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Optimal portfolios

- We explore the properties of efficient portfolios using two assets and how efficient portfolios of more than two assets are determined.
- We will also introduce a risk-free asset to complement the risky assets in a portfolio.

Portfolio choice

- Assets can be characterised by their expected returns and risk
- A portfolio of assets will also require information on the correlation between assets
- Expected returns of a portfolio are given by the weights of each asset and their individual expected returns
- $\mu_P = \boldsymbol{\omega}^T \boldsymbol{\mu}$
- The risk of a portfolio is given by the weights of each asset and the variance-covariance matrix
- $\sigma_P^2 = \boldsymbol{\omega}^T \boldsymbol{\Sigma} \boldsymbol{\omega}$
- A portfolio of assets reduces risk

Portfolio choice

→ We will first look at how the properties of a portfolio emerge from that of the individual assets.

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  - Each asset can has its own outcome (expected return),
  - as well as its own variance (risk). We will require information on this for each asset we consider in our portfolio.
- Of importance will also be the correlation between all these assets, which will also have to be determined. It is common to use past returns to obtain such information for stocks, but individual assessments can also be used to obtain this information.
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  - The outcome of the portfolio (expected return) is given by the weight of each asset in the portfolio. As a normalisation we use the investment into the specific asset, divided by the total investment, which gives the weight of the asset in the portfolio.
  - We multiply this weight by the expected return of the asset. This gives us the weighted average of the expected returns, which is the expected return of the portfolio.
- *Formula*. The weight and expected returns are vectors, the superscript ^{T} denotes the transpose.
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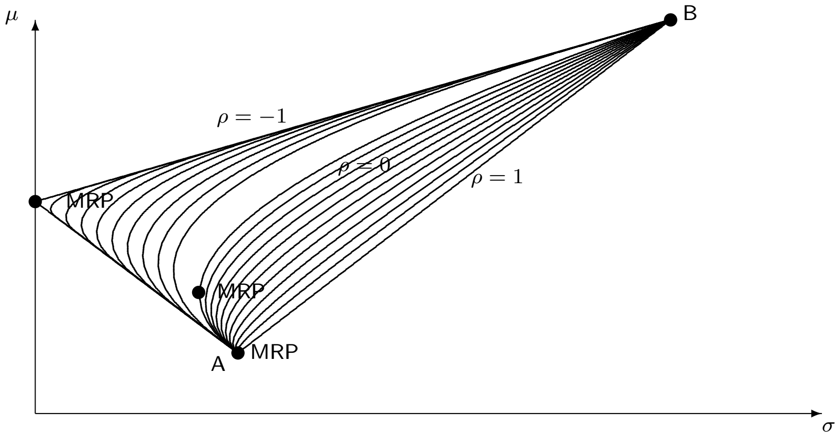
  - We do a similar approach to determine the risk (variance) of the portfolio. We use the weights of the asset in the portfolio.
  - In addition, we need the variance-covariance matrix. This matrix has as its main diagonal the variance of the returns and the off-diagonal elements are the covariances between the assets; the matrix is symmetric as the covariance between assets i and j is the same as the covariance between assets j and i.
- *Formula*. For two assets, this formula is equivalent to $\sigma_P^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_{12}$, with σ_{12} being the covariance of the two assets.
- What can be shown is that a portfolio reduces the compared to the risk of each asset. There will always exist a portfolio that has a lower risk than any of the assets included in it.

→ We will now look at the possible portfolios that can be formed from two assets.

Portfolios with two assets



Portfolios with two assets

- → Looking at the possible portfolios between two assets, we will focus on the impact that different correlations have on the risks of the portfolio.
- We look at the expected return and rather than the variance, the standard deviation, its square root, is used as risk measure; this use of the risk measure is done by convention.
- We consider the first asset to have a specific expected return and standard deviation.
- The second asset here has a higher return and a higher risk.
- We can show that is the two assets are perfectly correlated, all possible portfolios are located on a straight line between them.
- The minimum risk portfolio in this case will be the asset with the lower risk, here asset A
- We can now slowly decrease the correlation between the two assets. We see how the possible portfolios move towards the left; the minimum risk portfolio (MRP) moves to the left as well, implying a reduced risk, but it also moves up, providing a higher return.
- We can carry on reducing the correlation until we reach a correlation of zero.
- The minimum risk portfolio will have a significantly lower risk than either of the two assets.
- We can now reduce the correlation even further by looking at negative correlations. The portfolios move ever further to the left and the minimum risk portfolio has ever smaller risks.
- Once we reach a perfectly negative correlation, we can show that the portfolios are on a straight line to the vertical axis.
- The minimum risk portfolio has no risk in this case.
- → A lower correlation moves the efficient frontier, which would be the part above the minimum risk portfolio, to the upper left. This implies that a lower correlation gives a higher utility and individuals would prefer assets with lower correlations.

