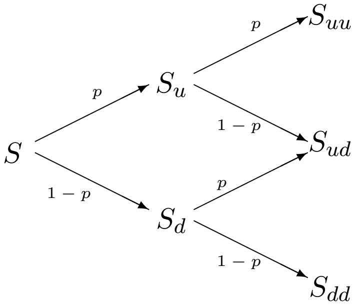
Andreas Krause

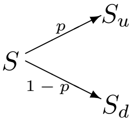
Binomial pricing of options

Price development of the underlying asset

- In each time period, the price of the asset will either increase to uS or decrease to dS
- The probability that the price increases is p



Arbitrage portfolio for a single time period



- The value of the option at maturity can be determined from the contract itself
- For a European call option this is the difference of the asset price and the strike price: $C_T = \max\left\{0; S_T - K\right\}$
- Assume the value of the option is given by a combination of the underlying asset and a risk-free asset
- $C = \Delta S + B$
- After one time step this portfolio is worth $C_u = \Delta S_u + (1+r)\,B$ if the asset value increases
- After one time step this portfolio is worth $C_d = \Delta S_d + (1+r)\,B$ if the asset value decreases

Option value for a single time period

- C_u and C_d are the possible payments of the option at maturity, which are known
- $C_u = \max\{0; uS - K\}$ for a call option
  $C_d = \max\{0; dS - K\}$ for a call option

$\Rightarrow$ $\Delta = \frac{C_u - C_d}{S(u-d)}$

$B = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d} < 0$

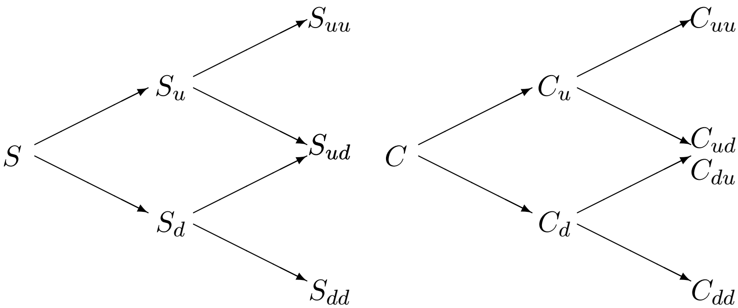
$\Rightarrow$ $C = \Delta S + B$

- The option value is given by holding the underlying asset, financed by a loan

Option value for a multiple time period

- Options mature generally only after multiple time periods
- Starting with the payments received at maturity, the option value in the previous time period can be determined
- Having established the option in the penultimate time period, these option values can be taken to determine the option value in the preceding time period
- The option price can be solved by backwards induction

Solving the binomial tree through backward induction



$$\Delta_u = \frac{C_{uu} - C_{ud}}{S_u(u-d)},\ B_u = \frac{1}{1+r}\frac{uC_{ud} - dC_{uu}}{u-d} \Rightarrow C_u = \Delta_u S_u + B_u$$

$$\Delta_d = \frac{C_{du} - C_{dd}}{S_d(u-d)},\ B_d = \frac{1}{1+r}\frac{uC_{dd} - dC_{du}}{u-d} \Rightarrow C_d = \Delta_d S_d + B_d$$

$$\Delta = \frac{C_u - C_d}{S(u-d)},\ B = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d} \Rightarrow C = \Delta S + B$$

Absence of expected return of the underlying asset

- The option price does not depend on the probability of the asset price increasing
- This implies the option price does not depend on the expected return of the underlying asset
- Arbitrage eliminates any risk, the value of the option is perfectly matched and no risk premium is payable
- The option value over time will be affected by the expected return as the underlying asset is included into its value

Andreas Krause
Department of Economics
University of Bath
Claverton Down
Bath BA2 7AY
United Kingdom

E-mail: `mnsak@bath.ac.uk`