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Mean-Variance criterion

- We look at the investment into a number of assets and how much money to invest into each asset, called a portfolio.
- We will use our knowledge on the preferences of individuals to reduce the possible portfolios that need to be considered further without requiring knowledge of the utility function.
- The final selection will then have to be made using the preferences of individuals.

Preferences for outcomes

- In finance it is common to express outcomes in terms of returns on the initial investment and associated risks as risks in the return
- Individuals will prefer higher expected returns to lower expected returns, *ceteris paribus*
- Individuals will prefer lower risks to higher risks, *ceteris paribus*
- Based on these two criteria we can compare choices and exclude many cases

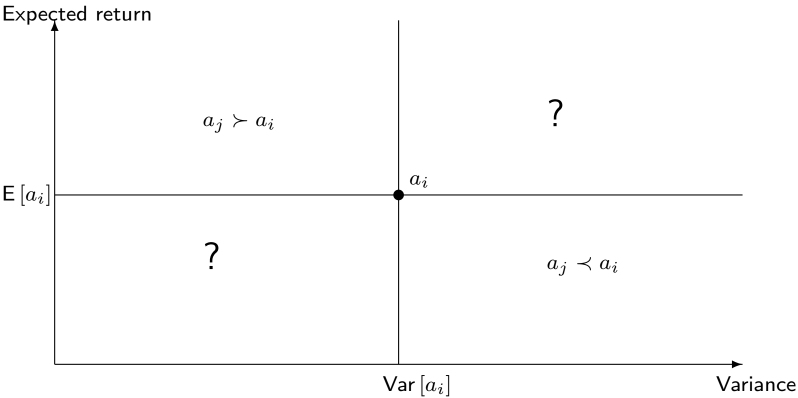
Preferences for outcomes

→ We will restate the risk preferences in terms of the investment into assets and what the consequences for choices are.

- - The outcome of a decision when investing into assets would be the final wealth of an individual after this investment is concluded. This can be more easily expressed with the return the investment generates. Knowing the initial investment, which is given, this then leads to the final wealth; thus looking at returns is equivalent to looking at final wealth.
  - The risks to the outcome are then analogously given by the risk of the returns, thus the variance of returns.
- With individuals preferring a higher outcome to a lower outcome, for a given risk, this translates to a higher return being preferred.
- For a given return, individuals will prefer lower risks.
- Knowing the returns and risks of portfolio choices, we can now exclude a large number of possible portfolios.

→ We will analyse the choices graphically.

Choice between two alternatives



Choice between two alternatives

→ Let us assume we have to make a choice between two portfolios; we will introduce more portfolios later once we have established how to make a choice between any two.

- We will express a portfolio in its characteristics, the expected return (outcome) and the variance (risk).
- Let us consider a portfolio i that has some known characteristics.
- We can now consider four cases that are determined relative to the existing portfolio.
- In the upper left area, the expected return is higher, preferred by individuals, and the risks are lower, also preferred by individuals. Therefore, any portfolio in the upper left is preferred.
- In the lower right area, the expected return is lower, not preferred by individuals, and the risks are higher, also not preferred by individuals. Therefore, any portfolio in the lower right is not preferred.
- In the upper right area, the expected return is higher, preferred by individuals, and the risks are higher, not preferred by individuals. Therefore, any portfolio in the upper right cannot directly be compared; for a full assessment the detailed preferences are required.
- In the lower left area, the expected return is lower, not preferred by individuals, and the risks are lower, preferred by individuals. Therefore, any portfolio in the lower left cannot directly be compared; for a full assessment the detailed preferences are required.

→ We thus see that we can exclude from our choices any portfolios that are to the lower right of a given portfolio. We cannot make a choice without detailed knowledge of the preferences of individuals. We see that in this area there is a positive trade-off between risk and return, and it depends on how steep this relationship is, whether a portfolio in this atea is preferred or not.

