Andreas Krause

Binomial pricing of options

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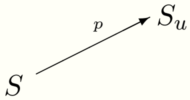

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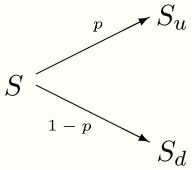

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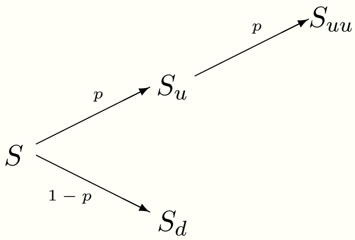

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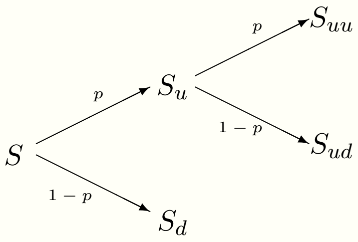

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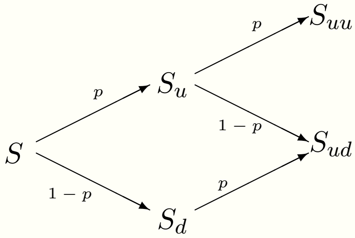

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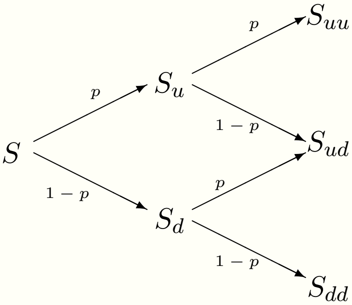

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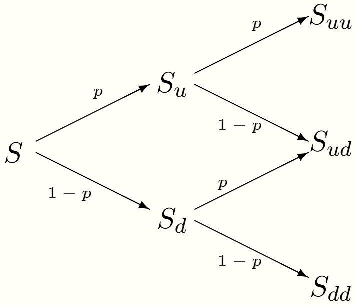

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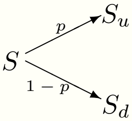

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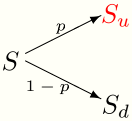

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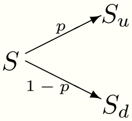

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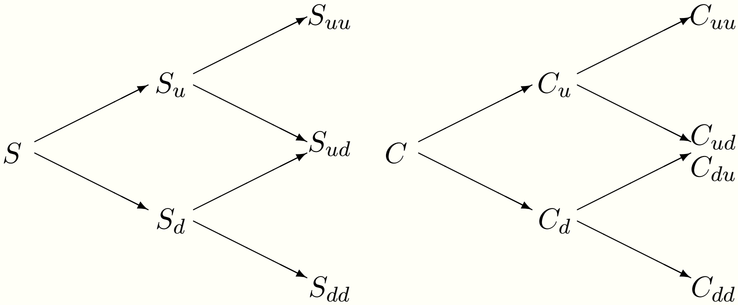

Solving the binomial tree through backward induction

$$
\begin{tikzcd}
 & & S_{uu} \\
 & S_u \arrow[ur] \arrow[dr] & \\
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 & & S_{dd}
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\qquad
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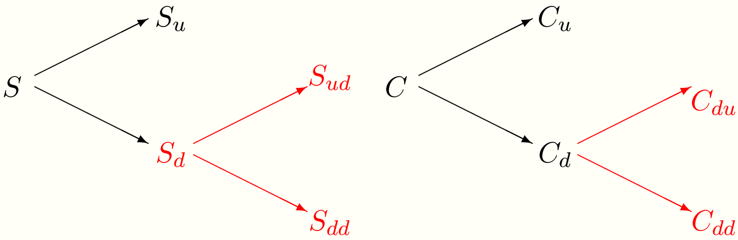

$$\Delta_d = \frac{C_{du}-C_{dd}}{S_d(u-d)},\ B_d = \frac{1}{1+r}\frac{uC_{dd}-dC_{du}}{u-d} \Rightarrow C_d = \Delta_d S_d + B_d$$

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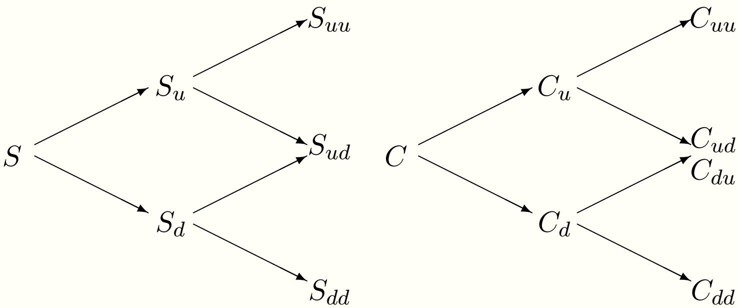

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