Value-at-risk

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Outline

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- The measurement of risk is a central aspect of managing investments in financial markets. The measurement of risk determines to a large degree the degree of diversification in a portfolio of assets.
- We will look at an alternative to measuring risk using the variance or volatility and then explore how this risk measure can be used to management risk ina portfolio of assets.

Outline

- We will first outline the limits of volatility as a risk measure and then suggest Value-at-Risk as an alternative risk measure.
- We will then see how Value-at-Risk can be used to manage the risk in a portfolio of assets.

- Measuring risk
- Value-at-Risk
- Managing portfolio risk
- Discussion

- We will first look at the properties of risk measures and how volatility fits into these properties.

Limits to volatility as a risk measure

- Risk is the possibility of a loss
- Volatility does not only capture negative outcomes below the mean, but also positive outcomes above the mean
- A risk measure is needed that focuses exclusively on losses
- The risk measure should capture the size losses better than volatility, especially fat tails

Limits to volatility as a risk measure

- → Volatility (standard deviation) or the variance of returns of assets are the standard risk measures used in finance. We will now outline why this risk measure is often not a good risk measure.
- In its normal definition outside of finance, risk is seen as a possibility to make a loss. Thus there has to be some uncertainty whether a loss will be incurred at all, and the risk focuses only on the losses that may emerge. Such losses have to be defined relative to a benchmark, which could be the status quo and imply a return of zero, or another benchmark return that might, for example, represent the opportunity costs of the investment; this could be the expected return, but also any other return that can be justified.
- Volatility does not only focus on the negative outcomes (losses), but also the positive outcomes (profits) and has a fixed benchmark in the form of the expected return as all deviations are assessed relative to the expected return.
- In order to become consistent with the common definition of risk, a risk measure should be focussing on the losses relative to a benchmark that is given.
-
    - A suitable risk measure should be able to better assess the side of losses,
    - especially taking into account so-called fat tails, that is distributions that have more events at the extremes of the outcomes than a normal distribution. It is common in financial markets to find distributions and large losses are significantly more often observed than implied by using a normal distribution.
- → We will now look at the idea of an alternative risk measure that takes these aspects into account.

Desirable properties of a risk measure

- A risk measure could be the reasonable amount that can be lost within a given time horizon
- What is reasonable will depend on the implications losses have and the risk aversion of the user
- The more severe the impact and the more risk averse the user is, the smaller the loss beyond what is reasonable should be
- Reporting risk as potential losses has the advantage that the result is intuitively understood by decision-makers

Desirable properties of a risk measure

- → We will provide now some thoughts on properties a risk measure should have.
- ▶
  - We suggest that a risk measure could provide an estimate of the loss that can reasonably incurred by the investor.
  - We will have to determine a time period in which such losses can be accumulated.
- ▶
  - The definition of what a reasonable loss is, will depend on the one hand on the impact any loss has, that is how much can be afforded to be lost. There might be regulatory implications of making losses or future obligations might require a minimum amount of value to be retained; this might affect the amount that can be lost.
  - On the other hand, the attitude of the investor to risk will also be important. It will depend on his risk aversion and how much losses he is willing to accept.
- ▶ If the impact of losses are severe, either because of their direct consequences on the investor or his preferences, losses beyond this reasonable level should be less likely to occur.
- ▶ We thus propose to use the potential reasonable losses as a risk measure. This has an additional advantage that the resulting risk measure (the loss) is much more intuitively understood that using volatility, or other similar measures. Having such an intuitive understanding helps in determining whether the risks actually taken are too high and measures need to be taken to reduce risks.
- → Based on these ideas, we are now able to define more formally a risk measure.

- Measuring risk
- Value-at-Risk
- Managing portfolio risk
- Discussion

- Value-at-Risk, which is based on the above ideas, was the main risk measure that has been used by banks until the early 2020s, when it was slowly superseded by expected shortfall.
- Value-at-Risk was introduced in the mid 1990s and become the standard risk measure to calcu;ate capital requirements based on the Basel regulations that were imposed on banks by most regulators.
- Since the 2020s this risk measure has fallen out of favour and is slowly being replaced with expected shortfall, a similar measure that uses the Value-at-Risk idea.
- Due to its widespread use in banks, Value-at-Risk has also been used in other areas of finance, such as investment management.

