The algebraic Riccati equation for infinite-dimensional systems

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***Abstract*— It is well-known that the algebraic Riccati equation for infinite-dimensional systems with very unbounded control operators in its classical form is not well-defined. Several alternatives have been proposed in the literature. We review some of these and give special emphasis to the recently proposed *operator node Riccati equation*, which is a generalization of the Lur'e form of the algebraic Riccati equation. We show how the other reviewed alternative Riccati equations can be obtained from this operator node Riccati equation.**

I. INTRODUCTION

The (control) algebraic Riccati equation associated to the continuous time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0, \qquad\qquad (1) \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

with $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{p \times n}$, $D \in \mathbb{C}^{p \times m}$ is

$$\begin{aligned} &A^*X + XA + C^*C \qquad (2) \\ &\quad = (XB + C^*D)(I + D^*D)^{-1}(B^*X + D^*C). \end{aligned}$$

One of the reasons for the importance of this algebraic Riccati equation is that its smallest nonnegative definite solution is the optimal cost operator for the classical linear quadratic optimal control problem:

$$\langle Xx_0, x_0 \rangle = \inf_{u \in L^2(0,\infty;\mathbb{C}^m)} \int_0^\infty \|u(t)\|^2 + \|y(t)\|^2 \, dt.$$

Infinite-dimensional systems (i.e. systems for which the state space is not $\mathbb{C}^n$, but is some possibly infinite-dimensional Hilbert space) arise from, for example, partial differential equations and delay differential equations. Also for infinite-dimensional systems the linear quadratic optimal control problem is of great importance. As in the finite-dimensional case, the algebraic Riccati equation often plays an important role in the numerical solution of the linear quadratic optimal control problem. If A generates a strongly continuous semigroup on the Hilbert space $\mathscr{X}$, $B : \mathscr{U} \to \mathscr{X}$, $C : \mathscr{X} \to \mathscr{Y}$ and $D : \mathscr{U} \to \mathscr{Y}$ are bounded operators and the finite cost condition holds, i.e. for all $x_0 \in \mathscr{X}$ there exists a $u \in L^2(0,\infty;\mathscr{U})$ such that $y \in L^2(0,\infty;\mathscr{Y})$, then the classical algebraic Riccati equation (2) is still the correct equation when considered in the weak form [1, Theorem 6.2.4]: for all $x \in D(A)$

$$\begin{aligned} &\langle Xx, Ax \rangle + \langle Ax, Xx \rangle + \langle Cx, Cx \rangle \qquad (3) \\ &= \langle (I + D^*D)^{-1}(B^*X + D^*C)x, (B^*X + D^*C)x \rangle. \end{aligned}$$

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However, if B is very unbounded, then it may happen that with X the optimal cost operator, $Xx \notin D(B^*)$ for some $x \in D(A)$ and therefore the weak form (3) of the algebraic Riccati equation doesn't even make sense. We briefly review the classical example where this occurs in Section II. Several alternative Riccati equations have been proposed in the literature. In this paper we concentrate on the *operator node Riccati equation* recently obtained in [2]. We show how it is the natural generalization of the Lur'e form of the Riccati equation (Section III), we illustrate it using the earlier considered example (Section IV) and we show how other alternative Riccati equations proposed in the literature may be obtained from the operator node Riccati equation (Section V).

II. REVIEW OF AN EXAMPLE

The classical example which shows that the weak form (3) of the classical form of the algebraic Riccati equation has insurmountable problems when B is very unbounded (while C is bounded) is given in [3]. We briefly present a trivial variation on this. Firstly, we consider the optimal control problem

$$\inf_{u \in L^2(0,\infty;\mathbb{C}^m)} \int_0^\infty \langle Ru(t), u(t) \rangle + \|y(t)\|^2 \, dt,$$

where $R > 0$ for which the weak form of the Riccati equation becomes

$$\begin{aligned} &\langle Xx, Ax \rangle + \langle Ax, Xx \rangle + \langle Cx, Cx \rangle \qquad (4) \\ &= \langle (R + D^*D)^{-1}(B^*X + D^*C)x, (B^*X + D^*C)x \rangle, \end{aligned}$$

