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Binomial pricing of options

- We determine the value of an option using the most flexible methodology.
- The binomial method can be applied to any type of option, including exotic options.
- The drawback of this method is that it doe snot result in a formula for the option value, but instead uses a numerical approximation of the option value.

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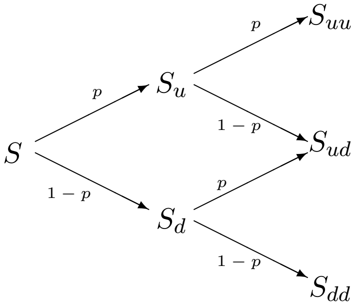
Price development of the underlying asset

- In each time period, the price of the asset will either increase to uS or decrease to dS
- The probability that the price increases is p



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Price development of the underlying asset

→ We first need to make an assumption about the way the price of the underlying asset evolves. This is necessary as the final payment the option provides will depend on this price dynamics.

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  - We assume that in each time period, the price of the underlying asset can increase by a factor u, which usually larger than 1. However, the model would still work even if the price would decrease.
  - Alternatively, the price of the underlying asset will decrease by a factor d, usually less than 1. However, the model would still work even if the price would not decrease but increase by a factor smaller than u. it is only relevant that the asset price can change in one of two possible ways. We can even allow the size of these changes to vary every time period.
- We assume that the higher return is obtained with some probability.
- We can represent the evolution of the asset price with a binomial tree, here shown for two time periods, but more time periods are possible. The current price of the asset is S.
- This price can go up to $S_u = uS$ with probability p,
- or it goes down to $S_d = dS$ with probability $1 - p$.
- The next time period this process is repeated, if the price is S_u, it can go up to $S_{uu} = uS_u = u^2 S$
- or it goes down to $S_u d = dS_u = udS$.
- If the price is S_d, it can go up to $S_{ud} uS_d = udS$
- or it goes down again to $_{dd} = dS_d = d^2 S$

→ We can use this evolution of the asset price to determine the value of an option on this underlying asset.

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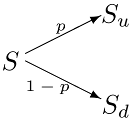
Arbitrage portfolio for a single time period



- The value of the option at maturity can be determined from the contract itself
- For a European call option this is the difference of the asset price and the strike price: $C_T = \max\{0; S_T - K\}$
- Assume the value of the option is given by a combination of the underlying asset and a risk-free asset
- $C = \Delta S + B$
- After one time step this portfolio is worth $C_u = \Delta S_u + (1+r)B$ if the asset value increases
- After one time step this portfolio is worth $C_d = \Delta S_d + (1+r)B$ if the asset value decreases

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→ We initially look at a model with only a single time period and will determine the value of an option in this case.

- The price of the underlying asset will either increase of decrease.
- The value of the option at maturity, here one time period, can be determined by investigating the option contract.
- A standard European call option pays the difference between the price of the underlying asset and the strike price, if it is positive and the option is exercised, or zero otherwise. Other payoff, from put options or exotic options, can be used as well.
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    - We now propose that the value of the option is given by a portfolio combining the underlying asset, of which we use Δ units,
    - and a risk-free asset.
- *Formula*
- At maturity this portfolio will have the value in the *formula* in case the asset price increases. The value of the asset has increased to S_u and the risk-free asset has accumulated interest.
- At maturity this portfolio will have the value in the *formula* in case the asset price decreases. The value of the asset has decreased to S_d and the risk-free asset has accumulated interest.

→ We can now match the value of this portfolio with the pay-offs of the option and determine the value of the option.

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Option value for a single time period

- C_u and C_d are the possible payments of the option at maturity, which are known
- $C_u = \max\{0; uS - K\}$ for a call option

  $C_d = \max\{0; dS - K\}$ for a call option

$\Rightarrow$ $\Delta = \frac{C_u - C_d}{S(u-d)}$

$B = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d} < 0$

$\Rightarrow$ $C = \Delta S + B$

- The option value is given by holding the underlying asset, financed by a loan

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Option value for a single time period

- → We first obtain the principle of pricing an option by using a single time period until maturity.
- ▶ At maturity the two possible values of the portfolio are C_u and C_d. We know the value of the option at maturity.
- ▶ The asset value is uS if the price increases and inserting this into the option value at maturity gives the *formula*.
- · The asset value is dS if the price decreases and inserting this into the option value at maturity gives the *formula*.
- ⇒ This now allows us to solve two equations in two unknowns, the Δ and the B. We can determine how many units of the underlying we need to choose
- · and how much to invest into the risk-free asset.
- ▶ We insert the valued for C_u and C_d and we then obtain the option value as in the *formula*.
- ▶ We can show that the amount of risk-free asset is always negative, hence we obtain a loan to finance the investment into the underlying asset.
- → It is straightforward to show that the value of the option is always positive. This must be the case as the option gives the buyer a right, but not an obligation, thus by holding an option the buyer can never make a loss, which gives it strictly positive value.

