# Representation of solutions of Riccati equations for well-posed systems

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***Abstract*— We give a representation of self-adjoint solutions of the control Riccati equation of a well-posed linear system. At this level of generality the appropriate Riccati equation is an integral Riccati equation. We assume that the Riccati equation has a strongly stabilizing and a strongly anti-stabilizing solution, and that the difference of these two solutions is coercive. We further assume that the uncontrolled dynamics are given by a strongly continuous group. Our representation is in terms of invariant subspaces of the stabilizing closed-loop semigroup.**

## I. Introduction

In this article we generalize a well-known result of Willems [8] –that gives a representation of self-adjoint solutions of the control Riccati equation– from finite-dimensional to infinite-dimensional systems. We do this for general time-invertible well-posed linear systems. Earlier efforts along the same lines for more restrictive classes of infinite-dimensional systems are [3], [2], [6] where also further references can be found.

At this level of generality the appropriate Riccati equation is an integral Riccati equation. We assume that the Riccati equation has a strongly stabilizing and a strongly anti-stabilizing solution, and that the difference of these two solutions is coercive. Further, a technical assumption on the so-called Popov-Toeplitz operators is needed. This technical assumption is automatically satisfied in most cases, including the case where the control operator is bounded.

Our main result is Theorem 13.

## II. Well-posed linear systems

In this section we briefly motivate the class of systems that we study, more information on this class of systems can be found in e.g. [7].

Initially consider the well-known finite-dimensional system for $t \geq 0$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0, \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

The solutions of this system are

$$\begin{aligned} x(t) &= \mathcal{A}(t)x_0 + \mathcal{B}(t)u(t), \\ y(t) &= \mathcal{C}(t)x_0 + \mathcal{D}(t)u(t), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{A}(t) \in \mathcal{L}(\mathscr{X}), \\ &\mathcal{B}(t) \in \mathcal{L}(L^2(\mathbb{R};\mathscr{U}), \mathscr{X}), \\ &\mathcal{C}(\cdot) \in \mathcal{L}(\mathscr{X}, L^2(\mathbb{R};\mathscr{Y})), \\ &\mathcal{D}(\cdot) \in \mathcal{L}(L^2(\mathbb{R};\mathscr{U}), L^2(\mathbb{R};\mathscr{Y})), \end{aligned}$$

are given by

$$\mathcal{A}(t) := \mathrm{e}^{tA}, \qquad \mathcal{B}(t)u := \int_0^t \mathrm{e}^{(t-\tau)A}Bu(\tau)d\tau,$$

$$[\mathcal{C}(t)x](\lambda) := \begin{cases} C\mathrm{e}^{A\lambda}x & \lambda \in [0,t] \\ 0 & \text{otherwise}, \end{cases}$$

$$[\mathcal{D}(t)u](\lambda) := \begin{cases} \int_0^\lambda C\mathrm{e}^{(\lambda-\tau)A}Bu(\tau)\,d\tau & \lambda \in [0,t] \\ 0 & \text{otherwise}. \end{cases}$$

These solution operators have certain properties. To describe these we need the following notation. Let $\mathbb{R}^- = (-\infty, 0)$, $\mathbb{R}^+ = [0,\infty)$, $J \subset \mathbb{R}$ and for any function $u$ defined on $\mathbb{R}$ denote

$$(\tau^t u)(s) = u(t+s) \quad \forall\ t,s \in \mathbb{R},$$

$$(\pi_J u)(s) = \begin{cases} u(s) & s \in J, \\ 0 & s \notin J. \end{cases}$$

The solution operators satisfy the following causality property: for all $t \geq 0$

$$\begin{aligned} \mathcal{B}(t)\pi_{[0,t)} &= \mathcal{B}(t), \qquad \pi_{[0,t)}\mathcal{C}(t) = \mathcal{C}(t), \\ \mathcal{D}(t)\pi_{[0,t)} &= \mathcal{D}(t), \qquad \pi_{[0,t)}\mathcal{D}(t) = \mathcal{D}(t), \end{aligned}$$

and time-invariance property: for all $s, t \geq 0$

$$\begin{aligned} \mathcal{A}(s+t) &= \mathcal{A}(s)\mathcal{A}(t), \\ \mathcal{B}(s+t) &= \mathcal{A}(s)\mathcal{B}(t) + \mathcal{B}(s)\tau^t, \\ \mathcal{C}(s+t) &= \mathcal{C}(t) + \tau^{-t}\mathcal{C}(s)\mathcal{A}(t), \\ \mathcal{D}(s+t) &= \mathcal{D}(t) + \tau^{-t}\mathcal{C}(s)\mathcal{B}(t) + \tau^{-t}\mathcal{D}(s)\tau^t. \end{aligned}$$

Moreover, $\mathcal{A}$ is strongly continuous: for all $x \in \mathscr{X}$

$$\mathcal{A}(\cdot)x \in C(\mathbb{R}^+;\mathscr{X}).$$

Now we allow $\mathscr{U}$, $\mathscr{Y}$ and $\mathscr{X}$ to be arbitrary Hilbert spaces. The operators $A$, $B$ and $C$ that appear in the differential equation (1) are usually unbounded and it is therefore more convenient to use an 'integral' representation of the system. A one-parameter family of four operators $\begin{bmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{bmatrix}$ that satisfies the above continuity, causality and time-invariance properties is called an $L^2$ **well-posed linear system**. This is the class of systems that we will consider. Many delay differential equations and partial differential equations fit into this framework.

