Distribution semigroups and control systems

Mark R. Opmeer

J. Evol. Equ. 6 (2006), no. 1, 145–159

Abstract

We introduce the new concept of a distributional control system. This class of systems is the natural generalization of distribution semigroups to input/state/output systems. We show that, under the Laplace transform, this new class of systems is equivalent to the class of distributional resolvent linear systems which we introduced in an earlier article. There we showed that this latter class of systems is the correct abstract setting in which to study many non-well-posed control systems such as the heat equation with Dirichlet control and Neumann observation. In this article we further show that any holomorphic function defined and polynomially bounded on some right half-plane can be realized as the transfer function of some exponentially bounded distributional resolvent linear system.

1 Introduction

In [12] Lions introduced the concept of a distribution semigroup. This was subsequently studied by many authors (among them Chazarain [2], Arendt et. al. [1], Kunstmann [10] and Kisynski [8, 9]). The reason for the introduction of distribution semigroups was the study of Cauchy problems that are not well-posed. A well-known example of such a non-well-posed Cauchy problem is the Schrödinger equation on L^p with $p \neq 2$. In [13] we pointed out that distribution semigroups are a very useful tool for another type of problem as well. This application is in the field of control theory, where not Cauchy problems but problems of the following type are studied:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad y(t) = Cx(t) + Du(t).$$

Here A,B,C,D are (in general unbounded) operators on Banach spaces. Such a control system is called well-posed if, for every $t > 0$, the map $[x_0, u] \mapsto [x(t), y]$ is bounded from $X \times L^2(0, t; U)$ to $X \times L^2(0, t; Y)$ (for sake of brevity we are being a little bit sloppy here, see [18] for the precise details). This class can be seen as the natural analogue of well-posed Cauchy problems and the theory of strongly continuous semigroups plays an important role in the theory of these well-posed linear systems. Not all interesting control systems are however well-posed. This of course happens when the associated Cauchy problem is not well-posed, but it can also happen when the associated Cauchy problem is well-posed. A particular example of this latter case is the heat equation with Dirichlet control and Neumann observation (see [13, Section 4] for details). The non-well-posedness here is due to the effect of the control u and the observation y. In this article we continue the investigation into non-well-posed control systems initiated in [13]. In Section 2 we introduce the new concept of a distributional control system (this is the natural generalization of the concept of distribution semigroup). In Section 3 we recall the concept of a resolvent linear system (this is the natural generalization of the concept of resolvent) and the subclass of distributional resolvent linear systems from [13]. In Section 4 we recall the Laplace transformation for certain classes of distributions. Section 5 shows that the class of distributional control systems and the class of distributional resolvent linear systems are equivalent via the Laplace transform. Finally, in Section 6 we take up the issue of realization theory.

1.1 Some notation

Denote by $\mathcal{D}$ the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{C}$ with compact support equipped with its usual inductive limit topology. Let $*_0$ denote the convolution-like mapping

$$f *_0 g(t) = \int_0^t f(s)g(t-s)\,ds. \tag{1}$$

Notice that if f is differentiable, then $f *_0 g$ is too, and

$$(f *_0 g)' = f' *_0 g + f(0)g. \tag{2}$$

Denote by $\mathcal{D}'(\mathcal{X})$ the space of all continuous linear mappings $\mathcal{D} \to \mathcal{X}$ supplied with the topology of uniform convergence on the bounded subsets of $\mathcal{D}$ (here

$\mathcal{X}$ is a Banach space). We denote the subset of elements with support in $[0,\infty)$ by $\mathcal{D}'_0(\mathcal{X})$ and the subset of elements with support bounded to the left by $\mathcal{D}'_l(\mathcal{X})$. By $\mathcal{D}_+$ we denote the set of restrictions of functions in $\mathcal{D}$ to $[0,\infty)$.

2 Distributional control systems

In control theory one is interested in systems that can be described as

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t) + Du(t). \tag{3}$$

In the classical case A, B, C and D are matrices of compatible dimensions and the solutions are given explicitly by

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\,ds, \quad y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-s)}Bu(s)\,ds + Du(t). \tag{4}$$

Or in terms of Laplace transforms the solutions are given by

$$\begin{aligned} \hat{x}(s) &= (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s) \\ \hat{y}(s) &= C(sI - A)^{-1}x_0 + \big(C(sI - A)^{-1}B + D\big)\,\hat{u}(s). \end{aligned} \tag{5}$$

Many systems described by partial differential equations or delay-differential equations can also be described (in some sense) in the forms (3), (4), (5). One does have to impose certain continuity assumptions and the main goal is to impose assumptions that are weak enough to cover all interesting examples and strong enough to be able to solve all interesting problems. Of course such a set of continuity assumptions does not exist and one has to find a compromise between a "large class of systems" and a "large class of problems". We refer to [3] and [18] for more background on infinite-dimensional systems.

Our motivation in writing [13] was to find a class of systems that included (almost) all interesting examples and for which some interesting problems could be solved at such a high level of generality. This article was based on the representation (5). We called our class *distributional resolvent linear systems*. In [13] and [4], [5], [14] some interesting control problems were indeed solved at this level of generality. The purpose of this article is to show how, starting from (a generalization of) the representation (4), one can obtain the same class of systems (up to Laplace transformation of course).

