Decay of singular values of the Gramians of infinite-dimensional systems

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***Abstract*— We show that the Gramians of a control system with an analytic semigroup, control and observation operators that are not too unbounded and which have finite-dimensional input and output spaces have singular values which decay exponentially in the square root. As a corollary it is shown that the Hankel singular values of such control systems also decay exponentially in the square root. Another corollary shows that solutions of algebraic Riccati equations for such systems also have singular values which decay exponentially in the square root.**

I. Introduction

The Hankel singular values of a control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

provide fundamental information about how well the system can be approximated by lower dimensional ones. If G is a transfer function with Hankel singular values $(\sigma_k)_{k=1}^{\infty}$, then for any transfer function G_n of McMillan degree n there holds:

$$\sigma_{n+1} \leq \|G - G_n\|_{\infty},$$

where $\|\cdot\|_{\infty}$ is the H^{∞} norm. Moreover, there exists a transfer function G_n of McMillan degree n such that

$$\|G - G_n\|_{\infty} \leq \sum_{k=n+1}^{\infty} \sigma_k.$$

See for example [10, page 346 and Lemma 10.5.3] for the finite-dimensional case and [12] for the general case.

A popular model reduction method is balanced truncation [15], [6], [8], [10, Chapter 9], [28, Chapter 7]. To implement this method, two Lyapunov equations have to be solved to obtain the controllability and observability Gramians of the system. For large scale systems, for computational feasibility, usually low rank approximations are computed instead [13], [3]. A fundamental question is when such low rank approximations exist. To answer this question, it is necessary and sufficient to analyze the decay of the singular values of the Gramians. The same question regarding low rank approximations is relevant for algebraic Riccati equations instead of Lyapunov equations.

The above two questions are closely related since the Hankel singular values of a system are equal to the square roots of the singular values of the product of the controllability and observability Gramians. Also the singular values of the solution of a Riccati equation can be related to the singular values of a Gramian.

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There are various articles which deal with the above questions in the context of finite-dimensional systems [21], [1], [9]. However, the obtained bounds depend on the condition number of the matrix A. Since for discretizations of partial differential equations this condition number converges to infinity as the discretization is refined, these bounds are not very satisfactory. In [18], the infinite-dimensional case was considered and it was shown that for a large class of infinite-dimensional systems the Hankel singular values decay superpolynomially. This implies a result on superpolynomial decay of the singular values of the Gramians. Recently, in [11], it was shown that the singular values of the Gramians of certain infinite-dimensional systems decay at a rate exponential in the square root. However, in this result from [11], a nonstandard definition of singular value is used. This coincides with the standard definition of singular value only if B and C are bounded. In particular, the results of [11] can only be used to deduce decay at a rate exponential in the square root for the Hankel singular values in case B and C are bounded.

In this article we prove decay of the singular values (in the standard sense) of the Gramians at a rate exponential in the square root for a large class of infinite-dimensional systems (which includes many systems of interest with unbounded B and C), improving the results of [11]. From this we conclude decay at a rate exponential in the square root for the Hankel singular values for this class of systems, thereby improving the results of [18]. We also show that for a wide range of algebraic Riccati equations the singular values of the positive semidefinite solution decay at a rate exponential in the square root.

We now briefly outline the method of proof. The observability Gramian L is explicitly given by

$$L = \int_0^{\infty} \mathrm{e}^{A^* t} C^* C \mathrm{e}^{At}\, dt. \tag{1}$$

As in [11], we use sinc quadrature on this integral to obtain an explicit approximation of the Gramian of the required rank. In the case of unbounded C, the integrand is unbounded at $t = 0$. What is essentially done in [11] is that instead

$$A^{-\alpha} L A^{-\alpha} = \int_0^{\infty} \mathrm{e}^{A^* t} A^{-\alpha} C^* C A^{-\alpha} \mathrm{e}^{At}\, dt,$$

is considered. Under the assumption that $CA^{-\alpha}$ is bounded for some $\alpha \in [0, 1]$ (which is a reasonable assumption for applications to PDEs), the integrand is no longer unbounded at $t = 0$. However, the additional factors $A^{-\alpha}$ lead to the nonstandard singular values of L considered in [11]. Contrary to what is done in [11], we deal directly with the singularity of the integrand at $t = 0$ in (1). Using the theory of analytic semigroups it can be shown that, for the class of systems considered, the integrand has what is called an integrable algebraic singularity at $t = 0$, i.e. it behaves like $t^{-\gamma}$ for $\gamma \in [0, 1)$ near $t = 0$. Since sinc quadrature still gives a convergence rate exponential in the square root in the presence of such a singularity, the decay rate follows.