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## III. Riccati equations

The well-known control algebraic Riccati equation for the finite-dimensional system (1) is

$$A^*Q + QA + C^*C = QBB^*Q. \tag{2}$$

In terms of the solution operators this equation can be rewritten as for $t \geq 0$

$$\mathcal{K}(t)^*\mathcal{S}(t)\mathcal{K}(t) = \mathcal{A}(t)^*Q\mathcal{A}(t) - Q + \mathcal{C}(t)^*\mathcal{C}(t), \tag{3}$$

$$\mathcal{S}(t) = I + \mathcal{D}(t)^*\mathcal{D}(t) + \mathcal{B}(t)^*Q\mathcal{B}(t), \tag{4}$$

$$\mathcal{S}(t)\mathcal{K}(t) = -\mathcal{D}(t)^*\mathcal{C}(t) - \mathcal{B}(t)^*Q\mathcal{A}(t). \tag{5}$$

It is this (integral) control Riccati equation that we study for (possibly infinite-dimensional) well-posed linear systems. The operators $\mathcal{K}(t)$ appearing in this formulation of the Riccati equation are state feedback operators that satisfy $\mathcal{K}(\cdot) \in \mathcal{L}(\mathscr{X}, L^2(\mathbb{R};\mathscr{U}))$ and which in the finite-dimensional case is given by

$$[\mathcal{K}(t)x](\lambda) := \begin{cases} -B^*Q\mathrm{e}^{(A-BB^*Q)\lambda}x & \lambda \in [0,t] \\ 0 & \text{otherwise.} \end{cases}$$

The operators $\mathcal{S}(t)$ are known as truncated Popov-Toeplitz operators and (4) serves as their definition. We refer to Mikkola [4] for more information on these (integral) Riccati equations.

We note that the Riccati equation (3)–(5) is equivalent to (6) and also to (7).

## IV. Time-inversion

In order to provide a representation of solutions of the Riccati equation, we will also need the time-inverted system. This is the system with generating operators $-A$, $-B$, $C$ and again $t \geq 0$. We now describe this time-inverted system in terms of the solution operators. For this we first define the family of operators $\mathcal{R}(t)$ with $t \geq 0$ by

$$[\mathcal{R}(t)h](\lambda) = h(t-\lambda), \quad \lambda \in [0,t].$$

It can be shown that the solution operators of the time-inverted system are

$$\begin{aligned} &\mathcal{A}_\leftarrow(t) := \mathcal{A}(-t), \\ &\mathcal{B}_\leftarrow(t) := -\mathcal{A}(-t)\mathcal{B}(t)\mathcal{R}(t), \\ &\mathcal{C}_\leftarrow(t) := \mathcal{R}(t)\mathcal{C}(t)\mathcal{A}(-t), \\ &\mathcal{D}_\leftarrow(t) := \mathcal{R}(t)\mathcal{D}(t)\mathcal{R}(t) - \mathcal{R}(t)\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)\mathcal{R}(t). \end{aligned} \tag{8}$$

In the infinite-dimensional case it is an assumption that $\mathcal{A}$ extends to a group (i.e. that $\mathcal{A}(-t)$ exists for all $t > 0$). Under this assumption we take (8) as the definition of the time-inverted system.

We now relate the time-inverted system to solutions of the Riccati equation. The following lemma is very obvious when we are allowed to use the algebraic Riccati equation (2), but is seems surprisingly difficult when we have to use the Riccati equations (3)–(5). For a solution $Q$ of the Riccati equation we define its time-inverted Popov-Toeplitz operators by

$$\mathcal{S}_\leftarrow(t) := I + \mathcal{D}_\leftarrow(t)^*\mathcal{D}_\leftarrow(t) - \mathcal{B}_\leftarrow(t)^*Q\mathcal{B}_\leftarrow(t).$$

*Proposition 1:* Assume that $\mathcal{A}$ is a group. If $Q$ is a solution of the Riccati equation (3)–(5) with $\mathcal{S}$ and $\mathcal{S}_\leftarrow$ coercive, then $-Q$ is a solution of Riccati equation (3)–(5) corresponding to the time-inverted system.