As starting point we take the "impulse responses" of the system:

$$\mathbb{A}(t) = \mathrm{e}^{At}, \quad \mathbb{B}(t) = \mathrm{e}^{At}B, \quad \mathbb{C}(t) = C\mathrm{e}^{At}, \quad \mathbb{D} = C\mathrm{e}^{At}B + D\delta.$$

These satisfy certain equations, e.g.

$$\begin{aligned}
\mathbb{A}(t+s) &= \mathbb{A}(t)\mathbb{A}(s), \\
\mathbb{B}(t+s) &= \mathbb{A}(t)\mathbb{B}(s), \\
\mathbb{C}(t+s) &= \mathbb{C}(t)\mathbb{A}(s).
\end{aligned}$$

Our generalization is as follows.

Definition 2.1 *A* distributional control system *on a triple of Banach spaces* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *is a distribution*

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{bmatrix} \in \mathcal{D}_0'(\mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{X} \times \mathcal{Y})),$$

that satisfies for all $\varphi, \psi \in \mathcal{D}$

$$\mathbb{A}(\varphi *_0 \psi) = \mathbb{A}(\varphi)\mathbb{A}(\psi), \tag{6}$$

$$\mathbb{B}(\varphi *_0 \psi) = \mathbb{A}(\varphi)\mathbb{B}(\psi), \tag{7}$$

$$\mathbb{C}(\varphi *_0 \psi) = \mathbb{C}(\varphi)\mathbb{A}(\psi), \tag{8}$$

$$\mathbb{D}(\varphi *_0 \psi) = \mathbb{C}(\varphi)\mathbb{B}(\psi). \tag{9}$$

Remark 2.2 *A distributional control system defines a time-invariant causal continuous operator from* $\mathcal{D}_l'(\mathcal{X} \times \mathcal{U})$ *to* $\mathcal{D}_l'(\mathcal{X} \times \mathcal{Y})$ *by convolution. It is this operator that generalizes the integral representation* (4)*.*

The distribution $\mathbb{A}$ with the functional equation (6) as above is a distribution semigroup. As defined above it is actually a generalization of the concept introduced by Lions [12]. This generalization was called a quasi-distribution semigroup by Wang [19], a pre-distribution semigroup by Kunstmann [10] and simply a distribution semigroup by Kisynski [9]. We refer to these articles and the references therein for more on distribution semigroups.

The following proposition gives an alternative characterization of distributional control systems.

Proposition 2.3 *The distribution*

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{bmatrix} \in \mathcal{D}_0'(\mathcal{L}\left(\mathcal{X} \times \mathcal{U}, \mathcal{X} \times \mathcal{Y}\right))$$

is a distributional control system if and only if for all $\varphi, \psi \in \mathcal{D}$

$$\psi(0)\mathbb{A}(\varphi) - \varphi(0)\mathbb{A}(\psi) = \mathbb{A}(\varphi')\mathbb{A}(\psi) - \mathbb{A}(\varphi)\mathbb{A}(\psi'), \tag{10}$$
$$\psi(0)\mathbb{B}(\varphi) - \varphi(0)\mathbb{B}(\psi) = \mathbb{A}(\varphi')\mathbb{B}(\psi) - \mathbb{A}(\varphi)\mathbb{B}(\psi'), \tag{11}$$
$$\psi(0)\mathbb{C}(\varphi) - \varphi(0)\mathbb{C}(\psi) = \mathbb{C}(\psi)\mathbb{A}(\varphi') - \mathbb{C}(\psi')\mathbb{A}(\varphi), \tag{12}$$
$$\psi(0)\mathbb{D}(\varphi) - \varphi(0)\mathbb{D}(\psi) = \mathbb{C}(\varphi')\mathbb{B}(\psi) - \mathbb{C}(\varphi)\mathbb{B}(\psi'). \tag{13}$$

Proof That the first equation given here is equivalent to the first equation in Definition 2.1 is proven in Kisynksi [8, Theorem 2.6]. The equivalence of the other equations given here to their counterparts in Definition 2.1 follow using exactly the same arguments. We will give the details for the fourth equation.

It follows from (2) that

$$\psi(0)\varphi - \varphi(0)\psi = \varphi' *_0 \psi - \varphi *_0 \psi'$$

since both $\varphi *_0 \psi' + \psi(0)\varphi$ and $\varphi' *_0 \psi + \varphi(0)\psi$ are equal to $(\varphi *_0 \psi)'$. Applying $\mathbb{D}$ to this equation and using (9) gives (13). To prove the converse we define the shift operators τ^t on the spaces of functions $\mathbb{R} \to \mathbb{C}$ to itself by $(\tau^t f)(s) = f(t+s)$. Let $\varphi, \psi \in \mathcal{D}$ and let a be such that $\varphi(s) = 0$ for all $s \geq a$. If $a < 0$ then, using that $\mathbb{C}$ and $\mathbb{D}$ have support in $[0, \infty)$, (13) reads $0 = 0$ and we are done. So suppose $a \geq 0$.