II. Analytic control systems

In this section we define the class of systems to which our results apply and prove some properties for this class of systems. We first recall the definition of an exponentially stable analytic semigroup. Standard references for this include [24], [2], [5], [20].

Definition 1: Let $\mathscr{X}$ be a Hilbert space, let $\delta \in (0, \pi/2]$, and let Δ_δ be the open sector

$$\Delta_\delta := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \delta\}.$$

The family of operators $\{T(t) \in \mathcal{L}(\mathscr{X}) : t \in \Delta_\delta\}$ is called an *exponentially stable analytic semigroup* if

1) $z \mapsto T(z)$ is analytic on Δ_δ,
2) $T(0) = I$ and $T(z+w) = T(z)T(w)$ for all $z, w \in \Delta_\delta$,
3) there exist $M \geq 1$ and $\omega > 0$ such that for all $z \in \Delta_\delta$

$$\|T(z)\| \leq M\mathrm{e}^{-\omega \mathrm{Re}(z)},$$

4) for all $x \in \mathscr{X}$ there holds $\lim\limits_{\substack{z \to 0 \\ z \in \Delta_\delta}} T(z)x = x$.

Remark 2: We note that most books consider the case $\omega = 0$ of *bounded analytic semigroups*. It is easily seen that if S is a bounded analytic semigroup, then $T(z) := \mathrm{e}^{-\omega z}S(z)$ is an exponentially stable analytic semigroup in the sense of Definition 1. Conversely, if T is an exponentially stable analytic semigroup in the sense of Definition 1, then $S(z) := \mathrm{e}^{\omega z}T(z)$ is a bounded analytic semigroup.

An analytic semigroup is strongly continuous and therefore has a generator A (in some sense the semigroup equals $z \mapsto \mathrm{e}^{Az}$). A crucial and well-known property of analytic semigroups is the following. Since most books only consider the case of real z in the desired estimate, we give the proof.

Lemma 3: Let T be an exponentially stable analytic semigroup as in Definition 1. Denote its generator by A. Then for all $a \in [0, 1)$ there exists a constant $M_a \geq 1$ such that for all $z \in \Delta_{a\delta}$

$$\|AT(z)\| \leq \frac{M_a}{|z|}\mathrm{e}^{-\omega \mathrm{Re}(z)}.$$

Proof: We adapt the proof from [2, Theorem 3.7.19] (which only considers the case of real z). We have $AT(z) = T'(z)$ and by Cauchy's integral formula for the derivative we therefore have

$$AT(z) = \frac{1}{i2\pi}\int_{|w-z|=r} \frac{T(w)}{(w-z)^2}\, dw,$$

provided that $r > 0$ is such that $\{w \in \mathbb{C} : |w-z| \leq r\} \subset \Delta_\delta$. Estimating this integral gives

$$\|AT(z)\| \leq \frac{M\mathrm{e}^{-\omega \mathrm{Re}(z)}}{r},$$

so that

$$\|zAT(z)\| \leq M\mathrm{e}^{-\omega \mathrm{Re}(z)}\frac{|z|}{r}.$$

We can choose $r := |z|\sin(\varepsilon\delta)$, where $\varepsilon := 1 - a \in (0, 1]$. We then have

$$\|zAT(z)\| \leq \frac{M}{\sin(\varepsilon\delta)}\mathrm{e}^{-\omega \mathrm{Re}(z)}.$$

Therefore, defining

$$M_a := \frac{M}{\sin(\delta - a\delta)},$$

we obtain the desired result. ∎

As in [24, Lemma 3.10.9] we obtain a corollary on fractional powers.