Probability of large losses

- The Value-at-Risk is a statement that the loss will not exceed this amount with a probability of c over the next T time periods.
- Losses will only be larger than the Value-at-Risk with probability $1 - c$
- A loss needs to be defined relative to a benchmark, which could be the status quo (absolute loss) or the expected outcome (relative loss)
- In financial markets returns are small and for simplicity the relative loss is commonly used

→ The key idea is that the Value-at-Risk is set such that larger losses are unlikely to occur.

- Value-at-Risk states that the loss will not exceed a certain amount, called 'Value-at-Risk', with an exogenously given probability in a given time period. Value-at-Risk thus measures the size of the loss that is not exceeded with some probability. This can be interpreted as the size of the 'reasonable' loss and reasonable is defined as this probability.
- Losses can exceed this Value-at-Risk with a probability of $1-c$, thus it does not determine the maximum loss that can be incurred. Expected shortfall builds on this idea and determines the expected size of the losses, given that the losses exceed the Value-at-Risk.
-
  - We have to determine a loss relative to a benchmark as we need to see which outcomes are classified as a loss.
  - The benchmark could be the current value of the investment, thus a return of zero,
  - this is often referred to as the absolute loss.
  - We might also use the expected outcome as a benchmark. In this case we take into account opportunity costs of the investment and interpret as a loss any shortfall to this expected return.
  - When using the expected return as a benchmark, the loss is referred to as the relative loss.
- We usually use short time horizons for investments in financial markets and over such short time periods the expected return is commonly close to zero and for computational and mathematical simplicity we usually will use the relative loss, thus the expected return as our benchmark. The difference in the risk measure will be minimal as the expected return is close to zero. Of course, we could use alternative benchmarks, but in most cases the differences to the expected returns will be very small.

→ We can now continue to formally define Value-at-Risk.

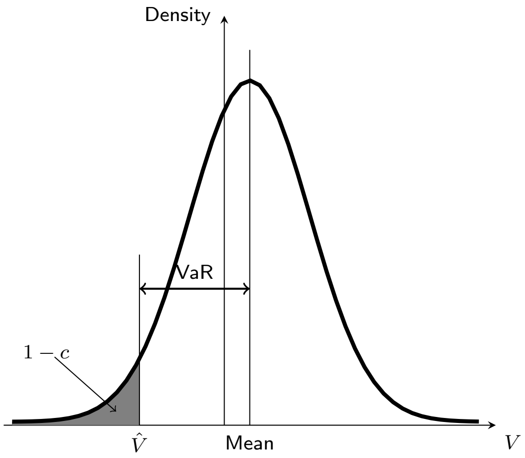
Definition of Value-at-Risk

- Define a threshold such that the probability that the outcome is below this threshold is given by a certain value
- $Prob(V < \hat{V}) = 1 - c$
- The Value-at-Risk is then given as the difference between this threshold and the expected outcome
- $\text{VaR} = E[V] - \hat{V}$
- Value-at-Risk is the estimation of the $1 - c$-quantile of the distribution of outcomes

Definition of Value-at-Risk

- → The formal definition of the Value-at-Risk will focus solely on the losses in the possible outcomes.
- ▶ We define a reference outcome such that the probability of the actual outcome being below this reference outcome is exactly $1 - c$. This reference outcome is the one that the actual outcome is not falling below 'too often'.
- ▶ *Formula*
- ▶ We can now use this reference outcome to determine the Value-at-Risk. We had chosen the relative loss and hence the reasonable losses will be the difference to any outcomes below this bechmark of the expected return.
- ▶ *Formula*
- ▶ In essence, we determine the Value-at-Risk as the estimation of the $1 - c$-quantile of outcomes. We will also have to estimate the expected outcome.
- → The Value-at-Risk is thus defined implicity through the quantile of the distribution of outcomes and we focus solely on the lower tail of the outcomes. Note that if we are holding a short position in the asset, the losses occur as the asset incerases in value and hence the Value-at-Risk would be defined at the upper tail of the distribution.

Value-at-Risk as a quantile



Value-at-Risk as a quantile

- → We will now illustrate the Value-at-Risk graphically.
- ▶ We consider the outcomes, which is usually measured as the return of an investment, but could also be the value of the assets directly. We will consider the density distribution of these possible outcomes.
- ▶ This distribution is shown here.
- ▶ We can easily determine the mean of this distribution.
- ▶ and then we determine the $1 - c$ quantile of this distribution.
- ▶ This quantile gives us reference outcome.
- ▶ The Value-at-Risk will then be difference between these two values.
- ▶ We can easily see that as the expected outcome (mean) is close to zero in financial markets, that the impact on the size of the Value-at-Risk will be minimal.
- → Determining the Value-at-Risk is most importantly about determining the reference outcome ($\hat{V}$) as that essentially determines result.

