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Capital Asset Pricing Model

- We will now explore the implications of portfolio theory for the expected return of assets.
- The result will be one of the best-known and most-widely used models in finance, the Capital Asset Pricing Model (CAPM)

Portfolio theory

- Portfolio theory suggests that investors hold a portfolio of risky assets (optimal risky portfolio) and combine this with the risk-free asset
- Based on these investment decisions, we are able to derive an equilibrium in which all assets are held
- This equilibrium will restrict the returns of assets as a too high (low) return would result in a too high (low) weight for this asset

Portfolio theory

- → We start from the ideas of portfolio selection theory and develop these further to derive the expected returns of assets.
- ▶
  - From portfolio theory we know that all investors hold the same portfolio of risky assets, which is independent of their preferences.
  - This portfolio was called the optimal risky portfolio (ORP).
  - This ORP is then combined with the risk-free asset, and it is only this combination that depends on the preferences of investors.
- ▶ Using these investment decisions, we will derive how the expected returns must be such that all assets available to investors are actually held.
- ▶
  - If the returns were higher than this equilibrium, the assets would be attractive and the demand by investors would be higher than the supply.
  - If the returns were lower than this equilibrium, the assets would not be attractive and the demand by investors would be lower than the supply.
- → We can now derive this equilibrium in a very simple way.

Sharpe ratio

- The slope of the Capital Market Line in portfolio selection theory is given by $s = \frac{\mu_p - r}{\sigma_p}$
- This is known as the Sharpe ratio
- The optimal portfolio will consist of the optimal risky portfolio and the risk-free asset
- $\mu_P = \boldsymbol{\omega}^T \boldsymbol{\mu} + \left(1 - \boldsymbol{\omega}^T \boldsymbol{\iota}\right) r$
- $\sigma_p^2 = \boldsymbol{\omega}^T \boldsymbol{\Sigma} \boldsymbol{\omega}$

Sharpe ratio

- → We can now use the slope of the line connecting the risk-free rate and the optimal risky portfolio to derive the equilibrium returns.
- ▶ The capital market line has a slope which is given by the height difference the optimal risky portfolio and the risk-free asset given by the differences in returns, $\mu_P - r$, and the difference in risk of σ_p.
- ▶ This slope is also known as the Sharpe ratio and can be used as a performance measure when assessing investment strategies.
- ▶
  - We know that the optimal portfolio consist of the optimal risky portfolio, which has weight $\boldsymbol{\omega}$ of the risky assets in the total portfolio.
  - This is combined with the risk-free asset.
- ▶ The expected return will consist of the expected return from the risky asset comprising the optimal risky portfolio, and the remainder being invested into the risk-free asset. $\boldsymbol{\iota}$ denotes a vector of 1s and hence $\boldsymbol{\omega}^T\boldsymbol{\iota}$ would be the sum of all the weights of the risky assets, leaving $1 - \boldsymbol{\omega}^T\boldsymbol{\iota}$ as the weight of the risk-free asset.
- ▶ The variance of the whole portfolio will only be determined by the risky assets.
- → We know that the optimal portfolio will also lie on the Capital Market Line, thus it will have the same slope os the optimal risky portfolio.

Maximizing the Sharpe ratio

- The Capital Market Line is tangential to the efficient frontier, this is equivalent to the slope being maximal

$\Rightarrow$ $\frac{\partial s}{\partial \boldsymbol{\omega}} = 0$

$\Rightarrow$ $\boldsymbol{\mu} = r\boldsymbol{\iota} + \frac{\boldsymbol{\Sigma}\boldsymbol{\omega}}{\sigma_P^2}(\mu_P - r)$

- We define $\boldsymbol{\beta} = \frac{\boldsymbol{\Sigma}\boldsymbol{\omega}}{\sigma_P^2}$

$\Rightarrow$ $\boldsymbol{\mu} = r\boldsymbol{\iota} + \boldsymbol{\beta}(\mu_P - r)$

Maximizing the Sharpe ratio

→ We can now obtain the equilibrium returns of the risky assets.

- ▶
  - We know that the Capital Market Line is touching the efficient frontier.
  - This implies that the Capital Market Line has the highest possible slope that touches a portfolio, the optimal risky portfolio.
- [⇒] We will therefore seek a portfolio that maximizes this slope, which implies maximizing the Sharpe ratio over the optimal portfolio weights.
- [⇒] Inserting for the expected return and variance and solving the first order condition gives us this *formula*
- We can now define a vector β for convenience.
- [⇒] This allows us the require the expected return of the risky assets in the more common form.

→ We have thus derived how the expected returns of the risky assets are determined in equilibrium. We can now interpret this result.