(in [3] the case $R = I$ was considered). Secondly, we consider a bounded spatial interval $(0, 2)$ rather than the semi-infinite spatial interval $(0, \infty)$ considered in [3] (the reason for making this modification is that a bounded spatial interval is more convenient for numerical calculations; these however are not presented here). The PDE considered is

$$\begin{aligned} &\frac{dw}{dt}(t,\xi) = \frac{dw}{d\xi}(t,\xi), \quad t > 0, \quad \xi \in (0,1) \cup (1,2), \\ &w(t,1^-) - w(t,1^+) = u(t), \quad t > 0, \\ &w(t,2) = 0, \quad t > 0, \\ &w(0,\xi) = w_0(\xi), \quad \xi \in (0,2), \\ &y(t) = \xi \mapsto w(t,\xi)|_{(0,1)}. \end{aligned}$$

The optimal control problem is easy to solve explicitly in a way entirely similar to that used in [3]. The optimal control is given by

$$u^{\mathrm{opt}}(t) = \begin{cases} \frac{-1}{1+R} w_0(t+1) & t \in (0,1), \\ 0 & t \geq 1. \end{cases}$$

The optimal trajectory is given by the following formulas. For $\xi \in (0,1)$ we have

$$w^{\mathrm{opt}}(t,\xi) = \begin{cases} w_0(\xi+t) & \xi+t \in (0,1), \\ \frac{R}{1+R} w_0(\xi+t) & \xi+t \in (1,2), \\ 0 & \xi+t > 2, \end{cases}$$

and for $\xi \in (1,2)$ we have

$$w^{\mathrm{opt}}(t,\xi) = \begin{cases} w_0(\xi+t) & \xi+t \in (1,2), \\ 0 & \xi+t > 2. \end{cases}$$

Note that we have for $t \in (0,1)$ that

$$\begin{aligned} w^{\mathrm{opt}}(t,1^-) &= \frac{R}{1+R} w_0(1+t), \\ w^{\mathrm{opt}}(t,1^+) &= w_0(1+t), \end{aligned}$$

in particular, $\xi \mapsto w^{\mathrm{opt}}(t,\xi)$ is not continuous at $\xi = 1$ for $t \in (0,1)$. The optimal cost operator $X : L^2(0,2) \to L^2(0,2)$ is given by

$$(Xf)(\xi) = p(\xi) f(\xi),$$

where

$$p(\xi) = \begin{cases} \xi & \xi \in (0,1), \\ \frac{R}{1+R} & \xi \in (1,2). \end{cases}$$

Note that p is not continuous at $\xi = 1$.

The considered PDE can be written in abstract state space form as follows. The state space is $\mathscr{X} := L^2(0,2)$, the input space $\mathscr{U}$ is one-dimensional, the output space is $\mathscr{Y} := L^2(0,1)$ and we have

$$\begin{aligned} &Af = f', \\ &D(A) = \left\{ f \in L^2(0,2) : f' \in L^2(0,2),\ f(2) = 0 \right\}, \\ &Cf = f|_{[0,1]}, \\ &D(C) = L^2(0,2), \\ &D = 0, \\ &B^* f = f(1), \\ &D(B^*) = \left\{ f \in L^2(0,2) : f' \in L^2(0,2),\ f(0) = 0 \right\}. \end{aligned}$$

Because p is discontinuous at $\xi = 1$, there exist $x \in D(A)$ such that $Xx \notin D(B^*)$. It follows that the weak form (4) of the Riccati equation is not well-defined.

We compute the left-hand side of (4) (which is well-defined): For $x \in D(A)$ we have

$$\begin{aligned} &\langle Xx, Ax \rangle + \langle Ax, Xx \rangle + \langle Cx, Cx \rangle \\ &= 2\mathrm{Re} \int_0^2 p(\xi) x(\xi) \overline{x'(\xi)}\, d\xi + \int_0^1 |x(\xi)|^2\, d\xi \\ &= 2\mathrm{Re} \int_0^1 \xi x(\xi) \overline{x'(\xi)}\, d\xi + 2\mathrm{Re} \int_1^2 \frac{R}{1+R} x(\xi) \overline{x'(\xi)}\, d\xi \\ &\qquad + \int_0^1 |x(\xi)|^2\, d\xi \\ &= \left[\xi |x(\xi)|^2 \right]_0^1 - \int_0^1 |x(\xi)|^2\, d\xi \\ &\qquad + \left[\frac{R}{1+R} |x(\xi)|^2 \right]_1^2 + \int_0^1 |x(\xi)|^2\, d\xi \\ &= |x(1)|^2 - \frac{R}{1+R} |x(1)|^2 = \frac{1}{1+R} |x(1)|^2. \end{aligned}$$