Value-at-Risk for normal distributions

- If the distribution is normal, the $1-c$-quantile can be determined using the quantiles of the standard normal distribution
- The standard normal distribution needs to be adjusted by the standard deviation of outcomes and the amount invested
- $VaR = \alpha\sigma V_0$
- The choice of quantile will depend on the risk aversion of the investor
- The more risk averse an investor is, the lower the quantile to cover a wider range of possible losses

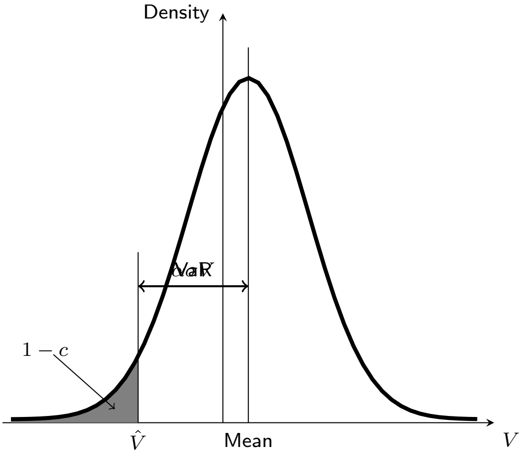
Value-at-Risk for normal distributions

→ We will now look at a special case of the distribution being a normal distribution; this will allow us to obtain some explicit formulae to determine the Value-at-Risk. Similar formulae and approaches can be used for other, more realistic return distributions.

- If we assume that outcomes are normally distributed, we can use the quantile of the standard normal distribution to determine the quantile of the distribution required.
- We only need to take into account the standard deviation of the actual distribution, in addition to takign into account the current value of the investment if the distribution is for the returns.
- As the Value-at-Risk is the difference between the qualtile and the expected outcome, we do not need to determine the expected outcome separately, the quantile of the standard normal distribution, suitably adjusted using α, gives us this difference (remember that the mean of the standard normal distribution is zero).
- It was argued above that the quantile chosen was the result of the risk aversion of the investor. This quantile now translates in a specific α.
- The more risk averse the investor is, the more possible outcomes he would want to consider and thus the quantile chosen will become ever smaller. This will result in α becoming larger.

→ We can now graphically illustrate the Value-at-Risk with normally distributed outcomes.

Quantiles with a normal distribution



Quantiles with a normal distribution

→ We will look at how the Value-at-Risk can be determined graphically if the distribution of outcomes is normal.

- This was the graphical definition of the Value-at-Risk as used previously
- We now know that the distance of the mean to the $1 - c$-quantile in a standard normal distribution is α. In a general distribution this will be $\alpha\sigma$, σ being the standard deviation.

→ We can now apply the idea of Value-at-Risk to an investment portfolio.

- Measuring risk
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- Thus var we only looked at the investments and their return distributions overall. We will now consider that investments are in most cases a portfolio of assets and use the Value-at-Risk to manage the risks in such a portfolio.

Impact of assets on portfolio risk

- Investors usually hold a portfolio of assets and using its standard deviation we obtain the Value-at-Risk: $VaR = \alpha \sigma_p V_0$
- Portfolio variance: $\sigma_P^2 = \sum_{i=1}^N \sum_{j=1}^N \omega_i \omega_j \sigma_{ij}$

$$= \sum_{i=1}^N \omega_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{j=1,j\neq i}^N \omega_i \omega_j \sigma_{ij}$$

$$\Rightarrow \frac{\partial \sigma_P^2}{\partial \omega_i} = 2\omega_i \sigma_i^2 + 2\sum_{j=1,j\neq i}^N \omega_j \sigma_{ij}$$

$$= 2Cov\left[R_i, \omega_i R_i + \sum_{j=1,j\neq i}^N \omega_j R_j\right]$$

$$= 2Cov[R_i, R_P] \equiv 2\sigma_{iP}$$

→ We will first seek to determine the impact a single asset has on the risk of a portfolio.