Correlation of assets and portfolio risk

- Lower correlations between assets reduces risks for a given return
- If asset returns are highly correlated, they mainly move in the same direction

⇒ Portfolio returns move widely

- With low correlations, the returns are often having different signs

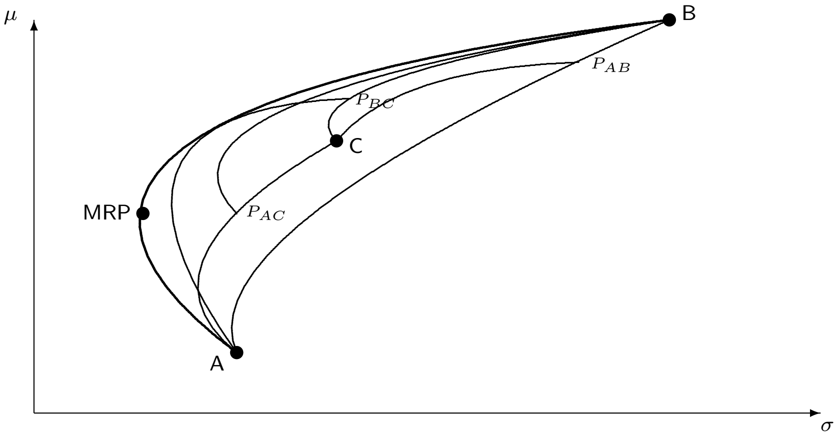
⇒ Portfolio returns are moving less

- If returns are perfectly negatively correlated, returns can be risk-free

Correlation of assets and portfolio risk

- → We can now explore the importance of correlations for the risk of a portfolio.
- ▶ We have sen that a lower correlation reduces the risks of portfolios be moving the opportunity set (the line as we only have two assets), to the left.
- ▶ To understand this relationship, assume that assets are highly correlated. In this case their movements will be mostly going in the same direction.
- ⇒ If both assets move in the same direction, then the portfolio will move a similar magnitude each time the assets themselves move. This means that the risk of the portfolio is similar to the risk of the assets.
- ▶ If correlations are low, then on many occasions the two assets will move in opposite directions. The consequence is that when adding the two movements in the portfolio, they partially will cancel each other out.
- ⇒ If returns are partially cancelling each other out, then the returns of the portfolio will move less, which implies that the risk is reduced.
- ▶ If returns are perfectly negatively correlated, then we can obtain a portfolio where these movements cancel each other out completely. With perfectly negatively correlated returns, the signs of these returns will be opposite on all occasions. It is then merely a case to obtain the magnitude of these movements from their respective variance to ensure the returns cancel each other out perfectly. If the portfolio has no risk, the return it generates should be the risk-free rate.
- → We can now expand our analysis from two assets to portfolios of more assets.

Portfolio with more assets



μ
B
P_{AB}
P_{BC}
C
MRP
P_{AC}
A
σ

Portfolio with more assets

→ To understand the principle of how the properties of larger portfolios can be obtained from individual assets, we consider the case of three assets. Extending this to more assets is straightforward.

- We again consider the expected return and risk, in the form of the standard deviation, of assets and portfolios.
- We consider the same asset A as before.
- Similarly, the same asset B is used.
- We now introduce a third asset, C
- We can now determine all the portfolios consisting of assets A and B, which would be on this line.
- We can now determine all the portfolios consisting of assets A and C, which would be on this line.
- We can now determine all the portfolios consisting of assets B and C, which would be on this line.
- We can now also combine asset A with any portfolio consisting of assets B and C. The line shows one of the many examples. We can treat the portfolio consisting of assets B and C as a single asset; as we are only concerned about the properties consisting of return, variance, and correlation, it is immaterial whether this is the result of a single asset or a combination of assets.
- We can now also combine asset C with any portfolio consisting of assets A and B. The line shows one of the many examples.
- We can now also combine asset B with any portfolio consisting of assets A and C. The line shows one of the many examples.
- This can be repeated for all possible combinations of assets and as we are only interested in the left-hand side of the possible portfolios, the shown red line is the furthest the possible portfolios can go.
- We can determine the minimum risk portfolio (MRP) and the efficient frontier would be located above this point on the red line.