Corollary 4: Let T be an exponentially stable analytic semigroup as in Definition 1. Denote its generator by A. Let $\alpha \in [0, 1]$. Then for all $a \in [0, 1)$ there exists a constant $\widetilde{M}_a \geq 1$ such that for all $z \in \Delta_{a\delta}$

$$\|A^\alpha T(z)\| \leq \frac{\widetilde{M}_a}{|z|^\alpha}\mathrm{e}^{-\omega \mathrm{Re}(z)}.$$

Proof: By [24, Lemma 3.9.8] there exists a constant $C > 0$ such that for all $y \in D(A)$

$$\|A^\alpha y\| \leq C\|y\|^{1-\alpha}\ \|Ay\|^\alpha.$$

Let $x \in \mathscr{X}$ and note that then $T(z)x \in D(A)$ and therefore by the above

$$\|A^\alpha T(z)x\| \leq C\|T(z)x\|^{1-\alpha}\ \|AT(z)x\|^\alpha.$$

Now using the bounds from Definition 1 and Lemma 3 we have

$$\|A^\alpha T(z)x\| \leq CM^{1-\alpha}\mathrm{e}^{-(1-\alpha)\omega \mathrm{Re}(z)}\frac{M_a^\alpha}{|z|^\alpha}\mathrm{e}^{-\alpha\omega \mathrm{Re}(z)},$$

so that with

$$\widetilde{M}_a := CM^{1-\alpha}M_a^\alpha,$$

we obtain the desired estimate. ∎

Theorem 5: Let T be an exponentially stable analytic semigroup as in Definition 1. Denote its generator by A. Let $\mathscr{Y}$ be a Hilbert space, let $\alpha \in [0, 1]$ and let $C : D(A) \to \mathscr{Y}$ be such that $CA^{-\alpha} \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$. Then for all $a \in [0, 1)$ there exists a constant $M_a \geq 1$ such that for all $z \in \Delta_{a\delta}$

$$\|CT(z)\|_{\mathcal{L}(\mathscr{X},\mathscr{Y})} \leq \frac{M_a}{|z|^\alpha}\mathrm{e}^{-\omega \mathrm{Re}(z)}.$$

Proof: We have, using Corollary 4 for the second inequality,

$$\begin{aligned}\|CT(z)\|_{\mathcal{L}(\mathscr{X},\mathscr{Y})} &\leq \|CA^{-\alpha}\|_{\mathcal{L}(\mathscr{X},\mathscr{Y})}\|A^\alpha T(z)\|_{\mathcal{L}(\mathscr{X})} \\ &\leq \|CA^{-\alpha}\|_{\mathcal{L}(\mathscr{X},\mathscr{Y})}\frac{\widetilde{M}_a}{|z|^\alpha}\mathrm{e}^{-\omega \mathrm{Re}(z)},\end{aligned}$$

so that with

$$M_a := \|CA^{-\alpha}\|_{\mathcal{L}(\mathscr{X},\mathscr{Y})}\widetilde{M}_a,$$

we obtained the desired inequality. ∎

Remark 6: From Theorem 5 it follows that, for A and C as in that theorem and with $\alpha \in [0,1/2)$, the *observability Gramian*

$$L := \int_0^\infty T(t)^* C^* C T(t)\, dt,$$

is a well-defined bounded operator on $\mathscr{X}$. See e.g. [24, Chapter 10] and [26, Chapter 5] for more information on Gramians for infinite-dimensional systems.

Remark 7: Let A and C be as in Theorem 5. For $x, y \in D(A)$ the function $f : \Delta_\delta \to \mathbb{C}$

$$f(z) := \langle CT(z)x, CT(\bar{z})y \rangle,$$

is holomorphic on Δ_δ and by Theorem 5 for all $a \in [0,1)$ there exists a constant $M_a \geq 1$ such that for all $z \in \Delta_{a\delta}$

$$|f(z)| \leq \frac{M_a^2}{|z|^{2\alpha}} \mathrm{e}^{-2\omega \mathrm{Re}(z)} \|x\| \, \|y\|.$$

Note that with L the observability Gramian from Remark 6

$$\langle Lx, y \rangle = \int_0^\infty f(t)\, dt.$$

III. Sinc quadrature

We quote the following result from [14] regarding sinc quadrature.