- • Let us assume that an investor holds a portfolio of assets and that its returns are jointly normally distributed. We then can get the Value-at-Risk of this portfolio using the standard deviation of the portfolio.
  • *Formula*
- With weight ω_i of an asset in the portfolio, we can easily get the variance of the portfolio as in the *Formula*.
- [] We can rewrite this expression be separating the variance and covariance terms out, remembering that $\sigma_{ii} = \sigma_i^2$
- [⇒] To assess the impact of a single asset on the portfolio variance, we look at how much this portfolio variance changes, if we change the weight of this asset in the partfolio marginally.
- This expression can be rewritten using the definition of variances and covariances. We now that $\omega_i \sigma_i^2 = \omega_i \text{Var}\left[R_i\right] = \omega_i \text{Cov}\left[R_i\right]$ and $\omega_j \sigma_{ij} = \omega_j \text{Cov}\left[R_j\right]$. We then use that $\text{Cov}\left[X, aX + bY\right] = \text{Cov}\left[X, aX\right] + \text{Cov}\left[X, bY\right] = a\text{Cov}\left[X, X\right] + b\text{Cov}\left[X, Y\right] = a\text{Var}\left[X\right] + b\text{Cov}\left[X, bY\right]$
- We then recognise that the final term represents the portfolio return as it is weighted average return of all assets. We write this as the Covariance of the asset with the portfolio the investor holds.

→ We can use this result to determine the impacta change in the partfolio weight has on the Value-at-Risk of the investor.

Marginal Value-at-Risk

- We are interested in how the Value-at-Risk changes as the weight of assets in the portfolio changes
- $$\frac{\partial \frac{VaR}{V_0}}{\partial \omega_i} = \alpha \frac{\partial \sigma_P}{\partial \omega_i} = \alpha \sigma_P \beta_i = \beta_i \frac{VaR}{V}$$
- This expression is referred to as the marginal Value-at-Risk, ∂VaR_i
- If we change the weight of asset i by a small amount, the Value-at-Risk changes by ∂VaR_i

- → Having obtained the marginal impact of an asset on the portfolio variance, we can now determine the marginal influence of the asset on the Value-at-Risk.
- Our aim is to obtain information on how much the Value-at-Risk of the portfolio changes if we change the weight of a single asset marginally.
- The marginal impact will be assessed for the Value-at-Risk relative to the current value of the portfolio; this is done solely to eliminate any effect the size of the portfolio has on our results.
- [] We insert from the result above for the marginal impact on the variance, noting that $\frac{\partial \sigma_P^2}{\partial \omega_i} = 2\sigma_P \frac{\partial \sigma_P}{\partial \omega_i}$ and we define $\beta_i = \frac{\sigma_{iP}}{\sigma_P^2}$.
- [] Using the definition of the Value-at-Risk abovem the first two terms can be replaced.
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  - As this expression is marginal impact the asset has on the Value-at-Risk of the partfolio, also called the 'marginal Value-at-Risk'.
  - The notation for this marginal Value-at-Risk is often given as shown here.
- The marginal Value-at-Risk determines by how much the value-at-Risk of the partfolio changes if the weight of an asset is changed.
- → We can now use this marginal value-at-Risk to determine how the Value-at-Risk will change if the portfolio composition changes.

Changes to the Value-at-Risk of a portfolio

- If we change the weight more than marginally, we can use a linear approximation of the change
- $\Delta VaR_i = \partial VaR_i V \Delta\omega_i = \beta_i \Delta\omega_i VaR$
- The total change in the Value-at-Risk is equal to the sum of the changes for each individual asset
- $\Delta VaR = \sum_{i=1}^{N} \Delta VaR_i = VaR \sum_{i=1}^{N} \beta_i \Delta\omega_i$
- If we only rearrange the weights, the total changes in the weights must be zero: $\sum_{i=1}^{N} \Delta\omega_i = 0$

Changes to the Value-at-Risk of a portfolio

→ We can now see how a change in the weights of several assets affects the value-at-Risk of the portfolio.

- We have determined the impact a marginal change in the weight of an asset has on the Value-at-Risk of the portfolio. We can now use a linear approximation to determine the impact on the Value-at-Risk of a larger change in the weight of a single asset.
- The first equation make this linear approximation and the second equation uses out result from the marginal Value-at-Risk.
- If we change the weights of several assets, these changes to the Value-at-Risk for each asset can be added up.
- The first equation shows this additive feature of the change of the Value-at-Risk and the second equation inserts from the previous expression.
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    - We now have to consider that when making changes to the portfolio, the total weights of the new portfolio still have to add up to 1. It is therefore that all weight changes together must be zero.
    - *Formula*

→ We can use this result to actively manage the risks of the portfolio.

