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Value-at-risk

Outline

- Measuring risk
- Value-at-Risk
- Managing portfolio risk
- Discussion

Limits to volatility as a risk measure

- Risk is the possibility of a loss
- Volatility does not only capture negative outcomes below the mean, but also positive outcomes above the mean
- A risk measure is needed that focuses exclusively on losses
- The risk measure should capture the size losses better than volatility, especially fat tails

Desirable properties of a risk measure

- A risk measure could be the reasonable amount that can be lost within a given time horizon
- What is reasonable will depend on the implications losses have and the risk aversion of the user
- The more severe the impact and the more risk averse the user is, the smaller the loss beyond what is reasonable should be
- Reporting risk as potential losses has the advantage that the result is intuitively understood by decision-makers

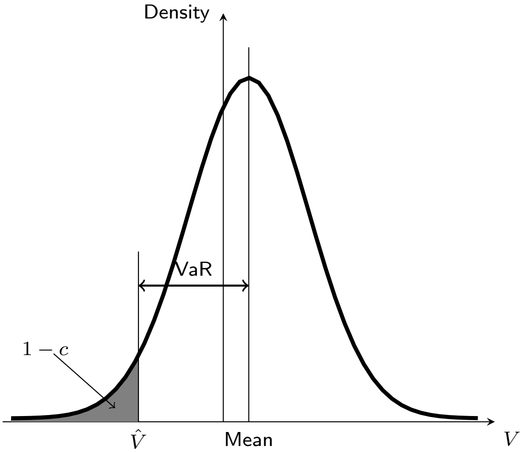
Probability of large losses

- The Value-at-Risk is a statement that the loss will not exceed this amount with a probability of c over the next T time periods.
- Losses will only be larger than the Value-at-Risk with probability $1 - c$
- A loss needs to be defined relative to a benchmark, which could be the status quo (absolute loss) or the expected outcome (relative loss)
- In financial markets returns are small and for simplicity the relative loss is commonly used

Definition of Value-at-Risk

- Define a threshold such that the probability that the outcome is below this threshold is given by a certain value
- $Prob(V < \hat{V}) = 1 - c$
- The Value-at-Risk is then given as the difference between this threshold and the expected outcome
- VaR $= E[V] - \hat{V}$
- Value-at-Risk is the estimation of the $1 - c$-quantile of the distribution of outcomes

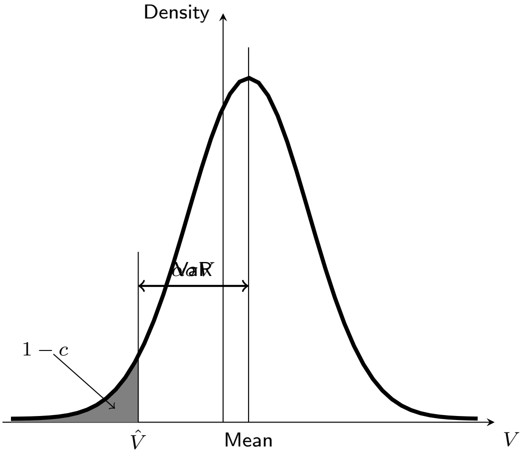
Value-at-Risk as a quantile



Value-at-Risk for normal distributions

- If the distribution is normal, the $1 - c$-quantile can be determined using the quantiles of the standard normal distribution
- The standard normal distribution needs to be adjusted by the standard deviation of outcomes and the amount invested
- $VaR = \alpha\sigma V_0$
- The choice of quantile will depend on the risk aversion of the investor
- The more risk averse an investor is, the lower the quantile to cover a wider range of possible losses