Theorem 8: Assume that for $\vartheta \in (0, \pi/2)$, the complex-valued function f is analytic on the sector

$$\overline{\Delta}_\vartheta := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| \leq \vartheta\},$$

and is such that

$$\sup_{r \geq 0} \int_{-\vartheta}^{\vartheta} |f(r\mathrm{e}^{i\theta})| r \, d\theta < \infty, \tag{2}$$

$$\int_0^\infty |f(r\mathrm{e}^{i\vartheta})|\, dr < \infty, \tag{3}$$

and such that there exist $M, \beta, \gamma > 0$, such that

$$|f(x)| \leq M \begin{cases} x^{\gamma - 1} & x \in (0,1), \\ \mathrm{e}^{-\beta x} & x \in (1, \infty). \end{cases} \tag{4}$$

Define for $m \in \mathbb{N}_0$

$$h := \sqrt{\frac{2\pi\vartheta}{\gamma m}}, \quad N := \left\lceil \frac{1}{h} \log\left(\frac{\gamma}{\beta} m h\right) + 1 \right\rceil.$$

Then there exists a $\widetilde{M} > 0$ such that for all $m \in \mathbb{N}_0$

$$\left| \int_0^\infty f(x)\, dx - h \sum_{k=-m}^{N} \mathrm{e}^{kh} f\left(\mathrm{e}^{kh}\right) \right| \leq \widetilde{M} \mathrm{e}^{-\sqrt{2\pi\vartheta\gamma m}}.$$

Proof: This is [14, Theorem 3.9] with the modification of N for exponential decay from [14, page 77]. ■

The following lemma gives a sufficient condition for the conditions from Theorem 8 to hold.

Lemma 9: If, for $\vartheta \in (0, \pi/2)$, the complex-valued function f is analytic on the sector

$$\overline{\Delta}_\vartheta := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| \leq \vartheta\},$$

and is such that there exist $M, \beta > 0$ and $\gamma \in (0,1]$ such that for all $z \in \overline{\Delta}_\vartheta$

$$|f(z)| \leq M |z|^{\gamma - 1} \mathrm{e}^{-\beta \mathrm{Re}(z)},$$

then f satisfies (2), (3) and (4).

Proof: That the condition implies (4) is clear. Note that for $\theta \in [-\vartheta, \vartheta]$ there holds

$$|f(r\mathrm{e}^{i\theta})| \leq M r^{\gamma-1} \mathrm{e}^{-\beta r \cos\theta} \leq M r^{\gamma-1} \mathrm{e}^{-\beta r \cos\vartheta}.$$

It follows that (2) holds since

$$\int_{-\vartheta}^{\vartheta} |f(r\mathrm{e}^{i\theta})| r \, d\theta \leq 2\vartheta M r^\gamma \mathrm{e}^{-\beta r \cos\vartheta},$$

which is uniformly bounded in $r \geq 0$ since $\gamma > 0$ and since $\vartheta \in (0, \pi/2)$ and $\beta > 0$ we have $\beta \cos\vartheta > 0$. It also follows that (3) holds since

$$\begin{aligned} \int_0^\infty |f(r\mathrm{e}^{i\vartheta})|\, dr &\leq \int_0^\infty M r^{\gamma-1} \mathrm{e}^{-\beta r \cos\vartheta}\, dr \\ &\leq M \int_0^1 r^{\gamma-1}\, dr + M \int_1^\infty \mathrm{e}^{-\beta r \cos\vartheta}\, dr, \end{aligned}$$

and the first integral on the right-hand side is finite since $\gamma > 0$ and the second integral on the right-hand side is finite since $\beta \cos\vartheta > 0$. ■

Remark 10: Alternatively, sinc quadrature with regard to a bullet-shaped region rather than with regard to a sector can be used to obtain the same convergence rate under slightly weaker assumptions (which are not relevant for our application); see [25, Example 4.2.11].