*Proof:* We take the equivalents of (4) and (5) (with $-Q$ instead of $Q$) as the definitions of $\mathcal{S}_\leftarrow$ and $\mathcal{K}_\leftarrow$ respectively. Since $\mathcal{S}_\leftarrow$ is by assumption coercive, (5) indeed uniquely defines $\mathcal{K}_\leftarrow$. It remains to show that with these definitions the equivalent of (3) for the time-inverted system (again with $-Q$ instead of $Q$) holds.

We first obtain an alternative formula for $\mathcal{S}_\leftarrow$. For the equality (9) we use the definitions of $\mathcal{S}_\leftarrow$ and $\mathcal{D}_\leftarrow$ and the facts that $\mathcal{R}(t)^2 = I$ and $\mathcal{R}(t)^* = \mathcal{R}(t)$. For the equality (10) we use (3). Simplification then gives (11). For equality (12) we use (5). Simplification then gives (13). For the equality (14) we use (4).

Secondly we obtain an alternative formula for $\mathcal{K}_\leftarrow$. For the equality (15) we use the definitions of $\mathcal{K}_\leftarrow$ and (8). For the equality (16) we use (3). For the equalities (17) and (18) we use (5). For equality (19) we use (4).

From (14) and (19) it follows that

$$\mathcal{K}_\leftarrow(t)^*\mathcal{S}_\leftarrow(t)\mathcal{K}_\leftarrow(t) = \mathcal{A}(-t)^*\mathcal{K}(t)^*\mathcal{S}(t)\mathcal{K}(t)\mathcal{A}(-t).$$

The equivalent of (3) for the time-inverted system (with $-Q$ instead of $Q$) follows immediately from this last equation, (3) and the definition of time-inverted system. ■

## V. Stable subspaces

For a strongly continuous semigroup $T$ the stable subspace is

$$\mathscr{S}(T) := \{x \in \mathscr{X} : \lim_{t\to\infty} T(t)x = 0\}.$$

For a strongly continuous group the unstable subspace equals the stable subspace of the time-inverted semigroup:

$$\begin{aligned} \mathscr{S}_\leftarrow(T) &= \{x \in \mathscr{X} : \lim_{t\to\infty} T_\leftarrow(t)x = 0\} \\ &= \{x \in \mathscr{X} : \lim_{t\to-\infty} T(t)x = 0\}. \end{aligned}$$

For a solution of the Riccati equation we define (with some abuse of notation) the stable and unstable subspaces to be the stable subspaces of the closed-loop group $\mathcal{A} + \mathcal{B}\mathcal{K}$ (provided of course that this is indeed a group and not only a semigroup).

*Lemma 2:* Let $Q$ be a solution of the Riccati equation. On the stable subspace of $Q$ we have

$$\langle Qx_0, x_0\rangle = \lim_{t\to\infty} \|\mathcal{C}(t)x_0 + \mathcal{D}(t)\mathcal{K}(t)x_0\|^2_{L^2(0,t;\mathscr{Y})} + \|\mathcal{K}(t)x_0\|^2_{L^2(0,t;\mathscr{U})}.$$

*Proof:* Consider the Riccati equation (7) with $v = 0$:

$$\begin{aligned} &\|\mathcal{C}(t)x_0 + \mathcal{D}(t)\mathcal{K}(t)x_0\|^2_{L^2(0,t;\mathscr{Y})} + \|\mathcal{K}(t)x_0\|^2_{L^2(0,t;\mathscr{U})} = \\ &\langle Qx_0, x_0\rangle - \langle Q\left[\mathcal{A}(t) + \mathcal{B}(t)\mathcal{K}(t)\right]x_0, \left[\mathcal{A}(t) + \mathcal{B}(t)\mathcal{K}(t)\right]x_0\rangle, \end{aligned}$$

let $t \to \infty$ and use that the last term converges to zero since $x_0$ is in the stable subspace of $Q$. ■

*Proposition 3:* Assume that $Q_1$ and $Q_2$ are solutions of the Riccati equation such that $\mathcal{S}_1$ is nonnegative. Then $Q_1 \leq Q_2$ on the stable subspace of $Q_2$.