Quantiles with a normal distribution



- Measuring risk
- Value-at-Risk
- **Managing portfolio risk**
- Discussion

Impact of assets on portfolio risk

- Investors usually hold a portfolio of assets and using its standard deviation we obtain the Value-at-Risk: $VaR = \alpha \sigma_p V_0$
- Portfolio variance: $\sigma_P^2 = \sum_{i=1}^N \sum_{j=1}^N \omega_i \omega_j \sigma_{ij}$

$$= \sum_{i=1}^N \omega_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{j=1, j \neq i}^N \omega_i \omega_j \sigma_{ij}$$

$$\Rightarrow \frac{\partial \sigma_P^2}{\partial \omega_i} = 2\omega_i \sigma_i^2 + 2 \sum_{j=1, j \neq i}^N \omega_j \sigma_{ij}$$

$$= 2Cov\left[R_i, \omega_i R_i + \sum_{j=1, j \neq i}^N \omega_j R_j\right]$$

$$= 2Cov[R_i, R_P] \equiv 2\sigma_{iP}$$

Marginal Value-at-Risk

- We are interested in how the Value-at-Risk changes as the weight of assets in the portfolio changes
- $$\frac{\partial \frac{VaR}{V_0}}{\partial \omega_i} = \alpha \frac{\partial \sigma_P}{\partial \omega_i} = \alpha \sigma_P \beta_i = \beta_i \frac{VaR}{V}$$
- This expression is referred to as the marginal Value-at-Risk, ∂VaR_i
- If we change the weight of asset i by a small amount, the Value-at-Risk changes by ∂VaR_i

Changes to the Value-at-Risk of a portfolio

- If we change the weight more than marginally, we can use a linear approximation of the change
- $\Delta VaR_i = \partial VaR_i V \Delta\omega_i = \beta_i \Delta\omega_i VaR$
- The total change in the Value-at-Risk is equal to the sum of the changes for each individual asset
- $\Delta VaR = \sum_{i=1}^{N} \Delta VaR_i = VaR \sum_{i=1}^{N} \beta_i \Delta\omega_i$
- If we only rearrange the weights, the total changes in the weights must be zero: $\sum_{i=1}^{N} \Delta\omega_i = 0$

Changing portfolio risk

- To reduce the risk of a portfolio, reduce the weight of those assets with high marginal Value-at-Risks, high β_i, and increase those with low marginal Value-at-Risk, low β_i
- Reducing the weight of an asset with a high β_i reduces the Value-at-Risk considerably and increasing the weight of an asset with low β_i increases it by less, leading to a reduction in the Value-at-Risk
- The larger the difference between these two assets is, the bigger the impact on the Value-at-Risk
- For two assets the solution is unique, but for more assets many solutions exist
- Not always is it desirable or possible to change the weight of an asset, strategic investment decisions might become relevant
- The marginal Value-at-Risk gives indication which assets to choose most efficiently

The benefits of using Value-at-Risk

- Value-at-Risk can be used to measure risk in an intuitive way by focussing exclusively on losses
- It provides a framework in which risks of individual assets in a portfolio can be assessed
- Portfolios can be re-arranged to meet risk limits and the marginal Value-at-Risk can be used to identify assets that should change weights

Limits of Value-at-Risk

- You have a single loan worth £100m with a probability of default 0.9%, the amount lost in case of default is the full amount

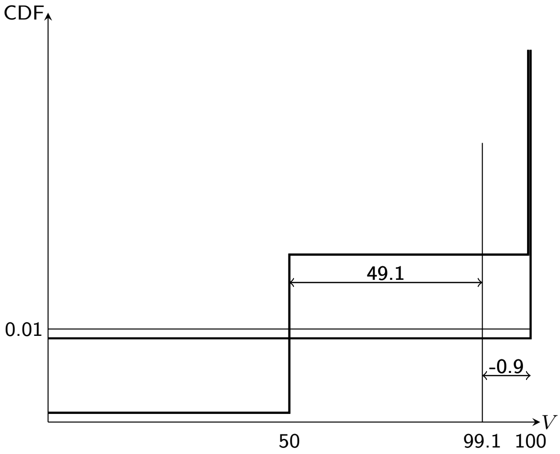
$\Rightarrow$ 99% VaR: £-0.9m

- Suppose now we have two loans of £50m each with the same default rate and defaults are independent
- $\text{Prob}\left(\text{1 default occurs}\right) = 2 \times \text{Prob}\left(\text{default}\right)\left(1 - \text{Prob}\left(\text{default}\right)\right) = 0.017838$
- $\text{Prob}\left(\text{2 defaults occur}\right) = \text{Prob}\left(\text{default}\right)^2 = 0.000081$

$\Rightarrow$ 99% VaR: £49.1m

$\Rightarrow$ The VaR increases with diversification

Diversification increasing Value-at-Risk



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