IV. The main result

We now combine the results from Sections II and III to prove our main result. Recall that the singular values of an operator $T : \mathscr{X} \to \mathscr{X}$ are defined for $k \in \mathbb{N}$ by

$$\sigma_k(T) := \inf\{\|T - T_k\| : \mathrm{rank}(T_k) < k\}.$$

Theorem 11: Let T be an exponentially stable analytic semigroup as in Definition 1. Denote its generator by A. Let $\mathscr{Y}$ be a finite-dimensional Hilbert space, let $\alpha \in [0, 1/2)$ and let $C : D(A) \to \mathscr{Y}$ be such that $CA^{-\alpha} \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$. Denote the observability Gramian of the pair A, C by L. Then there exist $M, c > 0$ such that for all $k \in \mathbb{N}$

$$\sigma_k(L) \leq M \mathrm{e}^{-c\sqrt{k}}.$$

Proof: Let $x, y \in D(A)$. By Remark 7, the function $f : \Delta_\delta \to \mathbb{C}$ defined by

$$f(z) := \langle CT(z)x, CT(\bar{z})y \rangle,$$

satisfies the conditions of Lemma 9 (with $\gamma := 1 - 2\alpha$, $\beta := 2\omega$ and $\vartheta := a\delta$ with $a \in (0,1)$) and therefore the conditions of Theorem 8.

With N and h as in Theorem 8, define for $m \in \mathbb{N}_0$ the operator $L_m : \mathscr{X} \to \mathscr{X}$ by

$$L_m := h \sum_{k=-m}^{N} \mathrm{e}^{kh} T\left(\mathrm{e}^{kh}\right)^* C^* C T\left(\mathrm{e}^{kh}\right).$$

Note that since $\mathrm{e}^{kh} > 0$ and the range of $T(t)$ is in $D(A)$ for $t > 0$, the operator L_m is well-defined and bounded.

As L_m is the sum of $m+N+1$ operators of rank at most $\dim \mathscr{Y}$, we have

$$\operatorname{rank}(L_m) \leq (m+N+1)\dim \mathscr{Y}.$$

Since N is bounded by a constant times m, it follows that there exists a $\ell \in \mathbb{N}$ such that for all $m \in \mathbb{N}$

$$\operatorname{rank}(L_m) < \ell m.$$

We note that

$$\langle L_m x, y\rangle = h \sum_{k=-m}^{N} \mathrm{e}^{kh} f\left(\mathrm{e}^{kh}\right),$$

and

$$\langle Lx, y\rangle = \int_0^\infty f(x)\, dx,$$

so that by Theorem 8, there exists a $\widetilde{M}$ such that

$$|\langle Lx, y\rangle - \langle L_m x, y\rangle| \leq \widetilde{M}\mathrm{e}^{-\sqrt{2\pi\vartheta\gamma m}}.$$

Using that the constant in Remark 7 is of the form a constant independent of x and y times $\|x\|$ times $\|y\|$ and keeping track of the constants in the proof of Theorem 8 in [14], we deduce that $\widetilde{M} = \widetilde{M}_1\|x\|\;\|y\|$, where $\widetilde{M}_1$ is independent of x and y. We infer that

$$|\langle Lx, y\rangle - \langle L_m x, y\rangle| \leq \widetilde{M}_1\mathrm{e}^{-\sqrt{2\pi\vartheta\gamma m}}\|x\|\;\|y\|.$$

Recall that for a self-adjoint operator T there holds

$$\|T\| = \sup_{x\in D:\|x\|=1} |\langle Tx, x\rangle|,$$

where D is a dense subspace. Applying this with $T := L - L_m$ and $D = D(A)$ gives

$$\|L - L_m\| \leq \widetilde{M}_1\mathrm{e}^{-\sqrt{2\pi\vartheta\gamma m}}.$$

Since, as mentioned above, $\operatorname{rank}(L_m) < \ell m$, it follows that

$$\sigma_{\ell m}(L) \leq \widetilde{M}_1\mathrm{e}^{-\sqrt{2\pi\vartheta\gamma m}}.$$

Standard arguments using monotonicity of the singular values then give that there exists $M, c > 0$ such that for all $k \in \mathbb{N}$

$$\sigma_k(L) \leq M\mathrm{e}^{-c\sqrt{k}}.$$

■

The following corollary proves decay of the Hankel singular values.