The operator B^* has two 'obvious' extensions $\tilde{B}^*$: $\tilde{B}^* f := f(1^-)$ and $\tilde{B}^* f := f(1^+)$. With these choices the right-hand side of (4) equals

$$\langle R^{-1} \tilde{B}^* X x, \tilde{B}^* X x \rangle = \begin{cases} \frac{1}{R} |x(1)|^2 & \tilde{B}^* f := f(1^-), \\ \frac{R}{(1+R)^2} |x(1)|^2 & \tilde{B}^* f := f(1^+). \end{cases}$$

We note that for no $R > 0$ does this equal the left-hand side of (4). As mentioned in [3], a correct Riccati equation is obtained by replacing B^* in (4) by $\tilde{B}^* f := f(1^-)$ (the Yosida extension of B^*) *and* replacing R in (4) by $R+1$. We come back to this issue later in this paper.

III. THE OPERATOR NODE RICCATI EQUATION

A Riccati equation theory using the concept of a system node (or more generally: an operator node) was recently developed in [2]. We first very briefly review the notion of system node; see [4, Section 4.7] for more information. The basic idea is to re-write the dynamical system (1) as

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

and consider $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as an operator from $\left[\begin{smallmatrix} \mathscr{X} \\ \mathscr{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathscr{X} \\ \mathscr{Y} \end{smallmatrix}\right]$ whose domain is not necessarily a product space. The following definition is taken from [4, Definition 1.1.1].

Definition 1: Let $\mathscr{X}, \mathscr{U}, \mathscr{Y}$ be Banach spaces. An operator $S : D(S) \subset \left[\begin{smallmatrix} \mathscr{X} \\ \mathscr{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathscr{X} \\ \mathscr{Y} \end{smallmatrix}\right]$ is called a *system node* if

1) S is a closed operator;
2) if we split $S = \left[\begin{smallmatrix} S_{\mathscr{X}} \\ S_{\mathscr{Y}} \end{smallmatrix}\right]$ in accordance with the splitting of the range space $\left[\begin{smallmatrix} \mathscr{X} \\ \mathscr{Y} \end{smallmatrix}\right]$, then $S_{\mathscr{X}}$ with domain $D(S)$ is closed;
3) the operator A defined by $Ax := S_{\mathscr{X}} \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ with domain $D(A) = \{x \in \mathscr{X} : \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in D(S)\}$ is the generator of a strongly continuous semigroup on $\mathscr{X}$;
4) for every $u \in \mathscr{U}$ there exists a x such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(S)$.

Like in [4], it is convenient to use the notation $S_{\mathscr{X}} =: A\&B$ and $S_{\mathscr{Y}} =: C\&D$.

For the PDE example from Section II, the system node is given by

$$S\begin{bmatrix}x\\u\end{bmatrix}=\begin{bmatrix}x'\\x|_{(0,1)}\end{bmatrix},$$
$$D(S)=\left\{\begin{bmatrix}x\\u\end{bmatrix}\in\begin{bmatrix}L^2(0,2)\\\mathbb{C}\end{bmatrix}: x|'_{(0,1)}\in L^2(0,1),\right.$$
$$\left. x|'_{(1,2)}\in L^2(1,2),\ x(1^-)-x(1^+)=u,\quad x(2)=0\right\},$$

where the derivative in the definition of S has to be understood as $x':=x'_{(0,1)}+x'_{(1,2)}$.

