Andreas Krause

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Interest rates will vary over time

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#### Bond portfolios

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Interest rate risk

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# Duration gap in banks

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Interest rate risk

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