Applying (13) to $\tau^t\varphi$ and $\tau^{-t}\psi$ we obtain

$$\psi(-t)\mathbb{D}(\tau^t\varphi) - \varphi(t)\mathbb{D}(\tau^{-t}\psi) = \mathbb{C}(\tau^t\varphi')\mathbb{B}(\tau^{-t}\psi) - \mathbb{C}(\tau^t\varphi)\mathbb{B}(\tau^{-t}\psi')$$

and we note that the last expression is equal to

$$\frac{d}{dt}\left[\mathbb{C}(\tau^t\varphi)\mathbb{B}(\tau^{-t}\psi)\right].$$

We have

$$\mathbb{C}(\varphi)\mathbb{B}(\psi) = -\int_0^a \frac{d}{dt}\left[\mathbb{C}(\tau^t\varphi)\mathbb{B}(\tau^{-t}\psi)\right]\, dt,$$

where we have used that $\mathbb{C}(\tau^a\varphi) = 0$ since $(\tau^a\varphi)(s) = 0$ for $s \geq 0$ and $\operatorname{supp} \mathbb{C} \subset [0, \infty)$. Hence we have

$$\mathbb{C}(\varphi)\mathbb{B}(\psi) = \int_0^a \varphi(t)\mathbb{D}(\tau^{-t}\psi) - \psi(-t)\mathbb{D}(\tau^t\varphi)\, dt,$$

the right-hand side of which equals $\mathbb{D}(\varphi *_0 \psi)$ since

$$\begin{aligned}\int_0^a \varphi(t)\psi(s-t) - \psi(-t)\varphi(s+t)\, dt &= \int_0^a \varphi(t)\psi(s-t)\, dt - \int_s^{s+a} \psi(t)\varphi(s+t)\, dt \\ &= \int_0^s \varphi(t)\psi(s-t)\, dt = \varphi *_0 \psi(s)\end{aligned}$$

by $\operatorname{supp} \varphi \subset (-\infty, a]$. □

3 Distributional resolvent linear systems

We now briefly review the situation (5) that was treated in [13]. We first consider the generalizations of the matrix-valued functions $(sI - A)^{-1}$, $(sI - A)^{-1}B$, $C(sI - A)^{-1}$ and $C(sI - A)^{-1}B + D$ as given in [13].

Definition 3.1 *A* resolvent linear system *on a triple of Banach spaces* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *consists of a nonempty open connected subset* Λ *of the complex plane and four operator valued function* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ *such that*
$\mathfrak{a} : \Lambda \to \mathcal{L}(\mathcal{X})$ *satisfies*

$$\mathfrak{a}(\beta) - \mathfrak{a}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{a}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{14}$$

$\mathfrak{b} : \Lambda \to \mathcal{L}(\mathcal{U}, \mathcal{X})$ *satisfies*

$$\mathfrak{b}(\beta) - \mathfrak{b}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{15}$$

$\mathfrak{c} : \Lambda \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *satisfies*

$$\mathfrak{c}(\beta) - \mathfrak{c}(\alpha) = (\alpha - \beta)\mathfrak{c}(\alpha)\mathfrak{a}(\beta) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{16}$$

$\mathfrak{d} : \Lambda \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ *satisfies*

$$\mathfrak{d}(\beta) - \mathfrak{d}(\alpha) = (\alpha - \beta)\mathfrak{c}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{17}$$

The function $\mathfrak{a}$ *is called the* pseudoresolvent*,* $\mathfrak{b}$ *the* incoming wave function*,* $\mathfrak{c}$ *the* outgoing wave function *and* $\mathfrak{d}$ *the* characteristic function *of the resolvent linear system.*

The following subclass was also identified in [13].

Definition 3.2 *A* distributional resolvent linear system *is a resolvent linear system with the additional propery that there exist constants $\alpha > 0, \beta \in \mathbb{R}$ and a polynomial p such that*

$$\Lambda_E(\alpha, \beta) := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq \beta, \quad |\mathrm{Im}\ s| \leq \mathrm{e}^{\alpha \mathrm{Re}\ s}\} \subset \Lambda \tag{18}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \forall\ s \in \Lambda_E. \tag{19}$$

A region Λ_E as above is called an exponential region (see [1]). Note that the wavefunctions and characteristic function of a distributional resolvent linear system are also polynomially bounded on Λ_E (this follows from the equations in Definition 3.1).

Equivalently we could assume that the pseudoresolvent is polynomially bounded on a logarithmic region. A logarithmic region is a region of the form

$$\Lambda_L(a, b, c) := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq c, \mathrm{Re}\ s \geq \frac{1}{a} \log |s| + b\} \tag{20}$$

with $a > 0$ and $b, c \in \mathbb{R}$. This is true since one can show that an exponential region is contained in a logarithmic region is contained in an exponential region (see [1]).

In [13] also a subclass of distributional resolvent linear systems was introduced, namely the following.

Definition 3.3 *A distributional resolvent linear system is called* exponentially bounded *if there exists a $\gamma \in \mathbb{R}$ and a polynomial p such that*

$$\Lambda_H(\gamma) := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq \gamma\} \subset \Lambda \tag{21}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \forall\ s \in \Lambda_H. \tag{22}$$

This subclass was called *integrated resolvent linear systems* in [13], but in Section 5 we will see that exponentially bounded distributional resolvent linear system is a better term.