$$\begin{aligned}&\|\mathcal{C}(t)x_0+\mathcal{D}(t)u\|^2_{L^2(0,t;\mathscr{Y})}+\|u\|^2_{L^2(0,t;\mathscr{U})}\\&=\langle Qx_0,x_0\rangle-\langle Q\left[\mathcal{A}(t)x_0+\mathcal{B}(t)u\right],\mathcal{A}(t)x_0+\mathcal{B}(t)u\rangle+\langle\mathcal{S}(t)\left[\mathcal{K}(t)x_0-u\right],\mathcal{K}(t)x_0-u\rangle_{L^2(0,t;\mathscr{U})}.\end{aligned}\tag{6}$$

$$\begin{aligned}&\|\left[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}(t)\right]x_0+\mathcal{D}(t)v\|^2_{L^2(0,t;\mathscr{Y})}+\|\mathcal{K}(t)x_0+v\|^2_{L^2(0,t;\mathscr{U})}\\&=\langle Qx_0,x_0\rangle-\langle Q\left[\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}(t)\right]x_0+\mathcal{B}(t)v\right],\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}(t)\right]x_0+\mathcal{B}(t)v\rangle+\langle\mathcal{S}(t)v,v\rangle_{L^2(0,t;\mathscr{U})}.\end{aligned}\tag{7}$$

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$$\begin{aligned}\mathcal{R}(t)\mathcal{S}_{\leftarrow}(t)\mathcal{R}(t)&=I+\mathcal{D}(t)^*\mathcal{D}(t)+\mathcal{B}(t)^*\mathcal{A}(-t)^*\left[\mathcal{C}(t)^*\mathcal{C}(t)-Q\right]\mathcal{A}(-t)\mathcal{B}(t)\\&\quad-\mathcal{D}(t)^*\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t). &(9)\\&=I+\mathcal{D}(t)^*\mathcal{D}(t)+\mathcal{B}(t)^*\mathcal{A}(-t)^*\left[\mathcal{K}(t)^*\mathcal{S}(t)\mathcal{K}(t)-\mathcal{A}(t)^*Q\mathcal{A}(t)\right]\mathcal{A}(-t)\mathcal{B}(t)\\&\quad-\mathcal{D}(t)^*\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t) &(10)\\&=I+\mathcal{D}(t)^*\mathcal{D}(t)+\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{K}(t)^*\mathcal{S}(t)\mathcal{K}(t)\mathcal{A}(-t)\mathcal{B}(t)-\mathcal{B}(t)^*Q\mathcal{B}(t)\\&\quad-\mathcal{D}(t)^*\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t) &(11)\\&=I+\mathcal{D}(t)^*\mathcal{D}(t)-\mathcal{D}(t)^*\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t)-\mathcal{B}(t)^*Q\mathcal{B}(t)\\&\quad+\mathcal{B}(t)^*\mathcal{A}(-t)^*[\mathcal{C}(t)^*\mathcal{D}(t)+\mathcal{A}(t)^*Q\mathcal{B}(t)]\mathcal{S}(t)^{-1}[\mathcal{D}(t)^*\mathcal{C}(t)+\mathcal{B}(t)^*Q\mathcal{A}(t)]\mathcal{A}(-t)\mathcal{B}(t) &(12)\\&=I+\mathcal{D}(t)^*\mathcal{D}(t)-\mathcal{D}(t)^*\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t)-\mathcal{B}(t)^*Q\mathcal{B}(t)\\&\quad+[\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t)+\mathcal{B}(t)^*Q\mathcal{B}(t)]\mathcal{S}(t)^{-1}[\mathcal{D}(t)^*\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)+\mathcal{B}(t)^*Q\mathcal{B}(t)]. &(13)\\&=\left[I+\mathcal{D}(t)^*\mathcal{D}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t)\right]\mathcal{S}(t)^{-1}\left[I+\mathcal{D}(t)^*\mathcal{D}(t)-\mathcal{D}(t)^*\mathcal{C}(t)\mathcal{A}(-t)\mathcal{B}(t)\right]. &(14)\end{aligned}$$

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$$\begin{aligned}\mathcal{R}(t)\mathcal{S}_{\leftarrow}(t)\mathcal{K}_{\leftarrow}(t)\mathcal{A}(t)&=-\mathcal{D}(t)^*\mathcal{C}(t)+\mathcal{B}(t)^*\mathcal{A}(-t)^*\left[\mathcal{C}(t)^*\mathcal{C}(t)-Q\right] &(15)\\&=-\mathcal{D}(t)^*\mathcal{C}(t)+\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{K}(t)^*\mathcal{S}(t)\mathcal{K}(t)-\mathcal{B}(t)^*Q\mathcal{A}(t) &(16)\\&=\mathcal{S}(t)\mathcal{K}(t)+\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{K}(t)^*\mathcal{S}(t)\mathcal{K}(t) &(17)\\&=\mathcal{S}(t)\mathcal{K}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\left[\mathcal{C}(t)^*\mathcal{D}(t)+\mathcal{A}(t)^*Q\mathcal{B}(t)\right]\mathcal{K}(t) &(18)\\&=\left[I+\mathcal{D}(t)^*\mathcal{D}(t)-\mathcal{B}(t)^*\mathcal{A}(-t)^*\mathcal{C}(t)^*\mathcal{D}(t)\right]\mathcal{K}(t). &(19)\end{aligned}$$