To motivate the operator node Riccati equation, we first recall what is essentially its finite-dimensional equivalent: the Lur'e form of the Riccati equation (usually only used for singular optimal control problems). The operator X is called a solution of the Lur'e form of the Riccati equation if there exist operators K and L such that

$$\begin{aligned}A^*X+XA+C^*C&=K^*K,\\B^*X+D^*C&=L^*K,\\R+D^*D&=L^*L.\end{aligned}$$

The operator L can be chosen as $(R+D^*D)^{1/2}$, and if $R+D^*D$ is invertible, then so is L and K must equal $(R+D^*D)^{-1/2}(B^*X+D^*C)$ by the second of the Lur'e equations. Upon substitution, the first of the Lur'e equations then becomes the standard Riccati equation. The Lur'e equations are mostly studied for the case when $R+D^*D$ is not invertible, but in our case they provide a convenient form even for the case where $R+D^*D$ is invertible since they generalize to system nodes. We first note that the Lur'e equations can equivalently be written in the weak matrix form

$$\left\langle\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix},Xx\right\rangle+\left\langle Xx,\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix}\right\rangle$$
$$+\|\begin{bmatrix}C & D\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix}\|^2+\langle Ru,u\rangle=\|\begin{bmatrix}K & L\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix}\|^2.$$

The generalization to system nodes considered in [2] is then immediate.

Definition 2: Let S be a system node and $R=R^*$. The operators $X=X^*\in\mathcal{L}(\mathscr{X})$ and $K\&L:D(S)\to\mathscr{U}$ are called a solution of the *operator node Riccati equation* for S if:

$$\left\langle A\&B\begin{bmatrix}x\\u\end{bmatrix},Xx\right\rangle+\left\langle Xx,A\&B\begin{bmatrix}x\\u\end{bmatrix}\right\rangle+\left\|C\&D\begin{bmatrix}x\\u\end{bmatrix}\right\|^2$$
$$+\langle Ru,u\rangle=\left\|K\&L\begin{bmatrix}x\\u\end{bmatrix}\right\|^2,\quad\begin{bmatrix}x\\u\end{bmatrix}\in D(S).$$

Remark 3: Note that compared to [2, Definition 5.1 and Remark 5.2] we have dropped the condition that for some α larger than the growth bound of the semigroup generated by A the operator

$$K\&L\begin{bmatrix}(\alpha-A)^{-1}B\\I\end{bmatrix},$$

is invertible in $\mathcal{L}(\mathscr{U})$. This condition is in fact satisfied for the example from Section II.

IV. THE OPERATOR NODE RICCATI EQUATION FOR THE EXAMPLE

It is proven in [2] that the optimal cost operator gives rise to a solution of the operator node Riccati equation. However, for the example from Section II it is easy and instructive to verify this directly. We have for $\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]\in D(S)$

$$\left\langle A\&B\begin{bmatrix}x\\u\end{bmatrix},Xx\right\rangle+\left\langle Xx,A\&B\begin{bmatrix}x\\u\end{bmatrix}\right\rangle$$
$$+\left\|C\&D\begin{bmatrix}x\\u\end{bmatrix}\right\|^2+\langle Ru,u\rangle$$
$$=2\mathrm{Re}\int_0^2 p(\xi)x(\xi)\overline{x'(\xi)}\,d\xi+\int_0^1|x(\xi)|^2\,d\xi+R|u|^2.$$

Elementary computations show that this equals

$$|x(1^-)|^2-\frac{R}{1+R}|x(1^+)|^2+R|u|^2.$$

Using that since $\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]\in D(S)$ we have $x(1^-)-x(1^+)=u$, the above can be written as

$$\left|\frac{1}{\sqrt{1+R}}x(1^+)+\sqrt{1+R}\ u\right|^2.$$

It follows that we can choose

$$K\&L\begin{bmatrix}x\\u\end{bmatrix}=\frac{1}{\sqrt{1+R}}x(1^+)+\sqrt{1+R}\ u,$$

to satisfy the operator node Riccati equation.

Note that setting $K\&L\begin{bmatrix}x\\u\end{bmatrix}=0$ leads to $\frac{1}{\sqrt{1+R}}x(1^+)+\sqrt{1+R}\ u=0$, which can be solved for u to give the feedback

$$u=\frac{-1}{1+R}x(1^+),$$

which is the same optimal feedback as obtained in [3] and as obtained by direct computation of the optimal control and optimal state in Section II.