→ We see that with more assets, the efficient frontier moves further to the upper left. Thus having more assets in a portfolio, will increase the utility level.

The effect of diversification

- More assets can reduce the risks of the efficient portfolios for a given return
- Risks cannot increase as additional assets could be given a weight of zero

⇒ Diversification reduces portfolio risk

The effect of diversification

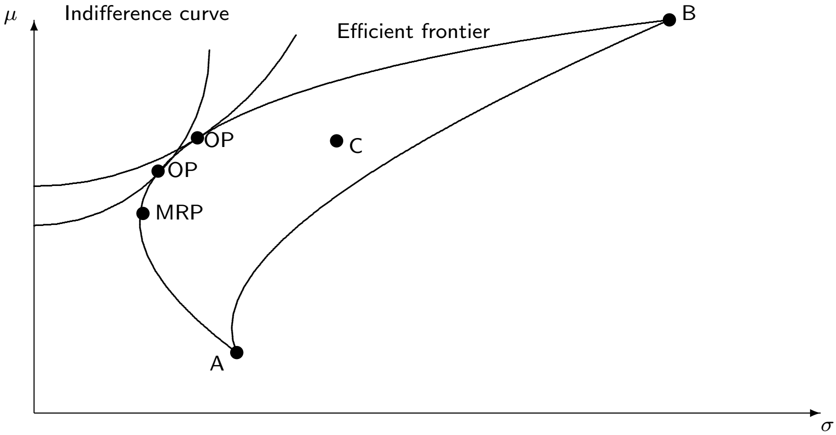
→ We can use these results to assess the impact diversification has on individuals.

- We have seen that increasing the number of assets reduces the risk of the efficient frontier.
- Risks cannot increase when having more assets as one option is to assign an additional asset a weight of zero, implying that it is not included in the portfolio and hence it will not affect the outcome. If increasing the weight of this additional asset does reduce risks, then the resulting portfolio is not efficient and the efficient frontier remains unchanged. If it reduces the risks, it will move the efficient frontier to the upper left and improve it the utility.

⇒ Diversification is what it is called when holding several assets in a portfolio and more diversification is when holding more assets. We see that diversification will reduce the risk of the portfolio and increase the utility of investors.

→ Having seen that diversification is beneficial, we can now determine the optimal portfolio an investor should choose.

The optimal portfolio


μ
Indifference curve
Efficient frontier
B
OP
C
OP
MRP
A
σ

The optimal portfolio

→ We can now determine graphically the optimal portfolio in our case of having three assets.

- We again consider the expected return and risk, in the form of the standard deviation, of assets and portfolios.
- We consider the same three assets as previously.
- The assets included form the opportunity set that consists of the risk-return combinations of all possible portfolios included in the bordered area. The minimum risk portfolio (MRP) is also indicated.
- The part of the border like above the MRP until asset B is the efficient frontier.
- To determine the optimal portfolio on the efficient frontier, we need to use the utility function. We know that the indifference curves have a positive slope and that the utility increases the higher the indifference curve is located. The highest possible point for the indifference point is when it is tangential to the efficient frontier.
- The optimal portfolio (OP) is then located at this tangential point.
- If we consider a different individual who is less risk averse, then its indifference curve will be less steep. The optimal portfolio will still be located where the indifference curve is tangential to the efficient frontier.
- The resulting optimal portfolio is further away from the minimum risk portfolio.

→ We see that the optimal portfolio is the point on the efficient frontier that is tangential to the indifference curve; a higher risk aversion will cause the optimal portfolio to be closer to the minimum risk portfolio.

Optimal portfolios with risky assets

- The optimal portfolio is where the indifference curve is tangential to the efficient frontier
- Higher risk aversion implies that the optimal portfolio is closer to the minimum risk portfolio
- Any optimal portfolio will be efficient