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$$\begin{aligned}&\|\mathcal{C}(t)x_0+\mathcal{D}(t)\mathcal{K}_2(t)x_0\|^2_{L^2(0,t;\mathscr{Y})}+\|\mathcal{K}_2(t)x_0\|^2_{L^2(0,t;\mathscr{U})}\\&=\langle Q_1x_0,x_0\rangle-\langle Q_1\left[\mathcal{A}(t)x_0+\mathcal{B}(t)\mathcal{K}_2(t)x_0\right],\mathcal{A}(t)x_0+\mathcal{B}(t)\mathcal{K}_2(t)x_0\rangle\\&\quad+\langle\mathcal{S}_1(t)\left[\mathcal{K}_1(t)x_0-\mathcal{K}_2(t)x_0\right],\mathcal{K}_1(t)x_0-\mathcal{K}_2(t)x_0\rangle_{L^2(0,t;\mathscr{U})}.\end{aligned}\tag{20}$$

$$\begin{aligned}&\|\mathcal{C}(t)x_0+\mathcal{D}(t)\mathcal{K}_2(t)x_0\|^2_{L^2(0,t;\mathscr{Y})}+\|\mathcal{K}_2(t)x_0\|^2_{L^2(0,t;\mathscr{U})}\\&\geq\langle Q_1x_0,x_0\rangle-\langle Q_1\left[\mathcal{A}(t)x_0+\mathcal{B}(t)\mathcal{K}_2(t)x_0\right],\mathcal{A}(t)x_0+\mathcal{B}(t)\mathcal{K}_2(t)x_0\rangle.\end{aligned}\tag{21}$$

$$\lim_{t\to\infty}\|\mathcal{C}(t)x_0+\mathcal{D}(t)\mathcal{K}_2(t)x_0\|^2_{L^2(0,t;\mathscr{Y})}+\|\mathcal{K}_2(t)x_0\|^2_{L^2(0,t;\mathscr{U})}\geq\langle Q_1x_0,x_0\rangle.\tag{22}$$

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*Proof:* Considering the Riccati equation (6) for $Q_1$ with $u=\mathcal{K}_2(t)x_0$ gives (20). Using that $\mathcal{S}_1$ is nonnegative then gives (21). Using that $x_0$ is in the stable subspace of $Q_2$ then gives (22). Invoking Lemma 2 we obtain $\langle Q_2x_0,x_0\rangle\geq\langle Q_1x_0,x_0\rangle$, as desired. ■

*Corollary 4:* If the Riccati equation has a stabilizing solution, then it has a maximal solution among all solutions with $\mathcal{S}$ nonnegative, and the two coincide.

*Proof:* This follows from Proposition 3 since the stable subspace of the stabilizing solution equals the whole state space. ■

*Corollary 5:* Assume that $\mathcal{A}$ is a group. If the Riccati equation has an anti-stabilizing solution (i.e. a solution for which the unstable subspace equals the state space), then it has a minimal solution among all solutions with both $\mathcal{S}$ and $\mathcal{S}_{\leftarrow}$ coercive, and the two coincide.

*Proof:* This follows from applying Corollary 4 to the time-inverted system and using Proposition 1. ■

*Corollary 6:* Assume that the Riccati equation has a stabilizing solution with nonnegative Popov-Toeplitz operators. Then any solution $Q$ with nonnegative Popov-Toeplitz operators coincides with this stabilizing solution on $\mathscr{S}(Q)$.

In particular, a stabilizing solution with nonnegative Popov-Toeplitz operators is unique.

*Proof:* Denote the mentioned stabilizing solution with nonnegative Popov-Toeplitz operators by $Q_+$. Apply Proposition 3 with $Q_1 = Q_+$ and $Q_2 = Q$. It follows that $Q_+ \leq Q$ on $\mathscr{S}(Q)$. By corollary 4 we have $Q \leq Q_+$ on $\mathscr{X}$ so that $Q = Q_+$ on $\mathscr{S}(Q)$. In particular if $Q$ itself is a stabilizing solution with nonnegative Popov-Toeplitz operators then $Q = Q_+$ on $\mathscr{X}$. ■

*Corollary 7:* Assume that $\mathcal{A}$ is a group. Further assume that the Riccati equation has an anti-stabilizing solution with coercive Popov-Toeplitz operators and coercive time-inverted Popov-Toeplitz operators. Then any solution $Q$ with coercive Popov-Toeplitz operators and coercive time-inverted Popov-Toeplitz operators coincides with this anti-stabilizing solution on $\mathscr{S}_\leftarrow(Q)$. In particular, an anti-stabilizing solution with coercive Popov-Toeplitz operators and coercive time-inverted Popov-Toeplitz operators is unique.