Determining possible choices

- By comparing any two alternatives, we can eliminate alternatives that have higher risk and lower return (lower right)
- We cannot make a choice if the alternative has higher risks and higher returns (upper right)
- We cannot make a choice if the alternative has lower risks and lower returns (lower left)
- The best choices have no alternatives with lower risks and higher returns (upper right)

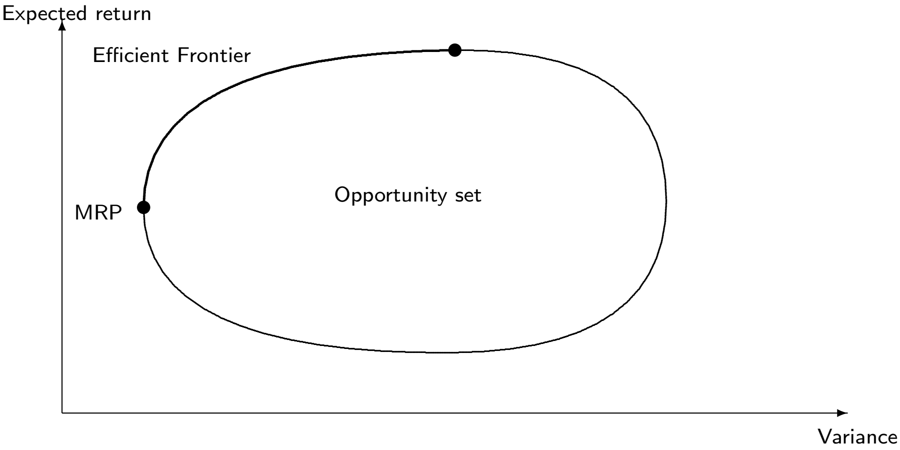
Determining possible choices

→

- ▶
  - Individuals will not choose a portfolio that has a lower return and higher risk as both are not preferred.
  - This eliminates all choices in the lower right area,
- ▶
  - If a portfolio has a higher risk, but also a higher return, then we cannot make a general assessment of the preferences. The reason is that a higher return is wanted, but a higher risk is not wanted. The details of the preferences are needed to determine the better choice.
  - This makes choices in the upper right area uncertain.
- ▶
  - If a portfolio has a lower risk, but also a lower return, then we cannot make a general assessment of the preferences. The reason is that a lower return is not wanted, but a lower risk is wanted. The details of the preferences are needed to determine the better choice.
  - This makes choices in the lower left area uncertain.
- ▶
  - If a portfolio has a lower risk, but also a higher return, then such a portfolio is preferred.
  - Thus portfolios is in the upper left area are preferred.

→ We can conclude that all portfolios to the upper left are preferred and hence the best portfolios will have no alternative choices in this area. Using this result we can now determine those portfolios that can be eliminated from the choice.

The efficient frontier



- → We will now look at determining all portfolios that can be eliminated, hence leaving us with those portfolios that an individual has to choose between using their preferences.
- We again look at portfolios just from their properties of expected return (outcome) and variance (risk)
- Let us now assume that all possible portfolios are located in an area as shown here. All possible portfolios are also called the opportunity set.
- We can now eliminate from our opportunity set most portfolios. We will be left only with those portfolios that do not have any other portfolio to the upper left. This means we have left only portfolios on the upper left boundary of the opportunity set.
- The portfolio furthest to the left has the lowest risk and in portfolio theory is therefore called the Minimum Risk Portfolio (MRP). This portfolio is the boundary of the potentially chosen portfolios.
- There exists another boundary portfolio, which has the highest possible return. This portfolio has not specific name.
- Those portfolios that may be chosen are called the efficient frontier.
- → From now on we will only consider portfolios on the efficient frontier as these are the only portfolios that need to be considered in the final choice.

Selecting the best choice

- The efficient frontier resembles all possible choices that do not have an alternative with lower risks and higher returns (upper right)
- So far we only assumed that individuals are risk averse
- The specific utility function or the level of risk aversion was not required
- The optimal choice will be on the efficient frontier, but the utility function is needed to select it

Selecting the best choice

→ We can summarize the finding of our graphical analysis.

- We have established that those portfolios on the efficient frontier are those that any individual will choose between. It represents those portfolios to which no alternative exists that has both a higher expected return and a lower risk.
- In deriving the efficient frontier we did not need to use the utility function of the individual, but were only assuming that he was risk-averse.
- The degree of risk aversion was irrelevant for this result, as long as it was positive.
-
    - We now know that the best portfolio for an individual will be on the efficient frontier.
    - While so far we did not need specific information about the individual's utility function, we will now need this information to make the final choice.

→ In order to make this choice, we will now look at the utility function of risk-averse individuals.