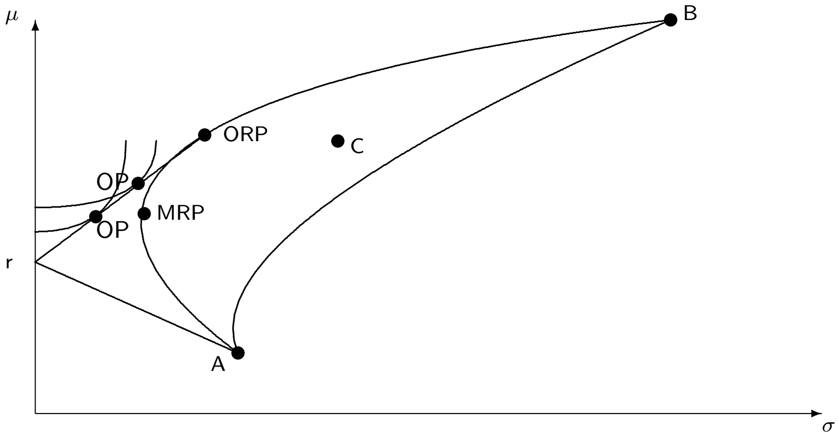
Optimal portfolios with risky assets

→ We can now summarize the key result on how to find the optimal portfolio.

- We determine the point where the slope of the efficient frontier is the same as the slope of the indifference curve. This is another way of saying they are tangential.
- The higher the risk aversion, the lower the risk of the optimal portfolio, and hence the optimal portfolio will move towards the minimum risk portfolio.
- As we have determined the optimal portfolio as one of the efficient portfolios, we can reverse this and state that any optimal portfolio will be efficient.

→ Thus far we have assumed that all assets in our portfolio are risky. Of course, we could also choose an asset that is risk-free.

Portfolios with a risk-free asset


μ
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Portfolios with a risk-free asset

→ A risk-free asset is one that has a certain return, thus its variance or standard deviation is zero. We will not introduce one such asset.

- We again consider the expected return and risk, in the form of the standard deviation, of assets and portfolios.
- We consider the same three risky assets and the opportunity set of these three assets remains unchanged.
- We now introduce a fourth asset, the risk-free asset. This asset will have a return equal to the risk-free return and will be located on the vertical axis as its standard deviation is zero. As it is risk-free, its covariance with all other assets is also zero.
- We can show that combining the risk-free asset with any other risky asset, will lead to the portfolios being located on a straight line between the risky asset and the risk-free asset. This is similar to the case where correlations were perfectly positive or perfectly negative, but has a different reason; here the correlation between the risk-free asset and the risky asset is zero. This straight line emerges due to the risk of the risk-free asset being zero. We have added the opportunity set with this risk-free asset. The lowest portfolios are on the line from the risk-free asset to asset A and no lower point can be reached. The highest point that can be reached is the point where the line from the risk-free asset touches the opportunity set of the risky assets as indicated here. No higher point can be reached. If course, any point between these two straight line can be achieved as well.
- The point at which the upper line touches the opportunity set of risky assets is called the optimal risky portfolio (ORP).
- The efficient frontier of this portfolio consisting of risky assets and the risk-free asset will not be the line connecting the risk-free asset and the ORP as these are the highest upper left points that can be reached. Beyond that the efficient frontier will follow the efficient frontier of the risky assets only. The optimal portfolio will now be again determined with help of the indifference curve being the one that is highest.
- The resulting tangential point is the optimal portfolio (OP). We now can determine the components of this optimal portfolio. As it is on the line between the risk-free asset and the ORP, it will be a combination of these two components; the constituents will be the risk-free asset and the ORP. In this context we can treat a portfolio as if it was a single asset; this is due to us only being concerned about the properties of the assets, regardless of how they have been achieved. Therefore, the ORP is the optimal portfolio, but only the part which is comprised of risky assets, hence its name. The optimal portfolio is a combination of the ORP and the risk-free asset.
- Similar to before, a higher risk aversion result in steeper indifference curves that again for the optimal portfolio have to be tangential to the efficient frontier.
- The optimal portfolio of a more risk-averse investor is closer to the overall minimum risk portfolio, which is the risk-free asset (the term 'minimum risk portfolio' is restricted to portfolios consisting only of risky assets). Thus a more risk averse investor would put more weight on the risk-free asset and less weight on the risky assets on for of the ORP.

→ We note that the ORP is independent of the risk aversion, it is the same for all investors.

Optimal risky portfolio

- A risk-free asset has no risk and no covariance with any asset

⇒ The efficient frontier with a risk-free assets is a straight line

- The efficient frontier is the line that is tangential to the efficient frontier of the risky assets
- This tangential point gives us the optimal risky portfolio
- The optimal risky portfolio does not depend on the preferences of individuals, but only on the characteristics of the assets

Optimal risky portfolio

→ We can now summarise the implications of introducing a risk-free asset into our portfolio.

