# A counter-example to "Positive realness preserving model reduction with $\mathcal{H}_\infty$ norm error bounds"

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**Abstract**

We provide a counter example to the $\mathcal{H}_\infty$ error bound for the difference of a positive real transfer function and its positive real balanced truncation stated in "Positive realness preserving model reduction with $\mathcal{H}_\infty$ norm error bounds" IEEE Trans. Circuits Systems I Fund. Theory Appl. 42 (1995), no. 1, 23–29. The proof of the error bound is based on a lemma from an earlier paper "A tighter relative-error bound for balanced stochastic truncation." Systems Control Lett. 14 (1990), no. 4, 307–317, which we also demonstrate is false by our counter example. The main result of this paper was already known in the literature to be false. We state a correct $\mathcal{H}^\infty$ error bound for the difference of a proper positive real transfer function and its positive real balanced truncation and also an error bound in the gap metric.

## 1 Counter-example

Consider the following continuous time, time invariant SISO linear system on the state-space $\mathbb{C}^4$ :

$$\begin{aligned} M\dot{\mathtt{x}}(t) &= K\mathtt{x}(t) + L\mathtt{u}(t), \\ \mathtt{y}(t) &= H\mathtt{x}(t) + J\mathtt{u}(t), \end{aligned} \tag{1}$$

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where

$$
\begin{aligned}
M &= \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} \end{bmatrix}, \qquad L = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
K &= \begin{bmatrix} -4 & 4 & 0 & 0 \\ 4 & -8 & 4 & 0 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix}, \quad H = L^*, \\
J &= 0.01.
\end{aligned} \tag{2}
$$

The physical motivation for studying (1) comes from a finite element approximation of the heat equation

$$
\left.\begin{aligned} w_t(t,x) &= w_{xx}(t,x), \\ w(0,x) &= w_0(x), \\ w(t,1) &= 0, \end{aligned}\right\} \quad t \geq 0,\ x \in [0,1], \tag{3}
$$

with input $\mathtt{u}$ and output $\mathtt{y}$ satisfying

$$
\begin{aligned} \mathtt{u}(t) &:= w_x(t,0), \\ \mathtt{y}(t) &:= -w(t,0) + J w_x(t,0). \end{aligned} \tag{4}
$$

By setting $A := M^{-1}K$, $G = M^{-1}L$, we can rewrite (1) as

$$
\begin{aligned} \dot{\mathtt{x}}(t) &= A\mathtt{x}(t) + G\mathtt{u}(t), \\ \mathtt{y}(t) &= H\mathtt{x}(t) + J\mathtt{u}(t), \end{aligned} \tag{5}
$$

with transfer function

$$
Z(s) = J + H(sI - A)^{-1}G. \tag{6}
$$

Observe that the system with transfer function $Z - J$ is positive real as $P = M = P^* > 0$, $N = \sqrt{-2K}$ and $R = 0$ satisfy the positive real linear matrix equalities

$$
\begin{aligned} A^*P + PA &= -N^*N, \\ PG - H^* &= -N^*R, \\ 0 &= R^*R. \end{aligned} \tag{7}
$$

Therefore for $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$,

$$
\begin{aligned} &[(Z-J)(s)]^* + (Z-J)(s) \geq 0, \\ &\qquad \Rightarrow [Z(s)]^* + Z(s) \geq 2J > 0, \end{aligned}
$$

and so the system (5) is *extended strictly positive real.* It is easy to verify also that (6) is a minimal, and hence controllable and observable, realisation of $Z$. The positive real singular values of $\Sigma$ are

$$
\sigma_1 = 0.6640,\ \sigma_2 = 0.2927,\ \sigma_3 = 0.0487,\ \sigma_4 = 0.0036. \tag{8}
$$

The first order positive real balanced truncation of $\Sigma$ is

$$\hat{Z}(s) = \frac{0.01s + 12.74}{s + 51.97},$$

and the approximation error $\|Z - \hat{Z}\|_{\mathcal{H}^\infty}$ is 0.7648. However, the error bound provided in [3, Theorem 2] is

$$2J \sum_{i=2}^{4} \frac{2\sigma_i}{(1-\sigma_i)^2} \left(1 + \sum_{j=1}^{i-1} \frac{2\sigma_j}{1-\sigma_j}\right)^2 = 0.6509,$$

which is smaller than the error. Hence [3, Theorem 2] is false.

