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Black-Scholes model

- The Black-Scholes model is one of the key theories in finance and option pricing in particular.
- The model assumption itself are very restrictive and the results only apply to standard European options and cannot easily be generalised.
- There are different ways to obtain the Black-Scholes formula of option pricing, through arbitrage, the use of partial differential equations and stochastic calculus, or by using an approximation from the binomial model; this latter approach we will be taking.

Using the binomial option pricing model

- In each time step, the price of the asset will either increase to uS or decrease to dS
- The probability that the price increases is p
- The price will therefore have a binomial distribution
- For a given time period, the number of steps is increased with the size of asset price increases and decreases becoming smaller
- Asset increases and decreases are symmetric: $u = \frac{1}{d}$

$\Rightarrow$ Asset prices become log-normally distributed

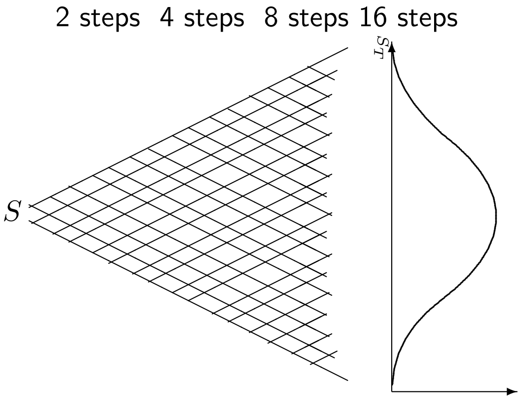
→ We will use the ideas of the binomial model and add some retsrictions before than increasing the number of steps per time period.

- • As with the binomial model, we assume that in each step the price of the underlying asset can either decrease
  • or increase.
- Again, the probability of an increase is set to be p, although the value is irrelevant as we derive the results.
- We now see that the distribution of the price is binomial.
- • We now increase the number of time steps per time period, that is we are making the time periods shorter and shorter.
  • This necessitates that the increases and decreases of the asset price also reduces to maintain a reasonable price range at maturity of the option.
- We further impose that increases and decreases of the asset price are symmetric, thus having the same percentage change.

⇒ Statistics shows that with these constraints, the asset prices at maturity become log-noramlly distributed; the logarith of the asset price will follow a normal distribution.

→ We can now illustrate these assumption in a simple graphic arising from the binomial model

Increasing the number of steps per time period



Increasing the number of steps per time period

→ We use the idea of the binomial model and increase the number of steps that we consider.

- The starting point for the price dynamics is the current price.
- We may start with 2 steps in the time to maturity.
- Then we can increase this to 4 steps; note that the time to maturity does not increase, we only introduce more intermediate steps.
- We can double the time stapes again to 8,
- and again to 16. This can now be increased ever more.
- The distribution of the asset price at maturity will converge towards the log-normal distribution.

→ Using this approach we can use the binomial model and it will converge towards the Black-Scholes formula.

The Black-Scholes formula

- For a large number of steps in the binomial option pricing formula then converges to
- $C = SN(d_1) - Ke^{-rT}N(d_2)$
  $d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$
  $d_2 = d_1 - \sigma\sqrt{T}$
- This is known as the Black-Scholes formula of option pricing
- The variance of the asset replaces the size of the increases and decreases of the asset values: $u = \frac{1}{d} = e^{\sigma\sqrt{\frac{T}{N}}}$
- The asset price changes are replaced by the cumulative normal distribution

The Black-Scholes formula

→ Using this convergence, we can obtain the Black-Scholes formula, although its formal derivation requires the use of more advanced stochastic theory on the convergence of distributions as the number of steps (the sample size) increases.

- We increase the number of steps for the time period towards infinity, and the result that emerges is the Black-Scholes formula.
- it will make use of the cumulative normal distribution $N(\cdot)$, and the volatility of the underlying asset σ, in addition to the current price S, the strike price K, the risk-free rate r, and the time to maturity T.
- This is the famous Black-Scholes formula for standard European options.
-
  - The variance (or volatility) replaces the increases and decreases of the asset price and there is a specific relationship between the two variables.
  - This is shown in the *formula*, with N denoting the number of steps taken per time period.
- The asset price changes in the binomial model are here replaced with their distribution as this conveys the relevant information at maturity.

→ We can now make a few comments on the differences and similarities between the Black-Scholes model and the binomial model.

Similarity of Black-Scholes formula and binomial option pricing

- The structure of the results from binomial option pricing are retained
- Black-Scholes formula: $C = SN(d_1) - Ke^{-rT}N(d_2)$
- Binomial option pricing formula: $C = \Delta S + B$
- The option price is composed of a number of underlying assets and a loan to finance the holding of these assets

Similarity of Black-Scholes formula and binomial option pricing

→ The essence of the results from the binomial model are retained in the Black-Scholes model.

- The basic idea arising from the formulae does not change.
- The Black-Scholes formula was this *formula*.
- The binomial formula was this *formula*.
-
    - The option price is in both cases given by investing into the underlying asset
    - and taking a loan to finance these underlying assets.

→ The key idea of the binomial model is retained in the Black-Scholes formula.

Replicating options

- Options are written by some investors to obtain the premium as revenue, often banks, insurance companies, or hedge funds
- This exposes them to risks if the price movement of the underlying asset causes the option to be exercised
- Option writers may want to hedge their risks
- If creating their own option, they can develop an off-setting position
- They can do so by holding $N(d_1)$ of the underlying asset and obtain a loan of $Ke^{-rT}N(d_2)$
- This is known as option replication

Replicating options

- → We can now use the Black-Scholes formula to create options using the underlying asset and a risk-free asset.
-
  - For some investors it is attractive to sell options as this generates income in the form of the premium.
  - The writers of options are commonly banks,
  - but also large institutional investors like insurance companies
  - or hedge funds.
- While sometimes this might be part of their investment strategy, it will often expose them to risks they do not necessarily seek to be exposed as writing options can expose them to large losses.
- They might want to hedge these risks they are exposed to; buying the same option would offset the risk, but it would also cost them the same in premium which they have received, eliminating any profits.
- Instead they can create their own options and thereby exploit any small deviations of the market price for an option from its value.
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  - They create their own option by holding the underlying asset
  - and obtaining a loan as detailed in the option pricing formula
- This investment strategy that creates the payoffs of an option at maturity is known as an option replication.
- → The Black-Scholes formula is a restricted version of the binomial model, only applicable to standard European options if the underlying asset's price distribution is log-normal. Despite its limitations due to these assumptions, it is convenient to use as it allows us to use an analytical solution (a formula), rather than having to reply on a stepwise procedure that realistically can only be conducted with computers.

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