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  - By definition, a risk-free asset has no risk, hence a variance or standard deviation of zero.
  - As it is risk-free it also does not move with any risky asset, giving it a correlation of zero with all assets.

⇒ We can show mathematically that the efficient frontier between a risk-free asset and a risky asset is a straight line.

- The efficient frontier in the case of having a risk-free and risky assets, would be the straight line that is tangential to the efficient frontier of the risky assets.
- This tangential point is called the optimal risky portfolio.
-
  - The optimal risky portfolio is the same for all investors, regardless of their degree of risk aversion.
  - The ORP only depends on the characteristics of the risky assets, their expected returns, standard deviation, and correlations, as well as the risk-free rate (the return of the risk-free asset).

→ We can use these points to obtain one of the most fundamental results in portfolio theory.

Separation theorem

- The optimal portfolio is a combination of optimal risky portfolio and the risk-free asset
- The more risk averse an individual is, the higher the weight of the risk-free asset

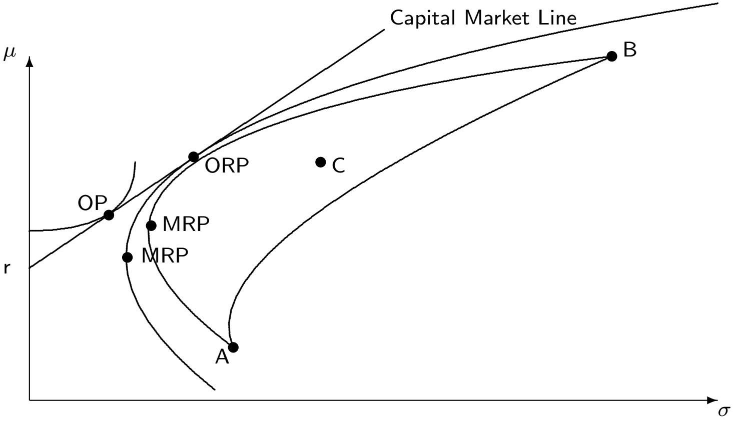
⇒ All individuals combine the optimal risky portfolio and the risk-free assets

- Only the weights of the risk-free asset and the optimal risky portfolio depends on individual preferences

Separation theorem

- → We have now established one of the key results in portfolio theory, the fact we can select portfolios in two steps.
- ▶ We have see that the optimal portfolio is a combination of the risk-free asset with a specific portfolio of risky asset, the optimal risky portfolio.
- ▶ We have seen that all investors use this combination of the risk-free asset with the same ORP, what varies with the degree of risk version are the relative weights of these two components. A higher risk aversion results in a larger weight of the risk-free asset.
- ⇒ Hence all investors combine the same two components and thus the portfolio selection process can be separated into two steps. Firstly, determine the optimal risky portfolio; secondly, combine this optimal risky portfolio with the risk-free asset. That portfolio selection can be decomposed into these two steps is known as the separation theorem.
- ▶ It is only the weight of the risk-free asset and the ORP that depend on the risk aversion of investors, not the composition of the ORP itself, which is identical for all investors, as long as they agree on the properties of assets.
- → Thus far we made the implicit assumption that the weights of all assets in a portfolio have to be positive, or zero. We will now introduce the possibility of having negative weights.

Optimal portfolio with short sales


Capital Market Line
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Optimal portfolio with short sales

→ We now allow for negative weights of assets in the portfolio; such negative weights are referred to as short sales. We can interpret short sales as selling assets we do not own. Consider the following procedure: You obtain a loan and have to repay this with the same way it has been granted. Usually a loan is given in the form of currency, cash or deposits; now consider that the loan is not given in this form but instead the loan is some security. You obtain such a loan and sell the security at its current price to make other investments. Now you have an obligation to repay the loan in this security (negative weight). When repaying the loan, the investor has to purchase the security again and hand it over to the lender. If the asset has increased in value, the investor would have made a loss as he has to repurchase the asset at a higher price than he has received; if the asset has decreased in value, the investor would have made a profit as he has to repurchase the asset at a lower price than he has received. It has to be noted that the total weight of the assets in a portfolio need to sum up to 1, thus a short sale in some asset(s) will necessitate a larger long-position (the normal positive weights) in other asset(s).