*Remark* 1. We remark that there is some confusion in the literature regarding the nomenclature balanced stochastic truncation (the term that was used in [3]). Originally balanced stochastic truncation of a positive real function $Z$ meant a reduced order triple $Z_r$, $V_r$, $W_r$ with $V_r$ and $W_r$ left and right spectral factors of $Z_r + Z_r^*$ respectively, which are obtained by balancing the minimal nonnegative definite solutions of the (primal and dual) positive real equations and truncating. Nowadays ([1, p. 229] or [4]) the term positive real balanced truncation is used for obtaining only $Z_r$ in this way, and the term balanced stochastic truncation is reserved for a generalization of obtaining $V_r$ from a function $V$ which can be seen as a left spectral factor of $Z + Z^*$. The matlab function bstmr (balanced stochastic truncation model reduction) for example only does the latter. The article [3] however pertains to what is now called positive real balanced truncation.

## 2 Explanation

The proof of [3, Theorem 2] fails because for our above example the bound (18) in [3] is false. Using the notation of [3] (note here only one state is truncated from $\Sigma$) it follows that

$$\|T_1\|_\infty = 4.0389 > 1.7692 = 2\sum_{i=1}^{3} \frac{\sigma_i^2}{1-\sigma_i^2}. \tag{9}$$

Their proof of bound (18) uses [6, Lemma 5], which is only proven in [6] under the assumptions (51) and (53) (using the numbering of [6]). However, the authors state that [6, Lemma 5] also holds when (51) and (54) are satisfied. The above example shows that this is false. Letting

$$S = T_1, \; P(s) = Q(s) = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3) =: \hat{\Pi},$$

then equations (51) and (54) from [6] hold with $A, B$ and $C$ replaced by $\hat{A}_1, \hat{B}_1$ and $\hat{C}_1$ (again, notation from [3]), but the conclusion fails as inequality (9) shows. In this instance,

$$\hat{A}_1^* \hat{\Pi} + \hat{\Pi} \hat{A}_1 + \hat{C}_1^* \hat{C}_1 \neq 0,$$

and so equation (53) of [6] does not hold.

Counter-examples to [6, Theorem 1], which also uses the flawed [6, Lemma 5] in its proof, can be found in Chen and Zhou [2] and Zhou *et al.* [7, p. 171]. It is not pointed out there, however, that the flaw to [6, Theorem 1] occurs in [6, Lemma 5].

# 3 A new error bound

We prove the following error bounds in [5]. The gap metric error-bound was proven independently by Timo Reis as well.

**Theorem 2.** *Let $G$ and $G_r$ denote the transfer functions of a minimal, asymptotically stable, positive real input-state-output system of McMillan degree $m$ and its positive real balanced truncation of McMillan degree $k$ respectively. Then*

$$\delta(G, G_r) \leq 2 \sum_{i=k+1}^{m} \sigma_i,$$

*where $\delta$ is the gap metric and*

$$\|G - G_r\|_{\mathcal{H}^\infty} \leq 2 \min \Big\{ (1 + \|G\|_{\mathcal{H}^\infty}^2)(1 + \|G_r\|_{\mathcal{H}^\infty}), \\ (1 + \|G\|_{\mathcal{H}^\infty})(1 + \|G_r\|_{\mathcal{H}^\infty}^2) \Big\} \sum_{i=k+1}^{m} \sigma_i,$$

*where $\sigma_i$ are the positive real singular values.*

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