Remark 3.4 *We briefly consider a relation between resolvent linear systems and descriptor systems (also known as generalized state space systems, singular systems, differential algebraic systems; see [6] for an account of finite-dimensional descriptor systems, infinite-dimensional descriptor systems do not seem to have been intensively studied). Instead of* (3) *we could also have considered*

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t) + Du(t). \tag{23}$$

After Laplace transforming we would then have obtained

$$\begin{aligned} \hat{x}(s) &= (sE - A)^{-1}Ex_0 + (sE - A)^{-1}B\hat{u}(s) \\ \hat{y}(s) &= C(sE - A)^{-1}Ex_0 + \big(C(sE - A)^{-1}B + D\big)\,\hat{u}(s). \end{aligned} \tag{24}$$

In the matrix case it is easy to verify that this also gives a resolvent linear system. The same holds true in the infinite-dimensional case for bounded E. A partial converse exists in the sense that any pseudoresolvent is of the form $(sE - A)^{-1}E$ for some operators A and E (see e.g. Kisynski [8, Section 1] or [9, Section 8]). For a resolvent linear system one can however not always define generators A, B, C, D and E as the following (finite-dimensional) example shows. The following descriptor system is a realization of s

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad C = [0\ 1], \quad D = 0.$$

This realization is minimal in the sense that there does not exist a 1-dimensional descriptor system that realizes s. However, the resolvent linear system

$$\mathfrak{a} = 0, \quad \mathfrak{b} = -1, \quad \mathfrak{c} = 1, \quad \mathfrak{d} = s$$

is a 1-dimensional realization of s. This shows that the latter system cannot be a descriptor system (which can also be verified directly). The conclusion is that descriptor systems are resolvent linear systems, but that the converse is not true.

4 Laplace transformation of distributions

To explain the relation between distributional control systems and distributional resolvent linear systems we study the Laplace transformation on some spaces of distributions.

The Laplace transform is defined for $L^1(0,\infty)$ functions as

$$\hat{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt,$$

seeing f as defining a regular distribution T_f we see that $\hat{f}(s) = T_f(e_\lambda)$, where the function $e_\lambda : \mathbb{R} \to \mathbb{C}$ is defined by $e_\lambda(t) = \mathrm{e}^{-\lambda t} 1_{[0,\infty)}(t)$. So what we want to do is make sense out of $T(e_\lambda)$ for certain distributions T, note that since $e_\lambda \notin \mathcal{D}$ it is not obvious that it does make sense.

We first consider the "classical" case of exponentially bounded distributions. We recall the relevant spaces of test functions and distributions. Define for $k \in \mathbb{N}$ the space $\mathcal{K}_{1,+,k}$ as

$$\mathcal{K}_{1,+,k} := \{f \in C^k([0,\infty)) : \sup_{t\geq 0, i=0,\ldots,k} \mathrm{e}^{kt} |f^{(i)}(t)| < \infty\},$$

with the norm

$$\|f\|_{\mathcal{K}_{1,+,k}} := \sup_{t\geq 0, i=0,\ldots,k} \mathrm{e}^{kt} |f^{(i)}(t)|$$

this space becomes a Banach space. We denote, for E a Banach space, $\mathcal{K}'_{1,+,k}(E) := \mathcal{L}(\mathcal{K}_{1,+,k}, E)$ which is obviously a Banach space. We note that $\mathcal{K}_{1,+,k+1}$ is contained in $\mathcal{K}_{1,+,k}$ with a continuous embedding. The space $\mathcal{K}_{1,+}$ is defined as the inverse limit (also known as the projective limit) of the spaces $\mathcal{K}_{1,+,k}$, i.e. the intersection of these spaces with as topology the least upper bound topology. Since the $\mathcal{K}_{1,+,k}$ are Banach spaces it follows that $\mathcal{K}_{1,+}$ is a Frechet space.

The space of test functions $\mathcal{K}_{1,+}$ is closely related to $\mathcal{K}_1$ (which is used in for example [20]). The difference is that the elements of $\mathcal{K}_{1,+}$ are defined on the nonnegative axis only and the elements of $\mathcal{K}_1$ are defined on the whole real axis.

We define the space $\mathcal{K}'_{1,+}(E) := \mathcal{L}(\mathcal{K}_{1,+}, E)$.

Alternatively we could have defined $\mathcal{K}_{1,+}$ in the following way. Define for $k \in \mathbb{N}$ the space $\tilde{\mathcal{K}}_{1,+,k}$ as

$$\tilde{\mathcal{K}}_{1,+,k} := \{f \in C^k([0,\infty)) : \sup_{i=0,\ldots,k} \int_0^\infty \mathrm{e}^{kt} |f^{(i)}(t)|\, dt < \infty\},$$

with the norm

$$\|f\|_{\tilde{\mathcal{K}}_{1,+,k}} := \sup_{i=0,\ldots,k} \int_0^\infty \mathrm{e}^{kt} |f^{(i)}(t)|\, dt$$

this space becomes a Banach space. Since

$$\|f\|_{\tilde{\mathcal{K}}_{1,+,k}} \leq \|f\|_{\mathcal{K}_{1,+,k+1}} \leq \|f\|_{\tilde{\mathcal{K}}_{1,+,k+2}}$$

it follows that the inverse limit of the spaces $\tilde{\mathcal{K}}_{1,+,k}$ equals $\mathcal{K}_{1,+}$. We note that $\mathcal{D}_+$ is not dense in any of the $\mathcal{K}_{1,+,k}$, but that it is dense in all the $\tilde{\mathcal{K}}_{1,+,k}$ and in $\mathcal{K}_{1,+}$ (this last fact follows using exactly the same reasoning that shows that $\mathcal{D}$ is dense in $\mathcal{S}$ as given in e.g. Rudin [15, Theorem 7.10]). For each element of $\mathcal{K}'_{1,+}(E) = \mathcal{L}(\mathcal{K}_{1,+}, E)$ there exists a k such that this element is continuous in the topology of $\tilde{\mathcal{K}}_{1,+,k}$ (this follows from [16], which characterizes continuous linear maps between locally convex spaces, using that the semi-norms used in the definition of $\mathcal{K}_{1,+}$ are directed). Since $\mathcal{D}_+$, and hence $\mathcal{K}_{1,+}$, is dense in $\tilde{\mathcal{K}}_{1,+,k}$ the given element of $\mathcal{L}(\mathcal{K}_{1,+}, E)$ extends to an element in $\tilde{\mathcal{K}}'_{1,+,k}(E)$ and this extension is unique. Since there is no possibility of confusion we will denote the extension by the same symbol.