V. THE CONNECTION WITH OTHER RICCATI EQUATIONS

To make the connection with other alternative Riccati equations proposed in the literature, we consider the more general operator node Riccati equation

$$\left\langle A\&B\begin{bmatrix}x\\u\end{bmatrix},Xx\right\rangle+\left\langle Xx,A\&B\begin{bmatrix}x\\u\end{bmatrix}\right\rangle+$$
$$\left\langle\begin{bmatrix}Q & N\\N^* & R\end{bmatrix}\begin{bmatrix}C\&D\begin{bmatrix}x\\u\end{bmatrix}\\u\end{bmatrix},\begin{bmatrix}C\&D\begin{bmatrix}x\\u\end{bmatrix}\\u\end{bmatrix}\right\rangle$$
$$=\left\|K\&L\begin{bmatrix}x\\u\end{bmatrix}\right\|^2,\quad\begin{bmatrix}x\\u\end{bmatrix}\in D(S).$$

Here $S:=\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is a system node; $Q=Q^*\in\mathcal{L}(\mathscr{Y})$, $N\in\mathcal{L}(\mathscr{U},\mathscr{Y})$ and $R=R^*\in\mathcal{L}(\mathscr{U})$ are weighting operators (above we had $Q=I$ and $N=0$). The unknowns are $X=X^*\in\mathcal{L}(\mathscr{X})$ and $K\&L:D(S)\to\mathscr{U}$.

Apart from Section V-A we also assume in the remainder that the system node is L^2 *well-posed* and that the *Popov function*

$$\Phi(s_1, s_2) := \begin{bmatrix} G(s_2) \\ I \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} G(s_1) \\ I \end{bmatrix}.$$

is coercive, since these are the assumptions under which the discussed alternative Riccati equations were derived.

A. The resolvent Riccati equation

We note that every system node has a control operator $B \in \mathcal{L}(\mathscr{U}, \mathscr{X}_{-1})$, where $\mathscr{X}_{-1} \supset \mathscr{X}$ is a certain extrapolation space (this follows from [4, Lemma 4.7.7]). It is easily shown that for any $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D(S)$ and $s \in \rho(A)$ there exists a unique $z \in D(A)$ such that $x = z + (s - A)^{-1}Bu$. It is trivially seen that all elements $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ of this form are in $D(S)$. This all follows from [4, Lemma 4.7.3viii].

We also note that the transfer function of a system node is defined by $G(s) := C\&D \begin{bmatrix} (s - A)^{-1}B \\ I \end{bmatrix}$.

We substitute elements $\begin{bmatrix} z_i + (s - A)^{-1}Bu_i \\ u_i \end{bmatrix}$ (with $i = 1, 2$) in the operator node Riccati equation to obtain three equations that are together equivalent to the operator node Riccati equation.

The first of these corresponds to the choice $u_1 = u_2 = 0$ and is a Lyapunov equation:

$$\langle Az, Xz \rangle + \langle Xz, Az \rangle + \langle QCz, Cz \rangle = \|Kz\|^2, \quad z \in D(A). \tag{5}$$

The second corresponds to the choice $z_1 = z_2 = 0$ and is

$$\begin{aligned} &(s + \bar{s})\langle (s - A)^{-1}Bu, X(s - A)^{-1}Bu \rangle \\ &\quad + \left\langle \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} G(s)u \\ u \end{bmatrix}, \begin{bmatrix} G(s)u \\ u \end{bmatrix} \right\rangle \\ &\quad = \|\mathcal{X}(s)u\|^2, \quad u \in \mathscr{U}, \end{aligned} \tag{6}$$

where we have used $A(s - A)^{-1}B + B = s(s - A)^{-1}B$ and where $\mathcal{X}$ is the transfer function of $\left[\begin{smallmatrix} A\&B \\ K\&L \end{smallmatrix}\right]$.

