : f(0) = 0\}, \qquad Cg = \int_0^1 g(\xi)\, d\xi.$$

We then have for $t \leq 1$

$$(\mathfrak{C}f)(t) = CT(t)f = \int_0^1 [T(t)f](\xi)\, d\xi = \int_t^1 f(\xi - t)\, d\xi = \int_0^{1-t} f(\eta)\, d\eta,$$

whereas $(\mathfrak{C}f)(t) = 0$ for $t > 1$. Therefore $\mathfrak{C} : L^2(0,1) \to L^2(0,\infty) = \begin{bmatrix} L^2(0,1) \\ L^2(1,\infty) \end{bmatrix}$ can be written as

$$\mathfrak{C} = \begin{bmatrix} RV \\ 0 \end{bmatrix},$$

where $R : L^2(0,1) \to L^2(0,1)$ is defined by $(Rf)(t) = f(1-t)$ and $V : L^2(0,1) \to L^2(0,1)$ is the Volterra operator defined by

$$(Vg)(\eta) = \int_0^\eta g(\xi)\, d\xi.$$

It is known that

$$\sigma_n(V^*V) = \frac{4}{(2n-1)^2\pi^2},$$

see e.g. [26], and using that R is an isometry it follows that with $Q = \mathfrak{C}^*\mathfrak{C}$ the observability Gramian

$$\sigma_n(Q) = \frac{4}{(2n-1)^2\pi^2}.$$

We note that $Cg = \langle c, g\rangle$ where $c \in \mathcal{X}$ is defined by $c(\xi) = 1$. We have $c \notin D(A)$, which implies that C^* is not a Gevrey operator for A^* and therefore the assumptions of Theorem 22 are not satisfied.

Example 32. Consider on $\mathcal{X} = L^2(0,1)$

$$Af = f'', \qquad D(A) = \{f \in H^2(0,1) : f(0) = f(1) = 0\}, \qquad Cf = f.$$

Then the observability Gramian Q equals $-\frac{1}{2}A^{-1}$. In particular, using the known eigenvalues of the Dirichlet Laplacian, $\sigma_n(Q) = \frac{1}{2n^2\pi^2}$. This does not contradict Theorem 22 since $\dim \mathcal{Y} = \infty$.

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