*Proof:* This follows from applying Corollary 6 to the time-inverted system and using Proposition 1. ■

## VI. Solutions from projections

*Lemma 8:* Let $Q_1$ and $Q_2$ be solutions of the Riccati equation. Assume that the projection $P$ satisfies:

1) $(Q_2 - Q_1)P = P^*(Q_2 - Q_1)$,
2) the image of $P$ is invariant under $\mathcal{A} + \mathcal{B}\mathcal{K}_2$,
3) the image of $I - P$ is invariant under $\mathcal{A} + \mathcal{B}\mathcal{K}_1$,

Then $Q := Q_1(I-P)+Q_2P$ defines a solution of the Riccati equation.

*Proof:* Using the first property of the projection $P$ it can be shown that $Q$ is self-adjoint.

Now we show that the Riccati equation holds. We do this by using the formulation (7) and treating the quadratic term in $x_0$ and the cross-term in $x_0$ and $v$ separately. The quadratic term in $v$ simply defines $\mathcal{S}$.

We first consider the cross-term. Since $Q_1$ and $Q_2$ are solutions of the Riccati equation (23) holds for $i = 1, 2$. We want to show that the same holds for $Q$. We define $\mathcal{K}(t) := \mathcal{K}_1(t)(I-P)+\mathcal{K}_2(t)P$ and consider (24) which we want to show is equal to zero. The equality (25) follows from writing $x_0 = Px_0 + (I - P)x_0$. The equality (26) follows from the fact that $\mathcal{K}P = \mathcal{K}_2P$ and $\mathcal{K}(I - P) = \mathcal{K}_1(I - P)$, which follow immediately form the definition of $\mathcal{K}$. Equality (27) follows from properties 2 and 3 of the projection $P$. Using that $QP = Q_2$, $Q(I-P) = Q_1$ which follows directly from the definition of $Q$ and (23) we see that (27) is indeed equal to zero.

We now consider the quadratic term in $x_0$. Since $Q_1$ and $Q_2$ are solutions of the Riccati equation we have (28) for $i = 1, 2$. We want to show that the same holds for $Q$. So we consider (29), which we want to show is equal to zero. We write $x_0 = Px_0 + (I - P)x_0$ and consider the $P$-terms, the $I - P$-terms and the cross-terms separately. The $P$-term is the expression (30) where we have again used that $\mathcal{K}P = \mathcal{K}_2P$, property 2 of the projection and $QP = Q_2$. By (28) this indeed equals zero. The case for the $I - P$-term is similar. The cross-term equals (31). This equals zero by the polarization of the Riccati equation (6) for $Q_1$ with $x_0^1 = (I - P)x_0$, $x_0^2 = Px_0$, $u_1 = \mathcal{K}_1(t)x_0^1$ and $u_2 = \mathcal{K}_2(t)x_0^2$.

It follows that $Q$ is indeed a self-adjoint solution of the Riccati equation. ■

*Corollary 9:* If in Lemma 8, $Q_2 \geq Q_1$ and $\mathcal{S}_1$ is coercive, then $\mathcal{S}$ is coercive.

*Proof:* Using property 1 of the projection from Lemma 8 at the last step we have

$$\begin{aligned} Q &= Q_1(I - P) + Q_2P = Q_1 + (Q_2 - Q_1)P \\ &= Q_1 + (Q_2 - Q_1)P^2 = Q_1 + P^*(Q_2 - Q_1)P. \end{aligned}$$

By definition of the Popov-Toeplitz operators we have

$$\begin{aligned} &\langle \mathcal{S}(t)v, v\rangle_{L^2(0,t;\mathscr{U})} \\ &= \|\mathcal{D}(t)v\|^2_{L^2(0,t;\mathscr{Y})} + \|v\|^2_{L^2(0,t;\mathscr{U})} + \langle Q\mathcal{B}(t)v, \mathcal{B}(t)v\rangle. \end{aligned}$$

Using the above formula for $Q$ we see that this equals

$$\begin{aligned} &\|\mathcal{D}(t)v\|^2_{L^2(0,t;\mathscr{Y})} + \|v\|^2_{L^2(0,t;\mathscr{U})} \\ &\quad + \langle Q_1\mathcal{B}(t)v, \mathcal{B}(t)v\rangle + \langle (Q_2 - Q_1)P\mathcal{B}(t)v, P\mathcal{B}(t)v\rangle \\ &= \langle \mathcal{S}_1(t)v, v\rangle_{L^2(0,t;\mathscr{U})} + \langle (Q_2 - Q_1)P\mathcal{B}(t)v, P\mathcal{B}(t)v\rangle, \end{aligned}$$

which from the assumptions is coercive. ■

The following lemma shows that, with respect to the (possibly indefinite and degenerate) inner-product induced by the gap $Q_2 - Q_1$, the adjoints of the closed-loop operators $\mathcal{A} + \mathcal{B}\mathcal{K}_1$ are the inverses of the closed-loop operators $\mathcal{A} + \mathcal{B}\mathcal{K}_2$.