Note that $e_\lambda \in \mathcal{K}_{1,+,k}$ if and only if $\mathrm{Re}\lambda \geq k$, that $e_\lambda \in \tilde{\mathcal{K}}_{1,+,k}$ if and only if $\mathrm{Re}\lambda > k$ and that $e_\lambda \notin \mathcal{K}_{1,+}$ for any $\lambda \in \mathbb{C}$.

Let $S \in \mathcal{K}'_{1,+}(E)$, from the above mentioned result it follows that there exists a $k \in \mathbb{N}$ such that $S \in \tilde{\mathcal{K}}'_{1,+,k}(E)$. We define the Laplace transform of S as $\lambda \mapsto S(e_\lambda)$, this Laplace transform is defined on the right half-plane $\mathrm{Re}\lambda > k$. The particular value of k used only affects the domain of the Laplace transform, not its values.

The above definition is equivalent to the one originally given by Schwartz who reduced the Laplace transform to the Fourier transform. One of the advantages of the above formulation is its similarity to a lesser known extension that we shall recall below. We summarize some well-known properties of this Laplace transform in the following proposition.

Proposition 4.1 *Every element S of $\mathcal{K}'_{1,+}(E)$ has a Laplace transform $\hat{S}$ which is obtained by evaluating at exponentials. This Laplace transform is holomorphic on some right half-plane and there exists a polynomial p such that $\|\hat{S}(\lambda)\| \leq p(|\lambda|)$ for all λ in this right half-plane. Conversely, every polynomially bounded holomorphic E-valued function F on some right half-plane is the Laplace transform of some element of $\mathcal{K}'_{1,+}(E)$. This distribution is given by*

$$S(\varphi) = \frac{1}{2\pi i} \int_\Gamma \hat{\varphi}(i\lambda) F(\lambda) \, d\lambda,$$

where Γ is any vertical line in the mentioned right half-plane and $\hat{\varphi}$ is the Fourier transform of φ.

We review an extension of the above concept of Laplace transform due to Kunstmann [11]. To this end we recall the following spaces that were introduced in [11]. Define for $\alpha > 0, \beta, \gamma \in \mathbb{R}$ the space $\mathcal{W}(\alpha, \beta, \gamma)$ as

$$\mathcal{W}(\alpha, \beta, \gamma) := \{f \in \bigcap_{j \geq 0} C^j([\alpha j, \infty)) : \sup_{j \geq 0, t \geq \alpha j} \mathrm{e}^{(\beta-\gamma)\alpha j + \gamma t} |f^{(j)}(t)| < \infty\},$$

with the norm

$$\|f\|_{\mathcal{W}(\alpha,\beta,\gamma)} := \sup_{j \geq 0, t \geq \alpha j} \mathrm{e}^{(\beta-\gamma)\alpha j + \gamma t} |f^{(j)}(t)|$$

this becomes a Banach space. The space $\mathcal{W}^k(\alpha, \beta, \gamma)$ is defined as follows: $f \in \mathcal{W}^k(\alpha, \beta, \gamma)$ if and only if $f^{(i)} \in \mathcal{W}(\alpha, \beta, \gamma)$ for $i = 0, \ldots, k$. With the norm

$$\|f\|_{\mathcal{W}^k(\alpha,\beta,\gamma)} := \sup_{i=0,\ldots,k} \|f^{(i)}\|_{\mathcal{W}(\alpha,\beta,\gamma)}$$

this becomes a Banach space. We have

$$\mathcal{W}^k(\alpha, \beta, \gamma) \subset \mathcal{W}^{\tilde{k}}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$$

with a continuous embedding if and only if $\alpha \leq \tilde{\alpha}$, $\beta \geq \tilde{\beta}$, $\gamma \geq \tilde{\gamma}$, $k \geq \tilde{k}$. So if we define $(\alpha, \beta, \gamma, k) \leq (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{k})$ if the above holds then the inverse limit of the spaces $\mathcal{W}^k(\alpha, \beta, \gamma)$ is well-defined. This inverse limit is denoted by $\mathcal{W}$. Since the inverse limit over the countable set $\alpha \in \mathbb{Q}$, $\beta, \gamma, k \in \mathbb{N}$ gives the same space the inverse limit $\mathcal{W}$ is a Frechet space. We could have defined $\mathcal{W}$ as the inverse limit of "L^1-type" spaces as was done for $\mathcal{K}_{1,+}$ above. The space $\tilde{W}(\alpha, \beta, \gamma)$ consists of all measurable functions from $[0, \infty)$ to $\mathbb{C}$ such that

$$\sum_{j=0}^{\infty} \mathrm{e}^{j\alpha(\beta-\gamma)} \int_{\alpha j}^{\infty} \mathrm{e}^{\gamma t} |f^{(j)}(t)| \, dt < \infty,$$

where the derivative is taken in the weak sense and the norm is the expression on the left-hand side. The space $\mathcal{D}_+$ is dense in all the spaces $\tilde{W}^k(\alpha, \beta, \gamma)$ and in $\mathcal{W}$. We note that $e_\lambda \in \mathcal{W}^k(\alpha, \beta, \gamma)$ if and only if $\lambda \in \overline{\Lambda_L(\alpha, \beta, \gamma)}$ (see