Changing portfolio risk

- To reduce the risk of a portfolio, reduce the weight of those assets with high marginal Value-at-Risks, high β_i, and increase those with low marginal Value-at-Risk, low β_i
- Reducing the weight of an asset with a high β_i reduces the Value-at-Risk considerably and increasing the weight of an asset with low β_i increases it by less, leading to a reduction in the Value-at-Risk
- The larger the difference between these two assets is, the bigger the impact on the Value-at-Risk
- For two assets the solution is unique, but for more assets many solutions exist
- Not always is it desirable or possible to change the weight of an asset, strategic investment decisions might become relevant
- The marginal Value-at-Risk gives indication which assets to choose most efficiently

Changing portfolio risk

→ Suppose you wanted to reduce the Value-at-Risk of the portfolio and achieve this by changing your composition of the portfolio, thus changing the weights of the assets in the portfolio.

- - We propose that in order to achieve the reduction in the Value-at-Risk assets with a high marginal Value-at-Risk are reduced.
  - We have seen that the marginal Value-at-Risk will be affected by $beta_i$, hence we choose an asset with a high value for β_i.
  - As the total change of asset weights must be zero, we need to increase the weight of other assets. we propose that we choose assets with a low marginal Value-at-Risk.
  - This will imply assets with a low β_i.
- - If we reduce the weight of an asset with a high marginal Value-at-Risk, , the Value-at-Risk reduces by a large amount.
  - If we increase the weight of an asset with a low marginal Value-at-Risk, , the Value-at-Risk increases by a small amount.
  - The net effect is that we have a large reduction and a small increase, resulting in an overall reduction of the Value-at-Risk.
- If the differences between the marginal Value-at-Risks (and hence β_is) are larger, the impact in the Value-at-Risk will be larger.
- - If we have a portfolio of two assets, we can get a unique solution for each reduction in the Value-at-Risk. This is because we know that $\Delta\omega_i = 1 - \Delta\omega_i$ and we thus only need to determine one variable for a given ΔVaR
  - If we have more than two assets, multiple solutions exist and we might choose those assets with the highest and lowest marginal Value-at-Risks, respectively. We might also want to select weight such that the weight changes across all assets are minimal, or impose other restrictions.
- - When holding a portfolio, it is not always possible to change the weights of all assets freely, there might be additional restrictions that need to be considered.
  - It might be a strategic decision to remain invested into certain assets, or some assets cannot be invested into more due to regulatory constraints.
- The marginal Value-at-Risk gives an indication for which asset weights to change when adjusting the portfolio, but we need to take into account any other considerations that are outside the scope of risk assessment; this might include the returns of assets, for example, in addition to the above considerations.

→ We have seen how Value-at-Risk can not only be used to assess risk, but also manage and adjust risks in a portfolio of assets.

- Measuring risk
- Value-at-Risk
- Managing portfolio risk
- **Discussion**

- We will now briefly discuss the properties of Value-at-Risk, pointing out its advantages and drawbacks.

The benefits of using Value-at-Risk

- Value-at-Risk can be used to measure risk in an intuitive way by focussing exclusively on losses
- It provides a framework in which risks of individual assets in a portfolio can be assessed
- Portfolios can be re-arranged to meet risk limits and the marginal Value-at-Risk can be used to identify assets that should change weights

→ We will first point out the benefits from using value-at-Risk compared to other risk measures, such as the volatility.

- Value-at-Risk gives the amount that can reasonably be lost over the given time horizon. The risk measure will be expressed in currency units, rather than a more abstract concept such as volatility. We have also seen that Value-at-Risk focus on losses only, which are commonly seen as being the risk; it ignores any profits.
- We can also assess the risks of individual assets ina portfolio and how much they contribute to the overall risk of the portfolio.
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    - We have then seen how Value-at-Risk can be adjusted to reduce risks, for example to meet risk limits set by regulators or by the investor's own preferences.
    - We used the marginal Value-at-Risk to identify assets that can be used to make adjustments to the portfolio and were able to quantify these adjustments. This was done in the same framework as the original risk measure.

→ Value-at-Risk does not only have benefits, there are significant shortcomings, which we will discuss next.