*Lemma 10:* Let $Q_1$ and $Q_2$ be solutions of the Riccati equation. Then

$$Q_2 - Q_1 = [\mathcal{A}(t) + \mathcal{B}(t)\mathcal{K}_1(t)]^*(Q_2 - Q_1)[\mathcal{A}(t) + \mathcal{B}(t)\mathcal{K}_2(t)].$$

*Proof:* The proof is entirely similar to that of [5, Lemma B.5] with the discrete-time operators there replaced by the corresponding continuous-time solution operators. ■

*Corollary 11:* Let $Q_1$ and $Q_2$ be solutions of the Riccati equation such that $Q_2 - Q_1$ is coercive. Then the subspace $\mathscr{V}$ is $\mathcal{A} + \mathcal{B}\mathcal{K}_2$ invariant if and only if $[(Q_2 - Q_1)\mathscr{V}]^\perp$ is $\mathcal{A} + \mathcal{B}\mathcal{K}_1$ invariant.

*Proof:* This follows from Lemma 10 using [1, Exercise 2.30a] (applied with the gap inner-product induced by $Q_2 - Q_1$). ■

*Proposition 12:* Let $Q_1$ and $Q_2$ be solutions of the Riccati equation such that $Q_2 - Q_1$ is coercive. Let $\mathscr{V}$ be a closed invariant subspace of $\mathcal{A} + \mathcal{B}\mathcal{K}_2$ and let $P_\mathscr{V}$ be the projection onto $\mathscr{V}$ along the subspace $[(Q_2 - Q_1)\mathscr{V}]^\perp$. Then $Q := Q_1(I - P_\mathscr{V}) + Q_2P_\mathscr{V}$ defines a solution of the Riccati equation.

*Proof:* In the inner-product defined by the gap $Q_2 - Q_1$ the spaces $\mathscr{V}$ and $[(Q_2 - Q_1)\mathscr{V}]^\perp$ are orthogonal. Since the gap inner-product is equivalent to the given inner-product on $\mathscr{X}$ (because the gap is assumed to be coercive), $\mathscr{X} = \mathscr{V} \oplus [(Q_2 - Q_1)\mathscr{V}]^\perp$ (a direct sum which may not be orthogonal in the given inner product on $\mathscr{X}$). The projection $P_\mathscr{V}$ is the projection onto $\mathscr{V}$ induced by this direct sum decomposition. By Corollary 11 this projection satisfies the three assumptions from Lemma 8. The result follows. ■

$$\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_i(t)]\,x_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}_i(t)x_0,v\rangle+\langle Q_i\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_i(t)\right]x_0,\mathcal{B}(t)v\rangle=0 \tag{23}$$

$$\begin{aligned}
&\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}(t)]\,x_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}(t)x_0,v\rangle+\langle Q\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}(t)\right]x_0,\mathcal{B}(t)v\rangle && (24)\\
&=\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}(t)]\,Px_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}(t)Px_0,v\rangle+\langle Q\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}(t)\right]Px_0,\mathcal{B}(t)v\rangle && (25)\\
&\quad+\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}(t)]\,(I-P)x_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}(t)(I-P)x_0,v\rangle+\langle Q\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}(t)\right](I-P)x_0,\mathcal{B}(t)v\rangle\\
&=\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_2(t)]\,Px_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}_2(t)Px_0,v\rangle+\langle Q\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_2(t)\right]Px_0,\mathcal{B}(t)v\rangle && (26)\\
&\quad+\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_1(t)]\,(I-P)x_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}_1(t)(I-P)x_0,v\rangle+\langle Q\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_1(t)\right](I-P)x_0,\mathcal{B}(t)v\rangle\\
&=\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_2(t)]\,Px_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}_2(t)Px_0,v\rangle+\langle QP\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_2(t)\right]Px_0,\mathcal{B}(t)v\rangle && (27)\\
&\quad+\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_1(t)]\,(I-P)x_0,\mathcal{D}(t)v\rangle+\langle\mathcal{K}_1(t)(I-P)x_0,v\rangle+\langle Q(I-P)\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_1(t)\right](I-P)x_0,\mathcal{B}(t)v\rangle.
\end{aligned}$$

$$\|\left[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_i(t)\right]x_0\|^2+\|\mathcal{K}_i(t)x_0\|^2=\langle Q_ix_0,x_0\rangle-\langle Q_i\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_i(t)\right]x_0,\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_i(t)\right]x_0\rangle \tag{28}$$

$$\|\left[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}(t)\right]x_0\|^2+\|\mathcal{K}(t)x_0\|^2-\langle Qx_0,x_0\rangle+\langle Q\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}(t)\right]x_0,\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}(t)\right]x_0\rangle \tag{29}$$