(20) for the definition of Λ_L), $e_\lambda \in \tilde{\mathcal{W}}^k(\alpha,\beta,\gamma)$ if and only if $\lambda \in \Lambda_L(\alpha,\beta,\gamma)$ and e_λ is never in $\mathcal{W}$. We have that if $S \in \mathcal{W}'(E)$ then there exists $\alpha > 0, \beta, \gamma \in \mathbb{R}, k \in \mathbb{N}$ such that $S \in \tilde{\mathcal{W}}^k(\alpha,\beta,\gamma)'$ by the above mentioned result from [16] and we define the Laplace transform as $S(e_\lambda)$, where again only the region of definition and not the values depend on the choices made. We have the following proposition.

Proposition 4.2 *Every element S of $\mathcal{W}'(E)$ has a Laplace transform $\hat{S}$ which is obtained by evaluating at exponentials. This Laplace transform is holomorphic on some logarithmic region and there exists a polynomial p such that $\|\hat{S}(\lambda)\| \leq p(|\lambda|)$ for all λ in this logarithmic region. Conversely, every polynomially bounded holomorphic E-valued function F on some logarithmic region is the Laplace transform of some element of $\mathcal{W}'(E)$. This distribution is given by*

$$S(\varphi) = \frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)F(\lambda)\,d\lambda,$$

where Γ is the boundary of any logarithmic region contained in the mentioned logarithmic region and $\hat{\varphi}$ is the Fourier transform of φ.

The above proposition was proven in [11].

5 Equivalence of distributional control systems and distributional resolvent linear systems

The following was shown for special cases in Chazarain [2] and Kunstmann [10] and in the general case by Kisynksi [8, Theorem 4.18] [9, Section 10 Theorem 1]. Note that the result was not formulated in this way in the above three papers since they did not use the space of distributions $\mathcal{W}$, but that the formulation below easily follows (see also Kunstmann [11, Theorem 4.1]).

Proposition 5.1 *A distribution semigroup is an element of $\mathcal{W}'(\mathcal{L}(\mathcal{X}))$, so it has a Laplace transform which is holomorphic and polynomially bounded on a logarithmic region. This Laplace transform satisfies the resolvent equation. Conversely, every polynomially bounded holomorphic $\mathcal{L}(\mathcal{X})$ valued function* *on a logarithmic region that satisfies the resolvent equation is the Laplace transform of a distribution semigroup.*

We use the above proposition to prove the equivalence of distributional control systems and distributional resolvent linear systems. This proof is very similar (in fact, almost identical) to the proof of the above proposition.

Let $[\mathbb{A}, \mathbb{B}; \mathbb{C}, \mathbb{D}]$ be a distributional control system. We show that this distribution is an element of $\mathcal{W}'(\mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{X} \times \mathcal{Y}))$ and that its Laplace transform satisfies the desired equations. Let $\psi \in \mathcal{D}$ be such that $\psi(0) = 1$, let $\varphi \in \mathcal{W}$ and define

$$\begin{aligned}
\tilde{\mathbb{B}}(\varphi) &:= \varphi(0)\mathbb{B}(\psi) + \mathbb{A}(\varphi')\mathbb{B}(\psi) - \mathbb{A}(\varphi)\mathbb{B}(\psi'),\\
\tilde{\mathbb{C}}(\varphi) &:= \varphi(0)\mathbb{C}(\psi) + \mathbb{C}(\psi)\mathbb{A}(\varphi') - \mathbb{C}(\psi')\mathbb{A}(\varphi),\\
\tilde{\mathbb{D}}(\varphi) &:= \varphi(0)\mathbb{D}(\psi) + \tilde{\mathbb{C}}(\varphi')\mathbb{B}(\psi) - \tilde{\mathbb{C}}(\varphi)\mathbb{B}(\psi').
\end{aligned}$$

Since $\mathbb{A} \in \mathcal{W}'(\mathcal{L}(\mathcal{X}))$ by Proposition 5.1 these are elements of $\mathcal{W}'$. By Proposition 2.3 they coincide with $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$, respectively, on $\mathcal{D}$. Since $\mathcal{D}_+$ is dense in $\mathcal{W}$ they are the unique continuous extensions of these distributions. Hence we can view $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ as elements of $\mathcal{W}'$. Denote the Laplace transforms by $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$, respectively. The equations (10),(11),(12),(13) extend by continuity to $\varphi, \psi \in \mathcal{W}$. From this with $\varphi = e_\beta$ and $\psi = e_\alpha$ we obtain the equation

$$\mathfrak{b}(\beta) - \mathfrak{b}(\alpha) = -\beta\mathfrak{a}(\beta)\mathfrak{b}(\alpha) + \alpha\mathfrak{a}(\beta)\mathfrak{b}(\alpha),$$

which is equation (15) and similarly we obtain (16) and (17).