The third equation is obtain by the choice $u_1 = 0 = z_2$ and is

$$\begin{aligned} &\langle Az, X(s - A)^{-1}Bu \rangle + \langle Xz, s(s - A)^{-1}Bu \rangle \\ &\quad + \left\langle \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} Cz \\ 0 \end{bmatrix}, \begin{bmatrix} G(s)u \\ u \end{bmatrix} \right\rangle = \\ &\quad \langle Kz, \mathcal{X}(s)u \rangle, \quad z \in D(A), \ u \in \mathscr{U}. \end{aligned} \tag{7}$$

It follows that the operator node Riccati equation is equivalent to the three equations (5), (6), (7) combined.

B. The spectral factor

If the semigroup generated by A is exponentially stable, so that the imaginary axis is contained in the resolvent set of A, then (6) implies that

$$\Phi(i\omega, i\omega) = \mathcal{X}(i\omega)^* \mathcal{X}(i\omega),$$

so that if $\mathcal{X}$ and its inverse are in H^∞, $\mathcal{X}$ is a spectral factor of the Popov function. Since the Popov function is assumed coercive, $\mathcal{X}$ as defined in Section V-A is indeed a spectral factor.

C. The Weiss^2–Staffans Riccati equation

We note that a transfer function G of a system node is called *regular* if $\lim_{s\to\infty} G(s)$ exists, where the limit is taken along the positive real axis (for infinite-dimensional input and output spaces, we need to make the distinction between uniformly, strongly and weakly regular). See [4, Section 5.6].

If the spectral factor $\mathscr{X}$ is weakly regular and the system transfer function G is weakly regular, then we can let $s \to \infty$ in (7) to obtain

$$\begin{aligned} &\langle B_w^* Xx, u \rangle + \langle QCx, Du \rangle + \langle N^* Cx, u \rangle \\ &\quad = \langle Kx, Lu \rangle, \quad x \in D(A), \ u \in \mathscr{U}, \end{aligned} \tag{8}$$

where we have used that $(s - A)^{-1}B \to 0$ and the definition of the Yosida extension of B^* [4, Definition 5.4.1]:

$$B_w^* := \lim_{s\to\infty} B^* s(s - A^*)^{-1}.$$

In (6) we consider the case where on the left we use s_1 and on the right we use s_2 (the derivation of this from the operator node Riccati equation is completely analogues). This gives instead of (6)

$$\begin{aligned} &(s_1 + \bar{s}_2)\langle (s_1 - A)^{-1}Bu, X(s_2 - A)^{-1}Bu \rangle \\ &\quad + \left\langle \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} G(s_1)u \\ u \end{bmatrix}, \begin{bmatrix} G(s_2)u \\ u \end{bmatrix} \right\rangle \\ &\quad = \langle \mathcal{X}(s_1)u, \mathcal{X}(s_2)u \rangle, \quad u \in \mathscr{U}. \end{aligned}$$

We first let s_1 go to infinity and then s_2. Note that $\bar{s}_2\langle (s_1 - A)^{-1}Bu, X(s_2 - A)^{-1}Bu \rangle \to 0$ when $s_1 \to \infty$ since $(s_1 - A)^{-1}Bu \to 0$. We therefore obtain

$$\begin{aligned} &\lim_{s\to\infty} \langle u, B_w^* X(s - A)^{-1}Bu \rangle + \left\langle \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} Du \\ u \end{bmatrix}, \begin{bmatrix} Du \\ u \end{bmatrix} \right\rangle \\ &\quad = \langle Lu, Lu \rangle, \quad u \in \mathscr{U}. \end{aligned} \tag{9}$$

Note that the first term is generally not equal to zero. This amounts to "changing" R when compared to the standard case when writing down the Riccati equation.

Combining (5) and (8) we obtain, noting that L is invertible since the Popov function is coercive,

$$\begin{aligned} &\langle Az, Xz \rangle + \langle Xz, Az \rangle + \langle QCz, Cz \rangle = \\ &\quad \left\langle (LL^*)^{-1} T_X z, T_X z \right\rangle, \quad z \in D(A), \end{aligned}$$

where

$$T_X := B_w^* X + D^* QC + N^* C.$$

This Riccati equation was obtained in [5, Theorem 12.8] and in [6]. The equation (9) for the feedthrough of the spectral factor appeared in [6, Corollary 7.2].