$$\|\left[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_2(t)\right]Px_0\|^2+\|\mathcal{K}_2(t)Px_0\|^2-\langle Q_2Px_0,Px_0\rangle+\langle Q_2\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_2(t)\right]Px_0,\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_2(t)\right]Px_0\rangle \tag{30}$$

$$\begin{aligned}
&\langle[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_1(t)]\,(I-P)x_0,[\mathcal{C}(t)+\mathcal{D}(t)\mathcal{K}_2(t)]\,Px_0\rangle+\langle\mathcal{K}_1(t)(I-P)x_0,\mathcal{K}_2(t)Px_0\rangle\\
&-\langle Q_1(I-P)x_0,Px_0\rangle+\langle Q_1\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_1(t)\right](I-P)x_0,\left[\mathcal{A}(t)+\mathcal{B}(t)\mathcal{K}_2(t)\right]Px_0\rangle.
\end{aligned} \tag{31}$$

## VII. MAIN RESULT

*Theorem 13:* Assume that $\mathcal{A}$ is a group. Further assume that the Riccati equation has a stabilizing solution $Q_+$ and an anti-stabilizing solution $Q_-$, both with coercive Popov-Toeplitz operators and coercive time-inverted Popov-Toeplitz operators, and that $Q_+ - Q_-$ is coercive. Then a closed invariant subspace $\mathscr{V}$ of $\mathcal{A}+\mathcal{B}\mathcal{K}_+$ gives rise to a solution $Q$ of the Riccati equation (as in Proposition 12). The stable subspace of $Q$ coincides with $\mathscr{V}$ and its unstable subspace coincides with $[(Q_+-Q_-)\mathscr{V}]^\perp$. In particular its stable and unstable subspace sum to $\mathscr{X}$. The Popov-Toeplitz operators and inverted Popov-Toeplitz operators of $Q$ are coercive.

Conversely, any solution $Q$ of the Riccati equation with the properties

- $\mathscr{S}(Q)\oplus\mathscr{S}_\leftarrow(Q)=\mathscr{X}$,
- its Popov-Toeplitz operators and inverted Popov-Toeplitz operators are coercive,

is of the form $Q := Q_-(I-P)+Q_+P$, where $P$ is the projection onto $\mathscr{S}(Q)$ along $\mathscr{S}_\leftarrow(Q)$.

*Proof:* The existence of the solution $Q$ follows immediately from Proposition 12. From its definition it follows that $Q = Q_+$ on $\mathscr{V}$ and $Q = Q_-$ on $[(Q_+-Q_-)\mathscr{V}]^\perp$. It follows (by the proof of Lemma 8) that $\mathcal{A}+\mathcal{B}\mathcal{K}=\mathcal{A}+\mathcal{B}\mathcal{K}_+$ on $\mathscr{V}$ and $\mathcal{A}+\mathcal{B}\mathcal{K}=\mathcal{A}+\mathcal{B}\mathcal{K}_-$ on $[(Q_+-Q_-)\mathscr{V}]^\perp$. Using that $Q_+$ and $Q_-$ are stabilizing and anti-stabilizing respectively, it follows that $\mathcal{S}(Q)\supset\mathscr{V}$ and $\mathcal{S}_\leftarrow(Q)\supset[(Q_+-Q_-)\mathscr{V}]^\perp$. We actually have equality in both cases which can be shown as follows. We have for $x\in\mathscr{X}$ that $x=x_1+x_2$ with $x_1\in\mathscr{V}$ and $x_2\in[(Q_+-Q_-)\mathscr{V}]^\perp$. For $x\in\mathcal{S}(Q)$ we have $Qx=Q_+x=Q_+x_1+Q_+x_2$ by Corollary 6. On the other hand we have by the definition of $Q$ that $Qx=Q_+x_1+Q_-x_2$. It follows that $(Q_+-Q_-)x_2=0$ which since $Q_+-Q_-$ is positive implies $x_2=0$. It follows that $\mathcal{S}(Q)\subset\mathscr{V}$. That $\mathcal{S}_\leftarrow(Q)\subset[(Q_+-Q_-)\mathscr{V}]^\perp$ follows similarly.

From Corollary 9 and the fact that $\mathcal{S}_-$ is coercive it follows that $\mathcal{S}$ is coercive. Similarly, by applying Corollary 9 to the time-inverted system it follows that $\mathcal{S}_\leftarrow$ is coercive.

We now prove the converse assertion. By the first assumption, the projection is well-defined. It remains to show that $Q = Q_+$ on $\mathscr{S}(Q)$ and $Q=Q_-$ on $\mathscr{S}_\leftarrow(Q)$. But this follows from Corollaries 6 and 7 respectively. ■

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