Let $[\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d}]$ be a distributional resolvent linear system. Let Γ be a curve as in Proposition 4.2 and define for $\varphi \in \mathcal{D}$

$$\begin{aligned}
\mathbb{B}(\varphi) &= \frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\mathfrak{b}(\lambda)\,d\lambda,\\
\mathbb{C}(\varphi) &= \frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\mathfrak{c}(\lambda)\,d\lambda,\\
\mathbb{D}(\varphi) &= \frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\mathfrak{d}(\lambda)\,d\lambda,
\end{aligned}$$

it follows as in Chazarain [2] (see also Kisynski [8, Theorem 4.18]) that these expressions are independent of the particular path Γ and that they are elements of $\mathcal{D}_0'$. We have

$$\psi(0)\mathbb{B}(\varphi) - \varphi(0)\mathbb{B}(\psi) = \psi(0)\frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\mathfrak{b}(\lambda)\,d\lambda - \varphi(0)\frac{1}{2\pi i}\int_\Gamma \hat{\psi}(i\lambda)\mathfrak{b}(\lambda)\,d\lambda.$$

By using Cauchy's theorem and the Fourier inversion formula it follows as in Kisynski [8, page 32] that

$$\eta(0) = \frac{1}{2\pi i}\int_\Gamma \hat{\eta}(i\alpha)\,d\alpha.$$

Substituting this in the above we obtain

$$\frac{1}{2\pi i}\int_\Gamma \hat{\psi}(i\alpha)\,d\alpha\frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\mathfrak{b}(\lambda)\,d\lambda - \frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\alpha)\,d\alpha\frac{1}{2\pi i}\int_\Gamma \hat{\psi}(i\lambda)\mathfrak{b}(\lambda)\,d\lambda.$$

using (15) in the first term and renaming variables in the second we see that the above equals

$$\frac{1}{2\pi i}\int_\Gamma \hat{\psi}(i\alpha)\frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\left[\mathfrak{b}(\alpha) + (\alpha-\lambda)\mathfrak{a}(\lambda)\mathfrak{b}(\alpha)\right]\,d\lambda\,d\alpha$$

$$-\frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\,d\lambda\frac{1}{2\pi i}\int_\Gamma \hat{\psi}(i\alpha)\mathfrak{b}(\alpha)\,d\alpha,$$

canceling terms we obtain

$$\frac{1}{2\pi i}\int_\Gamma \hat{\psi}(i\alpha)\frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)(\alpha-\lambda)\mathfrak{a}(\lambda)\mathfrak{b}(\alpha)\,d\lambda\,d\alpha.$$

On the other hand we have

$$\mathbb{A}(\varphi')\mathbb{B}(\psi) - \mathbb{A}(\varphi)\mathbb{B}(\psi') =$$

$$-\frac{1}{2\pi i}\int_\Gamma \lambda\hat{\varphi}(i\lambda)\mathfrak{a}(\lambda)\,d\lambda\frac{1}{2\pi i}\int_\Gamma \hat{\psi}(is)\mathfrak{b}(s)\,ds + \frac{1}{2\pi i}\int_\Gamma \hat{\varphi}(i\lambda)\mathfrak{a}(\lambda)\,d\lambda\frac{1}{2\pi i}\int_\Gamma s\hat{\psi}(is)\mathfrak{b}(s)\,ds.$$

Hence we see that

$$\psi(0)\mathbb{B}(\varphi) - \varphi(0)\mathbb{B}(\psi) = \mathbb{A}(\varphi')\mathbb{B}(\psi) - \mathbb{A}(\varphi)\mathbb{B}(\psi').$$

The equations (12) and (13) are proved similarly. Hence we obtain the following.

Theorem 5.2 *A distributional control system is an element of $\mathcal{W}'(\mathcal{L}(\mathcal{X}\times\mathcal{U},\mathcal{X}\times\mathcal{Y}))$, its Laplace transform is a distributional resolvent linear system. Conversely, every distributional resolvent linear system is the Laplace transform of a distributional control system.*

We now return to the study of exponentially bounded systems. A distribution is called *exponentially bounded* if it is an element of $\mathcal{K}'_{1,+}$. We call a distributional control system *exponentially bounded* if the distribution $[\mathbb{A}, \mathbb{B}; \mathbb{C}, \mathbb{D}]$ is exponentially bounded. It is easily seen that a distributional control system is exponentially bounded if and only if $\mathbb{A}$ is (this is similar to the fact proven above that a distributional control system is in $\mathcal{W}'$ if and only if $\mathbb{A}$ is). The following theorem is then easily proven.

Theorem 5.3 *The Laplace transform of an exponentially bounded distributional control system is an exponentially bounded distributional resolvent linear system. Conversely, every exponentially bounded distributional resolvent linear system is the Laplace transform of an exponentially bounded distributional control system.*

Remark 5.4 *The example that we mentioned in the introduction, the heat equation with Dirichlet control and Neumann observation, was shown to be desribed by an exponentially bounded distributional resolvent linear system in [13]. It follows from the results obtained that it can also be described by an exponentially bounded distributional control system.*

6 Realization theory

It is easy to see that the characteristic function of an exponentially bounded distributional resolvent linear system is polynomially bounded on the right half-plane Λ_H. We will show that the converse is also true: any polynomially bounded holomorphic operator-valued function defined on some right half-plane is the characteristic function of some exponentially bounded distributional resolvent linear system.