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Option value for a multiple time period

- Options mature generally only after multiple time periods
- Starting with the payments received at maturity, the option value in the previous time period can be determined
- Having established the option in the penultimate time period, these option values can be taken to determine the option value in the preceding time period
- The option price can be solved by backwards induction

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Option value for a multiple time period

→ We can now extend the single-period model to multiple time periods.
- Generally we have more than one time period, for example trading days, until an option matures.
- We know the value of the option at maturity and we can now work backwards and determine the value of the option in the previous time period.
- If we now have the value of the option one time period before maturity, we can take this option value and determine the option value in the time period before that.
- We thus solve for the option price recursively by backward induction.

→ We can illustrate this methodology for a simple case in which we have two time periods.

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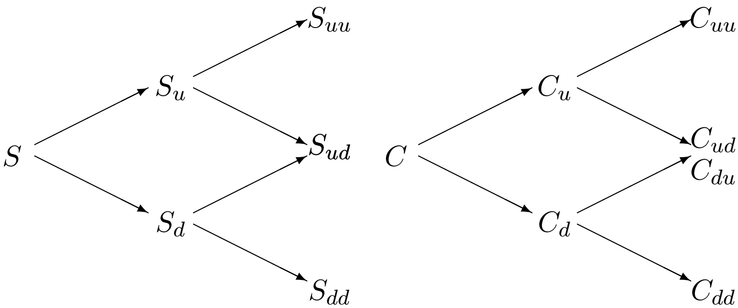
Solving the binomial tree through backward induction



$$\Delta_u = \frac{C_{uu}-C_{ud}}{S_u(u-d)},\ B_u = \frac{1}{1+r}\frac{uC_{ud}-dC_{uu}}{u-d} \Rightarrow C_u = \Delta_u S_u + B_u$$

$$\Delta_d = \frac{C_{du}-C_{dd}}{S_d(u-d)},\ B_d = \frac{1}{1+r}\frac{uC_{dd}-dC_{du}}{u-d} \Rightarrow C_d = \Delta_d S_d + B_d$$

$$\Delta = \frac{C_u-C_d}{S(u-d)},\ B = \frac{1}{1+r}\frac{uC_d-dC_u}{u-d} \Rightarrow C = \Delta S + B$$

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→ We will look at a case with two time periods as illustration of the way backwards induction can be applied.

- We can show the evolution of the price of the underlying asset and the corresponding option values. Here C_{uu}, C_{ud}, C_{du}, and C_{dd} represent the value of the option at maturity, C_u and C_d the value of the option one time period before that, and C the current value.
- We start by choosing two final payoff of the option and use this as the starting point for the option value. We can solve this for the option value, C_u, by making suitable replacements in the option pricing formula above, C_u becomes C_{uu}, C_d becomes C_{ud}, and C becomes C_u, similarly S_u becomes S_{uu}, S_d becomes S_{ud}, and S becomes S_u. This gives us a value for C_u.
- We repeat the same procedure for the lower outcomes.
- Once we have determined C_u and C_d, we can then repeat the procedure and obtain the initial option value

→ Thus we can solve for the current option value by working backwards step-by-step.

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Absence of expected return of the underlying asset

- The option price does not depend on the probability of the asset price increasing
- This implies the option price does not depend on the expected return of the underlying asset
- Arbitrage eliminates any risk, the value of the option is perfectly matched and no risk premium is payable
- The option value over time will be affected by the expected return as the underlying asset is included into its value

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Absence of expected return of the underlying asset

→ We have established the option value, but the formula is not including the expected return of the underlying asset, or the expected value of the underlying asset at maturity of the option.

- We see that the option price does not include an expression for the probability of the asset price increasing or decreasing.
- In the absence of the probability of these respective movements, the expected return of the underlying asset is not considered.
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    - The reason is that through arbitrage we have eliminated any risk,
    - the portfolio of the underlying asset and the risk-free asset match the option value at maturity perfectly.
    - For this reason there is no risk-premium required, given there is no risk; the risk premium would be included through the expected return, but this is therefore missing.
- Of course, indirectly the expected return of the underlying asset is included through the possible prices it can take, which are affected by the expected returns; a higher expected return will not only increase the probability of an upwards movement, but typically also the upwards movement itself.

→ We have now found a way to determine the value of an option. The binomial method can be applied to any type of option and any type of price movements of the underlying asset.

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