- We again consider the expected return and risk, in the form of the standard deviation, of assets and portfolios.
- We consider the opportunity set of the same three risky assets as previously.
- If we now allow for short sales, we increase the possibilities of forming portfolios; this increases the opportunity set. It will most likely also improve the efficient frontier; it cannot reduce the efficient frontier as portfolios with all weights being non-negative are still available. The opportunity set will also not be bounded by the individual assets anymore as short sales can be unlimited.
- Hence with short sales with are left with en efficient frontier that is enhanced compared to our previous portfolios not allowing short sales.
- As before we can now add the risk-free asset and as previously it will form a straight line with any risky asset. The best possible line is the one tangential to the new efficient frontier. The line does now not stop at the tangential point as short sales are allowed, but continues beyond this point. A short sale in the risk-free asset is what would be required on any point beyond the tangential point; a short sale in the risk-free asset would be an ordinary loan as the amount to be repaid is fixed.
- As before, the tangential point is again the optimal risky portfolio with the same rationale as above.
- The optimal portfolio is again determined such that the indifference curve is as high as possible.
- The tangential point of the indifference curve to the line connecting the risk-free asset and the ORP determines the optimal portfolio. For investors with low risk aversion, the optimal portfolio might well be beyond the ORP, implying they take out a loan (short sale of the risk-free asset) and invest more into the ORP.
- The straight line from the risk-free asset to the ORP and beyond is also known as the Capital Market Line.

→ Thus overall, the inclusion of short sales does not alter the ideas of portfolio selection. The increase in the opportunity set and the move of the efficient frontier towards the upper left increases the slope of the Capital Market Line; this in turn increases the utility level that can be achieved by investors. Hence investors would like to have short sales enabled.

The effect of short sales

- Short-sales are occurring if an asset is sold without holding it, giving a negative weight
- You obtain a loan in form of the asset, sell it for cash, then have to repay the asset in the future, buying it back for cash
- If the asset has increased in value, a loss accrues as it has to be bought back at a higher price
- If the asset has decreased in value, a profit accrues as it has to be bought back at a lower price
- Weights in the portfolio can now be negative, which increases the opportunity set and improves the efficient frontier
- The portfolio gives a higher utility than without short sales

The effect of short sales

→ We can now Summarize the key results of portfolio selection in the presence of short sales.

- ▶
  - Short sales occur is an investor sells an asset it does not hold.
  - This is equivalent to having negative weights on these assets.
- ▶
  - Short sales can be interpreted as a loan in the form of a (risky) asset.
  - After having obtained the loan, the asset is sold for cash and the proceeds invested into other assets.
  - When the loan is due to be repaid, it is in the form of the asset.
  - To make the repayment, the asset needs to be repurchased.
- An increase in the asset price, implies a loss to the investor as he has to repurchase the asset at a higher price.
- A decrease in the asset price, implies a profit to the investor as he has to repurchase the asset at a lower price. Thus the profu=its and losses are reversed to the normal situation. Thus a short position will have a negative correlation with other assets, assuming all assets have positive correlations. This negative correlation reduces the risks of the portfolio, providing the benefits from short sales.
- ▶
  - With such short sales, weights of assets can be negative.
  - This in itself, even neglecting the negative correlation of such positions, increases the opportunity set.
  - An increased opportunity set also moves the efficient frontier further to the upper left.
- The result is that using short sales increases the utility of investors.

→ While short sales are increasing the utility of investors, in reality the ability to make short sales will be limited. As they are loans to the investor, the availability of such loans will not be universal and the amount that can be borrowed will also be strictly controlled. Thus realistically, for most investors short sales are either not available at all or form a small part of the positions in a portfolio.

Summary

- The optimal portfolio consists of a risky portfolio, identical for all investors, and the risk-free asset
- Only the combinations between these two components vary across individuals
- Using expected returns, variances and covariances, the optimal risky portfolio can be determined accurately

Summary

→ We can now summarize the key results of portfolio selection theory.

- The have found that the optimal portfolio will consist of the risk-free asset and a risky port portfolio that is identical for all investors, the optimal risky portfolio. The optimal risky portfolio is identical for all investors as long as they agree on the characteristics of all assets.
- The combination of risk-free and optimal risky portfolio will differ between investors and will depend on their risk aversion.
- The optimal risky portfolio can be determined objectively, without reference to any utility function, by estimating expected returns, variances and correlations.

→ Portfolio selections seems a straight forward process. It is however difficult to determine the risk aversion of investors and the variances, returns and correlations of assets are also not stable over time, making their estimation difficult.

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