It is this Riccati equation which was used in [3] for the example considered in Section II (for the semi-infinite spatial interval). In that example R has to be replaced by $R + 1$ in the Riccati equation because, contrary to the finite dimensional case, $LL^* \neq R$ but instead $LL^* = R + 1$ for this example. Equivalently, in this example $\lim_{s\to\infty} \langle u, B_w^* X(s - A)^{-1}Bu \rangle = \|u\|^2$ and not equal to zero (as it would be for a finite dimensional system).

D. The Grabowski Riccati equation

Recently Grabowski [7] obtained a Riccati equation which is “astonishingly not the same” [7, page 38] as that obtained in [5]. We show that the Grabowski Riccati equation corresponds to taking $s = 0$ in the resolvent Riccati equation (5), (6), (7) (as noted, the Weiss^2–Staffans Riccati equation corresponds to $s = +\infty$). As Grabowski, we assume here that the semigroup generated by A is exponentially stable. For $s = 0$, equation (6) is

$$\mathcal{X}(0)^*\mathcal{X}(0) = R + G(0)^*N + NG(0) + G(0)^*QG(0),$$

where we note that the right-hand side is denoted R_- by Grabowski and is invertible since the Popov function is assumed to be coercive. From (7) with $s = 0$ we then obtain (this solution may not be unique, but the non-uniqueness would cancel out later)

$$K = R_-^{-1/2}(-B^*A^{-*}XA + G(0)^*QC + N^*C),$$

where we note that $A^{-1}B$ is denoted $\mathcal{D}$ by Grabowski and $N + QG(0)$ is denoted N_- by Grabowski. Substituting this in (5) gives for $z \in D(A)$

$$\begin{aligned}&\langle Az, Xz\rangle + \langle Xz, Az\rangle + \langle QCz, Cz\rangle\\ &\quad= \langle R_-^{-1}(-\mathcal{D}^*XA + N_-^*C)z, (-\mathcal{D}^*XA + N_-^*C)z\rangle,\end{aligned}$$

which is exactly the Riccati equation obtained by Grabowski: [7, Equation (2.7)].

VI. Conclusions

If the control operator is sufficiently unbounded, then it is known that the usual Riccati equation fails to be well-defined. There are several alternative Riccati equations available in the literature. We have shown how two of these (the Weiss^2–Staffans Riccati equation and the Grabowski Riccati equation) can be obtained from the recently proposed *operator node Riccati equation*. This operator node Riccati equation is a generalization of the Lur’e form of the Riccati equation and appears to be the most natural (and general) of the available Riccati equations which continue to hold for very unbounded control operators.

References

[1] R. F. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*. New York: Springer-Verlag, 1995.

[2] M. R. Opmeer and O. J. Staffans, “Optimal control on the doubly infinite continuous-time axis and coprime factorizations,” *SIAM J. Control Optim.*, accepted for publication.

[3] G. Weiss and H. Zwart, “An example in linear quadratic optimal control,” *Systems Control Lett.*, vol. 33, no. 5, pp. 339–349, 1998. [Online]. Available: http://dx.doi.org/10.1016/S0167-6911(97)00126-6

[4] O. J. Staffans, *Well-posed linear systems*. Cambridge: Cambridge University Press, 2005.

[5] M. Weiss and G. Weiss, “Optimal control of stable weakly regular linear systems,” *Math. Control Signals Systems*, vol. 10, no. 4, pp. 287–330, 1997. [Online]. Available: http://dx.doi.org/10.1007/BF01211550

[6] O. J. Staffans, “Quadratic optimal control of well-posed linear systems,” *SIAM J. Control Optim.*, vol. 37, no. 1, pp. 131–164, 1999. [Online]. Available: http://dx.doi.org/10.1137/S0363012996314257

[7] P. Grabowski, “The LQ-controller synthesis problem for infinite-dimensional systems in factor form,” *Opuscula Math.*, vol. 33, no. 1, pp. 29–79, 2013. [Online]. Available: http://dx.doi.org/10.7494/OpMath.2013.33.1.29