Corollary 12: Let T be an exponentially stable analytic semigroup as in Definition 1. Denote its generator by A. Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces, at least one of which is finite-dimensional. Let $\alpha \in [0, 1/2)$ and let $C : D(A) \to \mathscr{Y}$ be such that $CA^{-\alpha} \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$. Let $\beta \in [0, 1/2)$ and let $B : \mathscr{U} \to D(A^*)'$ be such that $B^*A^{-\beta *} \in \mathcal{L}(\mathscr{X}, \mathscr{U})$. Let the Hankel operator of the triple (A, B, C) be given by

$$H : L^2(0,\infty;\mathscr{U}) \to L^2(0,\infty;\mathscr{Y}),$$
$$(Hu)(t) = \int_0^\infty CT_{-1}(t+s)Bu(s)\, ds,$$

where T_{-1} is the extension of the semigroup T to the space $D(A^*)'$. Then there exist $M, c > 0$ such that for all $k \in \mathbb{N}$

$$\sigma_k(H) \leq M\mathrm{e}^{-c\sqrt{k}}.$$

Proof: Denote the observability Gramian of the system by L_C and the controllability Gramian of the system by L_B. Assume that $\mathscr{Y}$ is finite-dimensional (if $\mathscr{U}$ is finite-dimensional, then the roles of L_B and L_C below have to be reversed).

The assumptions imply that L_B is bounded and that L_C is compact. It follows that the product $L_B L_C$ is compact. It also follows from the assumptions that H is compact. Therefore the nonzero singular values of H equal the square roots of the nonzero eigenvalues of the product $L_B L_C$ (this can be proven as in [4, Lemma 8.2.9]). The nonzero eigenvalues of $L_B L_C$ equal the nonzero eigenvalues of $L_B^{1/2} L_C L_B^{1/2}$ and since this operator is self-adjoint, these are equal to the nonzero singular values of $L_B^{1/2} L_C L_B^{1/2}$. So for all $k \in \mathbb{N}$

$$\sigma_k(H) = \sqrt{\sigma_k(L_B^{1/2} L_C L_B^{1/2})}.$$

It therefore suffices to show that there exists $\widetilde{M}, \widetilde{c} > 0$ such that for all $k \in \mathbb{N}$

$$\sigma_k(L_B^{1/2} L_C L_B^{1/2}) \leq \widetilde{M}\mathrm{e}^{-\widetilde{c}\sqrt{k}}.$$

If $X \in \mathcal{L}(\mathscr{X})$ has rank at most k, then $L_B^{1/2} X L_B^{1/2}$ also has rank at most k. We further have

$$\begin{aligned}&\|L_B^{1/2} L_C L_B^{1/2} - L_B^{1/2} X L_B^{1/2}\| \\ &\quad\leq \|L_B^{1/2}\|\;\|L_C - X\|\;\|L_B^{1/2}\| = \|L_B\|\;\|L_C - X\|.\end{aligned}$$

We conclude that for all $k \in \mathbb{N}$

$$\sigma_k(L_B^{1/2} L_C L_B^{1/2}) \leq \|L_B\|\sigma_k(L_C).$$

The singular value bound for L_C proven in Theorem 11 then gives the desired result. ■

We now give a corollary regarding Riccati equations. Note that condition (5) below is in particular satisfied in the "standard case" $\left[\begin{smallmatrix} Q & N \\ N^* & R\end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I\end{smallmatrix}\right]$. It is also satisfied for "strictly bounded real" and "strictly positive real" systems.

Corollary 13: Let T be an exponentially stable analytic semigroup as in Definition 1. Denote its generator by A. Let $\mathscr{U}$ be a Hilbert space and let $\mathscr{Y}$ be a finite-dimensional Hilbert space. Let $\alpha \in [0, 1/2)$ and let $C : D(A) \to \mathscr{Y}$ be such that $CA^{-\alpha} \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$. Let $\beta \in [0, 1/2)$ and let $B : \mathscr{U} \to D(A^*)'$ be such that $B^*A^{-\beta *} \in \mathcal{L}(\mathscr{X}, \mathscr{U})$. Let $D \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$. Let the weighting operator $\left[\begin{smallmatrix} Q & N \\ N^* & R\end{smallmatrix}\right] \in \mathcal{L}(\mathscr{Y} \times \mathscr{U})$ be self-adjoint and such that there exists a $\varepsilon > 0$ such that for all $u \in L^2(0,\infty;\mathscr{U})$

$$\begin{aligned}&\int_0^\infty \left\langle \begin{bmatrix} Q & N \\ N^* & R\end{bmatrix}\begin{bmatrix} y(t) \\ u(t)\end{bmatrix}, \begin{bmatrix} y(t) \\ u(t)\end{bmatrix}\right\rangle dt \\ &\qquad\qquad \geq \varepsilon \int_0^\infty \|u(t)\|^2 + \|y(t)\|^2\, dt, \quad (5)\end{aligned}$$

where y is the output for initial condition zero and input u for the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t).$$

