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Interest rate risk

Bond value

- The value of a bond is the present value of all future payments, consisting of the coupon and the repayment of the face value
- $B = \sum_{\tau=1}^{T} \frac{C_\tau}{(1+r)^\tau} + \frac{B_0}{(1+r)^T}$
- Usually we assume that all coupons are identical, the interest the bond pays is fixed
- If interest rates are variable, usually by using to a reference interest rate, these will vary over time
- As bonds are assumed to be risk-free, the discount rate is the risk-free rate

Duration of bonds

- Interest rates will vary over time, while the coupon payment for the bond remain fixed
- If the discount rate changes, the value of the bond will change
- $$\begin{aligned}\frac{\partial B}{\partial r} &= \sum_{\tau=1}^{T} -\tau \frac{C_\tau}{(1+r)^{\tau+1}} - T\frac{B_0}{(1+r)^{T+1}} \\ &= -\frac{1}{1+r}\left(\sum_{\tau=1}^{T} \tau \frac{C_\tau}{(1+r)^{\tau}} + T\frac{B_0}{(1+r)^{T}}\right) \\ &= -\frac{DB}{1+r}\end{aligned}$$
- $D = \frac{\sum_{\tau=1}^{T} \tau \frac{C_\tau}{(1+r)^{\tau}} + T\frac{B_0}{(1+r)^{T}}}{B}$ is the duration of the bond

Interpretation of duration

- The duration of a bond measures the time it takes until an investor recovers its initial investment B
- The duration is always less then the maturity of the bond
- The duration of a zero-bond is identical to its time to maturity
- If the bond has variable coupons that track market rates, the time to maturity that is considered is only until the coupon is adjusted to the market rate
- As the coupon is reset to the market rate, the value of the bond will revert to its face value, similar to being repaid at maturity

Bond risks

- We can now approximate the change of the bond value if the interest rate changes
- $\Delta B = \frac{\partial B}{\partial r}\Delta r = -\frac{DB}{1+r}\Delta r$

$\Rightarrow$ $Var\left[\frac{\Delta B}{B}\right] = D^2 Var\left[\frac{\Delta r}{1+r}\right]$

- The variance of the changes to the bond value is dependent on the duration of bonds, and the variance of the interest rate
- The larger the duration of a bond, the more volatile its value will be

Bond portfolios

- We consider a portfolio of two bonds: $B = B_1 + B_2$

$\Rightarrow$ $\frac{\partial B}{\partial r} = \frac{\partial B_1}{\partial r} + \frac{\partial B_2}{\partial r} = -\frac{1}{1+r}\left(D_1 B_1 + D_2 B_2\right)$

$\Rightarrow$ $\Delta B = -\left(D_1 B_1 + D_2 B_2\right)\frac{\Delta r}{1+r}$

- We define the weight of a bond in the portfolio as $\omega_i = \frac{B_i}{B}$

$\Rightarrow$ $\frac{\Delta_B}{B} = -\left(\omega_1 D_1 + \omega_2 D_2\right)\frac{\Delta r}{1+r}$

- The duration of the bond portfolio is the weighted average of the durations of the individual bonds

Eliminating interest rate risks from bonds

- If we hold a bond and want to eliminate interest rate risk, we can form a portfolio such that $D = \omega_1 D_1 + \omega_2 D_2 = 0$

$\Rightarrow$ $\omega_2 = \frac{D_1}{D_1 - D_2}$

- By adding another bond with a different duration to our portfolio. we can eliminate interest rate risk
- If the other bond used to hedge our interest rate exposure has a longer duration than the bond we seek to hedge, $D_2 > D_1$, we will have to hold a short position
- If $D_2 > D_1$, then we easily see that the new bond needs a value of $\omega_2 > 1$ and hence a short position of the original bond, $\omega_1 < 0$ which is not feasible
- This hedge is only effective against equal changes in the interest rates for all maturities

Duration gap in banks

- Banks use the differences in duration between their assets (loans) and their liabilities (deposits) as a measure for their interest rate risk
- This is referred to as the duration gap
- Banks then manage this duration gap by granting more short-term loans or attracting more long-term deposits

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