Indifference curves

- To compensate for higher risks, individuals require a higher return

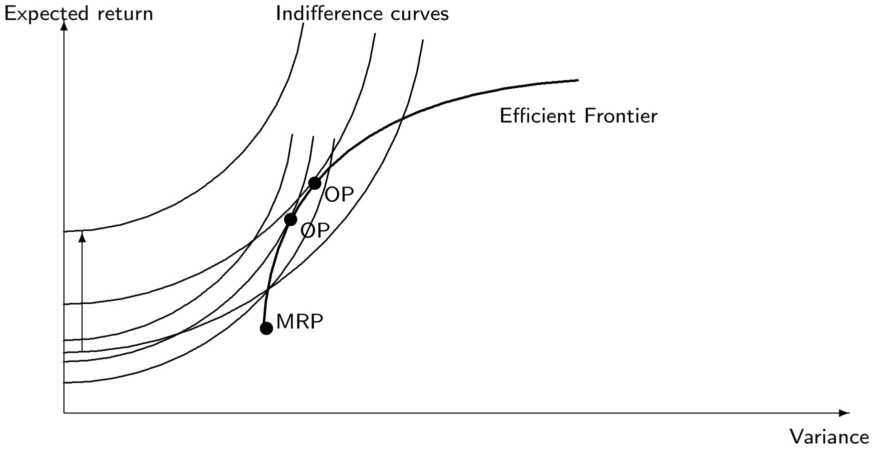
$\Rightarrow$ Indifference curves have a positive slope as risk with risk aversion

- A higher risk aversion implies that individuals require more compensation when taking on additional risk

$\Rightarrow$ Indifference curves have a higher slope the more risk averse an individual is

- → We will look at the implications of risk aversion for the shape of the indifference curves of individuals for risk and return.
- ▶ We know that higher risk reduce the utility of risk-averse individuals; this reduced risk can be compensated for by a higher outcome.
- ▶ [⇒] To be indifferent between two portfolios, a higher risk must be accompanied by a higher return. This implies a positive relationship between risk and return, thus indifference curves have a positive slope.
- ▶ If an individual is more risk averse, he needs a higher compensation for the same increase in risk
- ▶ [⇒] This stronger relationship between risk and return implies that the indifference curve is steeper for more risk-averse individuals.
- → We can now use these result to determine the optimal portfolio.

Determination of the optimal choice



- → We can now determine the optimal portfolio graphically.
- ▶ We again look at the expected return and risk of portfolios.
- ▶ Our focus will be only on the efficient frontier as depicted here as all portfolios chosen will have to be there.
- ▶ Indifference curves have a positive slope as shown here. The details of the slope will depend on the specific utility function.
- ▶ As individuals prefer higher outcomes (expected returns) to lower outcomes, a higher utility level can be found on indifference curves that are located higher. The highest possible indifference curve will be chosen.
- ▶ The highest possible indifference curve will be the one which just touches the efficient frontier. The efficient frontier is the equivalent of a budget constraint. The highest utility level is obtained where the indifference curve is tangential to the efficient frontier.
- ▶ At this point where the indifference curve and efficient frontier touch, the optimal portfolio (OP) can be found.
- ▶ We can now look at the optimal portfolio for an individual with a higher risk aversion. In this case, the indifference curve has a higher slope as indicated here.
- ▶ Once again a higher utility level is obtained the higher the indifference curve is.
- ▶ The highest possible indifference curve will be the one which just touches the efficient frontier.
- ▶ The optimal portfolio is again where the indifference curve is tangential to the efficient frontier.
- → We clearly see that the optimal portfolio moves closer the minimum risk portfolio (MRP), reducing the risk of the optimal portfolio.

Properties of the optimal portfolio

- The optimal choice (portfolio) is located where the indifference curve is tangential to the efficient frontier
- A higher risk aversion reduces the risk of the optimal portfolio
- The more risk averse an individual is, the closer the optimal portfolio moves to the minimum risk portfolio

Properties of the optimal portfolio

→ Having derived the optimal portfolio, we can now look at some properties of the optimal portfolio.

- We have seen that the optimal portfolio is located where the indifference curve and the efficiency frontier are tangential as this maximises the utility of the decision-maker.
- We have also seen that a higher risk aversion reduced the risk of the optimal portfolio. The stronger dislike of risk, would require a higher expected return, but as this is not possible on the efficient frontier, the only way to achieve a higher utility is by reducing the risk of the optimal portfolio.
- With increasing risk aversion the risk reduces and the optimal portfolio moves closer to the minimum risk portfolio.

→ Of course, the solution here only determines the risk-return combination of the optimal portfolio. We then need to 'reverse engineer' the portfolio to obtain the amounts that are to be invested into each asset. As the portfolio properties will have been determined in the first place by choosing an arbitrary combination of investments in the different assets, the optimal investments can be obtained by looking up which portfolio will match those properties.

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