Then the Riccati equation

$$\begin{aligned}XA + A^*X + C^*QC \\ &= (B^*X + D^*QC + N^*C)^* \cdot \\ &\quad (D^*QD + D^*N + N^*D + R)^{-1} \cdot \\ &\quad\quad (B^*X + D^*QC + N^*C),\end{aligned}$$

has a unique positive semidefinite solution $X \in \mathcal{L}(\mathscr{X})$ and this solution is such that there exist $M, c > 0$ such that for all $k \in \mathbb{N}$

$$\sigma_k(X) \leq M\mathrm{e}^{-c\sqrt{k}}.$$

Proof: It follows from [24, Theorem 5.7.3] that (A, B, C, D) generate a regular linear system. The existence of a unique positive semidefinite solution of the Riccati equation then follows from [27] or [23]. We note that for the special class of systems considered here, the Riccati equation given above coincides with the Riccati equations from [27] and [23] (see [22]). From [23, Proposition 7.2] we obtain that

$$X = \Psi^* \left[Q - (Q\mathbb{F} + N)\mathcal{R}^{-1}(\mathbb{F}^*Q + N^*) \right] \Psi, \qquad (6)$$

where $\Psi \in \mathcal{L}(\mathscr{X}, L^2(0, \infty; \mathscr{Y}))$ is the output map, $\mathbb{F} \in \mathcal{L}(L^2(0, \infty; \mathscr{U}), L^2(0, \infty; \mathscr{Y}))$ is the input-output map and $\mathcal{R} \in \mathcal{L}(L^2(0, \infty; \mathscr{U}), L^2(0, \infty; \mathscr{U}))$ is the Popov-Toeplitz operator defined by

$$\mathcal{R} := R + N^*\mathbb{F} + \mathbb{F}^*N + \mathbb{F}^*Q\mathbb{F}.$$

We note that the observability Gramian of the system equals $L := \Psi^*\Psi$. By Theorem 11 we have that the singular values of L decay exponentially in the square root. Since the nonzero singular values of Ψ equal the square roots of the nonzero eigenvalues of $\Psi^*\Psi$, and since $\Psi^*\Psi$ is self-adjoint implies that its eigenvalues equal its singular values, we see that the singular values of Ψ decay exponentially in the square root. By (6) we have that X equals a bounded operator times an operator whose singular values decay exponentially in the square root. It follows that the singular values of X decay exponentially in the square root. To see this, consider a bounded operator S and an operator T whose singular values decay exponentially in the square root. Then there exist $M, c > 0$ such that for all $k \in \mathbb{N}$ there exists an operator T_k of rank less than k such that

$$\|T - T_k\| \leq M\mathrm{e}^{-c\sqrt{k}}.$$

It follows that ST_k has rank less than k and that

$$\|ST - ST_k\| \leq \|S\| \; \|T - T_k\| \leq \|S\| M\mathrm{e}^{-c\sqrt{k}},$$

so that the singular values of ST decay exponentially in the square root. ■

Remark 14: We note that a result similar to Corollary 13 also holds when the system is exponentially stabilizable (in a suitable sense) rather than exponentially stable. However, a proof of that result requires some infinite-dimensional control theory which is beyond the scope of this article; therefore this result will be presented elsewhere.

V. Examples

For some examples where the conditions of Theorem 11 and Corollary 12 are satisfied we refer to [18, Section 3]. In this section we give some examples where the singular values do not decay at a rate exponential in the square root, highlighting the importance of the assumptions in Theorem 11 and Corollary 12.

We first consider what may happen if $\mathscr{U}$ and $\mathscr{Y}$ are allowed to be infinite-dimensional.