Theorem 6.1 *Let $\mathcal{U}$ and $\mathcal{Y}$ be Banach spaces and let $G : \Lambda_H(\gamma) \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ be holomorphic and satisfy*

$$\|G(s)\| \leq p(|s|) \quad \forall\ s \in \Lambda_H$$

for some polynomial p. Then there exists an exponentially bounded distributional resolvent linear system such that its characteristic function restricted to some right half-plane equals G.

Proof Since G is polynomially bounded there exists a k such that $G_0(s) := G(s)/s^k$ is uniformly bounded on some right half-plane. This G_0 has a realization as a well-posed linear system [18, Section 2.6 combined with Corollary 4.6.10] (note that every well-posed linear system is an exponentially bounded distributional resolvent linear system). That the function s has a realization as an exponentially bounded resolvent linear system was already mentioned in Remark 3.4: the resolvent linear system

$$\mathfrak{a} = 0, \quad \mathfrak{b} = -1, \quad \mathfrak{c} = 1, \quad \mathfrak{d} = s$$

is a 1-dimensional realization of s. As is easily checked, one obtains a realization for a product of two functions in terms of realizations for the individual functions as follows:

$$\mathfrak{a} = \begin{bmatrix} \mathfrak{a}_1 & 0 \\ \mathfrak{b}_2\mathfrak{c}_1 & \mathfrak{a}_2 \end{bmatrix}, \quad \mathfrak{b} = \begin{bmatrix} \mathfrak{b}_1 \\ \mathfrak{b}_2\mathfrak{d}_1 \end{bmatrix}, \quad \mathfrak{c} = [\mathfrak{d}_2\mathfrak{c}_1 \ \ \mathfrak{c}_2], \quad \mathfrak{d} = \mathfrak{d}_2\mathfrak{d}_1.$$

From these formulas we conclude that if the realizations for the factors are exponentially bounded distributional resolvent linear systems then the same is true for the product. Since we have exponentially bounded realizations for G_0 and s we can obtain an exponentially bounded realization for $G(s) = s^k G_0(s)$. $\Box$
Using Theorems 5.3 and 6.1 we then obtain the following.

Theorem 6.2 *Let $\mathcal{U}$ and $\mathcal{Y}$ be Banach spaces. Every exponentially bounded $\mathcal{L}(\mathcal{U},\mathcal{Y})$ valued distribution is the input-output impulse response of an exponentially bounded distributional control system.*

Finally, we formulate the following conjecture.

Conjecture 6.3 *Every distribution in $\mathcal{W}'(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ is the input-output impulse response of a distributional control system.*

Acknowledgement The author would like to thank Erik Thomas for sharing some of his insight into the theory of topological vector spaces.

References

[1] W. Arendt, O. El-Mennaoui, V. & Kéyantuo. Local integrated semigroups: evolution with jumps of regularity. *J. Math. Anal. Appl.*, 186(2):572–595, 1994.

[2] J. Chazarain. Problèmes de Cauchy abstraits et applications à quelques problèmes mixtes. *J. Functional Analysis*, 7:386–446, 1971.

[3] Ruth F. Curtain & Hans Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

[4] Ruth F. Curtain & Mark R. Opmeer. Normalized doubly coprime factorizations for infinite-dimensional linear systems, *Mathematics of control, systems and signals*, to appear.

[5] Ruth F. Curtain. Robustly stabilizing controllers with respect to left-coprime factor pertubations for infinite-dimensional linear systems, proceedings of the CDC 2005, to appear.

[6] L. Dai. *Singular control systems*, volume 118 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1989.

[7] Alexandre Grothendieck. *Topological vector spaces* Gordon and Breach Science Publishers, New York, 1973.

[8] Jan Kisyński. Pseudoresolvents and distribution semigroups *Inst. of mathematics. Pol. acad. sciences* preprint nr. 617, 2001, available at `www.impan.gov.pl/Preprints/p617.ps`.

[9] Jan Kisyński. Distribution semigroups and one parameter semigroups. *Bull. Polish Acad. Sci. Math.*, 50(2):189–216, 2002.

[10] Peer Christian Kunstmann. Distribution semigroups and abstract Cauchy problems. *Trans. Amer. Math. Soc.*, 351(2):837–856, 1999.

[11] Peer Christian Kunstmann. Laplace transform theory for logarithmic regions. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 125–138. Dekker, New York, 2001.

[12] J.-L. Lions. Les semi groupes distributions. *Portugal. Math.*, 19:141–164, 1960.

[13] Mark R. Opmeer. Infinite-dimensional linear systems: a distributional approach. *Proceedings of the London mathematical society*, 91(3): 738–760, 2005.

[14] Mark R. Opmeer. LQG balancing for continuous-time infinite-dimensional linear systems, submitted.

[15] Walter Rudin. *Functional analysis*. McGraw-Hill, New York, 1991.

[16] H.H. Schaefer and M.P. Wolff. *Topological vector spaces*. Springer, New York, 1999.

[17] Laurent Schwartz. *Théorie des distributions*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[18] Olof Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[19] Sheng Wang Wang. Quasi-distribution semigroups and integrated semigroups. *J. Funct. Anal.*, 146(2):352–381, 1997.

[20] Z. Zieleźny. On the space of convolution operators in $\mathcal{K}_1'$. *Studia Math.*, 31:111–124, 1968.