The CAPM equation

- The term $\Sigma\omega$ represents the covariance of the assets with the optimal risky portfolio
- The optimal risky portfolio is identical for all investors, it must be the market portfolio

$\Rightarrow$ $\mu_i = r + \beta_i \left(\mu_M - r\right)$

$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$

The CAPM equation

- → The result we have obtained is the Capital Asset Pricing Model (CAPM), determining the expected returns of risky assets.
- ▶ The term in the denominator of the $\boldsymbol{\beta}$ is the covariance of the assets with the whole portfolio the investor holds.
- ▶
  - We know that the optimal portfolio consists of the risk-free asset and the optimal risky portfolio; the optimal risky portfolio was shown to be identical for all investors.
  - If all investors hold the same portfolio in the risky assets, then this must be the market portfolio as otherwise not all assets would be held and other assets would have demand exceeding their supply. Thus the optimal risky portfolio in equilibrium has to be the market portfolio.
- ▶ [⇒] We can now write the CAPM in the more common form for an individual asset, using β_i for that asset and the portfolio return having been replaced by the market return.
- ▶ [] The β_i are now the ratio of the covariance of the asset with the market and the market variance.
- → The CAPM states that the expected excess returns of an asset, $\mu_i - r$ are proportional to the excess return of the market $\mu_M - r$. β_i serves as a risk measure here where for $\beta_i = 0$ the expected return is the risk free rate and it increases the higher beta is. The market itself has $\beta_M = 1$ as $\sigma_{MM} = \sigma_M^2$.

Problems with the market portfolio

- The market portfolio should include all possible investments, stocks, bonds, real estate, private equity, hedge funds, commodities, foreign exchange, cryptoassets, human capital, ...
- Many investments are not available to all investors, for others no data are available
- For the optimal risky portfolio to be the market portfolio, all investors need to agree on the properties of all assets

Problems with the market portfolio

→ The derivation of the CAPM did depend on the optimal risky portfolio being the market portfolio; we than have to asses the properties of the market portfolio, the expected return and variance, as well as the covariance of each asset with the market portfolio.

- ▶
  - As investors have a wide range of investment possibilities, all these possible investments have to be included into the market portfolio.
  - Investments include all the stocks accessible to investors, so in principle as stocks globally.
  - It will also have to include all bonds, whether corporate bonds or non-risk-free government bonds.
  - Investors can also purchase real estate, commercial and residential property.
  - Investors will in general also have access to non-listed companies through private equity funds or investing directly.
  - Another investment possibility are hedge funds, who often have very different investment strategies.
  - Another investment opportunity is into commodities, be it oil, precious metals, rare earth metals, or agricultural products.
  - Investors might also find investment into different currencies attractive as an investment.
  - A more recent investment are cryptoassets.
  - But not all investments bare securities or traded, an investment could be into increasing the investors knowledge and skills, usually referred to as human capital.
  - There are many other investment possibilities at least some investors have access to.
- ▶
  - When having to determine the market portfolio, the fact that not all investors have access to all assets can be a problem, for example investment in human capital is realistically only possible for their own person.
  - Obtaining data on human capital returns and risks is nearly impossible, the same is true for other non-traded assets.
- ▶ The assumption for the optimal risky portfolio to be identical was that all investors agree on the properties of the assets, and have access to all assets, is not easily fulfilled. Therefore the optimal risky portfolio will often not be the market portfolio.

→ It is common to apply the CAPM only to stocks and use a broad market index as the market portfolio. While in many cases this can be a good approximation of the CAPM, it has to be noted that it is also a substantial deviation from the basic of the model.

Systematic risk

- The CAPM only considers the covariance of an asset with the market, not its variance as a risk measure
- The covariance is regarded as the systematic risk of an asset and measures how much it varies with the market as a whole
- Unsystematic risk, or idiosyncratic risk, is the risk unique to the asset
- Idiosyncratic risk can be eliminated through diversification

- → We can briefly characterise the risk that determines the expected returns of assets.
- ▶
  - The risk measure in β_i consists of the covariance of the asset with the market portfolio.
  - The risk considered for the expected return of the asset is therefore not the total risk of the asset, which would be its variance.
- ▶
  - The covariance of the asset with the market is called the systematic risk and measures the risk relevant for investors.
  - The covariance measures how much the asset moves with the market.
- ▶
  - The other risk, unsystematic risk, is risk that is unique to each asset. Unsystematic risk makes asset returns vary around the movement implied by the systematic risk, that is the movement induced by the market. How strong this movement is relative to the market, will depend on the β_i.
  - Unsystematic risk is also called idiosyncratic risk.
- ▶ Idiosyncratic risk is a random movement of an asset in addition to the movement implied by the market. These movement will be independent of each other, because otherwise they would be captured as a movement by the market; therefore we have random independent movements of assets. If we hold a large number of assets, these random movements will cancel each other out and due to the law of large numbers in statistics, resulting in idiosyncratic risk to vanish. Thus diversification, increasing the number of assets held will therefore reduce idiosyncratic risk until it vanishes completely. Hence idiosyncratic risk can be eliminated through diversification and is therefore not rewarded with higher returns. The same is not possible for systematic risk; this risk cannot be eliminated as all assets move in the same direction with the market.
- → We have thus seen that the CAPM rewards systematic risk only and the expected asset returns reflect this.

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