Example 15: If A is a negative self-adjoint operator with a bounded inverse on a Hilbert space $\mathscr{X}$ and $C = I$ (with therefore $\mathscr{Y} = \mathscr{X}$), then the observability Gramian equals $-\frac{1}{2}A^{-1}$. Using this, by a suitable choice of $\mathscr{X}$ and A, the singular values of the observability Gramian can made equal to any desired sequence of nonnegative numbers. By choosing $B = I$, the same is true for the Hankel singular values.

Note that the assumptions of Theorem 11 and Corollary 12 are not satisfied in this case because both $\mathscr{Y}$ and $\mathscr{U}$ are allowed to be infinite-dimensional in this example. The other assumptions of Theorem 11 and Corollary 12 are satisfied.

As the following example shows, even for one-dimensional $\mathscr{Y}$ and $\mathscr{U}$ arbitrary singular values can be obtained.

Example 16: This example is taken from [17]. Let $(\sigma_k)_{k=1}^\infty$ be a strictly decreasing sequence of positive numbers. Define

$$c_k := \frac{\sigma_k}{k}, \quad A_{ij} := \frac{-c_i c_j}{\sigma_i + \sigma_j}.$$

Then c defines a bounded operator $C : \ell^2(\mathbb{N}) \to \mathbb{C}$ by $Cx := \langle x, c \rangle$ and A defines a bounded negative self-adjoint operator on $\ell^2(\mathbb{N})$. The observability Gramian of the pair A, C is represented by the diagonal matrix with diagonal entries σ_k. Therefore the singular values of the observability Gramian equal the given sequence $(\sigma_k)_{k=1}^\infty$. If we choose $B = C^*$, then the Hankel operator of the triple A, B, C has singular values equal to the given sequence $(\sigma_k)_{k=1}^\infty$.

In this example all the conditions of Theorem 11 and Corollary 12 are satisfied except for the exponential stability assumption. The semigroup generated by A is strongly stable, but not exponentially stable.

With a slightly more complicated construction the same conclusions can be drawn if $(\sigma_k)_{k=1}^\infty$ is a non-increasing sequence of positive numbers, see [16].

We note that this example can be easily adapted to show that any finite non-increasing sequence of positive numbers $(\sigma_k)_{k=1}^n$ can occur as the singular values of the Gramian of a finite-dimensional system with $\dim \mathscr{X} = n$ and $\dim \mathscr{U} = 1$ and as the sequence of nonzero Hankel singular values of a system with $\dim \mathscr{X} = n$ and $\dim \mathscr{U} = 1 = \dim \mathscr{Y}$.

The following example illustrates that B and C should not be "too unbounded" for the singular values to decay at a rate exponential in the square root.

Example 17: We consider the following one-dimensional heat equation on the unit interval

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2},$$
$$w(t,0) = -u(t), \quad w(t,1) = 0, \quad y(t) = \frac{\partial w}{\partial \xi}(t,0).$$

In this case $CA^{-\alpha}$ is not in $\mathcal{L}(\mathscr{X}, \mathscr{Y})$ for any $\alpha < 3/4$. Therefore, one of the conditions of Theorem 11 is not satisfied; the other conditions of Theorem 11 are satisfied. It can be shown that the observability Gramian is unbounded. The Hankel operator is also unbounded. In particular, all the singular values of the observability Gramian and all the Hankel singular values are infinite.

The next example is from [19]. It illustrates the importance of the analyticity of the semigroup.

Example 18: Consider the following first order damped hyperbolic PDE with dynamic boundary control and observation.

$$\frac{\partial w}{\partial t} = -\frac{\partial w}{\partial \xi} - \varepsilon w, \quad t > 0, \quad \xi \in (0,1),$$
$$w_t(t,0) + w(t,0) = u(t),$$
$$z_t(t) + z(t) = w(t,1),$$
$$y(t) = z(t).$$

The transfer function of this system is

$$G(s) = \frac{1}{(s+1)^2} \mathrm{e}^{-(s+\varepsilon)}.$$

This system satisfies all the assumptions of Corollary 12 (with $\alpha = \beta = 0$) except that the exponentially stable semigroup generated by A isn't analytic. Using the explicit description of the transfer function, it follows from [7] that for $(\sigma_k)_{k=1}^{\infty}$ the Hankel singular values

$$k^2 \sigma_k \to C \neq 0.$$

In particular, the Hankel singular values of this system do not decay at a rate exponential in $\sqrt{k}$.

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