Limits of Value-at-Risk

- You have a single loan worth £100m with a probability of default 0.9%, the amount lost in case of default is the full amount

$\Rightarrow$ 99% VaR: £-0.9m

- Suppose now we have two loans of £50m each with the same default rate and defaults are independent
- Prob (1 default occurs) $= 2 \times$ Prob (default) $(1 -$ Prob (default)$) = 0.017838$
- Prob (2 defaults occur) $=$ Prob (default)$^2 = 0.000081$

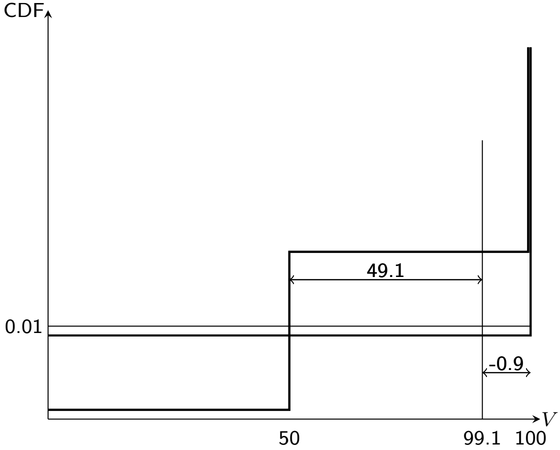
$\Rightarrow$ 99% VaR: £49.1m

$\Rightarrow$ The VaR increases with diversification

Limits of Value-at-Risk

- → Despite Value-at-Risk being intuitive and easy to apply, it might now always reflect the risks of a portfolio accurately. We will show a simple example to illustrate this problem.
- Let us assume that an investor has invested into a single loan with the properties outlined here.
- [⇒] The outcome of this loan has no normal distribution, hence we apply the quantiles directly. The expected value of the loan is $(1 - 0.009)\,100 = 99.1$ as in 99.1% of cases the loan is repaid in full and in the remaining 0.9% no repayment is made. The 1% quantile $(1 - c$, corresponding to a 99% Value-at-Risk) is 100, this is because there is no loss with a probability of 1% or higher. Therefore our Value-at-Risk is -0.9. Note that Value-at-Risk can be negative in some instances.
- The investor now diversifies and holds two such loans, each worth 50 and their defaults are independent of each other.
-
  - We have one default is loan 1 fails, but loan 2 does not fail, or vice versa,
  - giving the probability of this even as detailed here.
-
  - We have both loans defaulting if loan 1 fails and loan 2 fails,
  - giving the probability of this even as detailed here.
- [⇒] Having two fails and hence a loss of 100 has a probability below 1% and can hence be ignored, but having one loan fail has a probability of more than 1%, giving a loss of 50. With the expected value being unchanged at 99.1, the Value-at-Risk is 49.1.
- [⇒] We thus have an increase in the Value-at-Risk as we diversify our portfolio and intuitively it is clear that the risk will be reduced; this is not reflected in the risk measure. Value-at-Risk is therefore not always accounting correctly for diversification and it could be possible for an investor to reduce diversification and this might reduce the Value-at-Risk.
- → While such opportunities are not common, they might arise or with the help of derivatives can be created. This allows investors potentially to circumvent regulatory limits on their risk-taking and is a drawback of Value-at-Risk.

Diversification increasing Value-at-Risk



Diversification increasing Value-at-Risk

→ We can now illustrate this example graphically.

- We will look at the possible outcomes of the loan repayment. Instead of the density we here use the cumulative density (probability function) for easier representation of results.
- The expected value of the loan portfolio in both cases was found to be 99.1 and the full repayment would be 100.
- We use the 99% Value-at-Risk, hence we need to determine the 1% quantile (0.01).
- The single loan is either repaid fully or with a probability of 0.9% not repaid at all; thus the CDF of this loan is given here. We see that the CDF crosses the 1% quantile at 100, the full repayment of the loan.
- Given the expected value of 99.1, this gives a Va;ue-at-Risk of -0.9.
- Let us now consider the investment into two loans. With a very small probability they are not repaid at all, but witha probability of approximately 1.8% only one loan is repaid, leaving a repayment of 50. Both loans are repaid with the remainder probability. We now see that this portfolio CDF crosses the 1% quantile at 50.
- With the expected value unchanged, this gives a Value-at-Risk of 49.1, larger than in the undiversified portfolio.

→ It was this shortcoming of Value-at-Risk, combined with it ignoring the size of any losses below its reference outcomes, that lead to the development of expected shortfall and its ultimate adoption as a risk measure in banking regulation.

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