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Risk aversion

- A central assumption in much of finance is that individuals dislike risk. From this we obtain that when taking higher risks, this is to be compensated by higher returns.
- We will see how this dislike of risk, or risk aversion, can be measured and what implications this has for the specification of utility functions.

Dislike of risk

- Individuals prefer, *ceteris paribus*, higher outcomes to lower outcomes
- Individuals prefer, *ceteris paribus*, lower risks to higher risks
- The attitude towards risk is commonly referred to as risk aversion
- More formally, we define
  Individuals are risk averse if they always prefer to receive a fixed payment to a random payment of equal expected value.

  (B. Dumas and B. Allaz, Financial Securities: Market Equilibrium and Pricing Methods, Cengage Learning, London 1996)

Dislike of risk

- → Risk can be interpreted a 'bad' as it has negative value to individuals, hence any goods have two properties, one that is positive, in finance usually the return and one that is negative, the risk.
- In terms of the positive property, the outcome (return) of a good (security), more is preferred to less.
- In terms of the negative property, the risk of a good (security), less is preferred to more.
- We will seek to measure the degree to which individuals dislike risk; this is often referred to as risk aversion. Risk aversion seeks to collate the attitude towards risk into a single measure.
- An implication of the above properties is that if an individual has a choice between a fixed (risk-free) payment and a risky payment that on average yields the same outcome, the risk-free choice will be preferred.
- → We will use this property to derive a measure of risk aversion by compensating the risk with a higher outcome and how much compensation is required for taking on additional risk will determined the risk aversion.

Measuring risk and risk aversion

- The most common risk measure in finance is the variance of outcomes
- Variance is a measure of how much outcomes deviate from the expected value
- $\mathsf{Var}[V] = \mathsf{E}\left[(V - \mathsf{E}[V])^2\right]$
- Risk aversion is more difficult to measure as it will depend on the utility function of individuals

Measuring risk and risk aversion

- → We will first provide a measure for the risk of outcomes.
- The most common way to measure risks is through the variation of these outcomes; the more outcomes vary, the more risky they are. A measure that captures this property is the variance.
- The variance measures the deviation from the average (expected) outcome. As we are only interested in these deviations, we are not distinguishing between outcomes exceeding the average or falling short of it. This is in principle achieved by taking the absolute value. In the variance these deviations are squared, this has the effect that larger deviations become more prominent.
- *Formula*
- While a measure of risk can easily be obtained, risk aversion of individuals is more difficult ro measure; it will depend on the preferences of the individual. Thus the utility function will need to be used to obtain any such measure of risk aversion.
- → We will now see how such a measure can be derived.

Risk premium

- Using the definition of risk aversion, we see can compare the expected utility of risky outcomes and the utility non-risky outcomes
- The utility of the risky outcomes, should be less than that of a non-risky-outcome with the same expected value
- The amount to be deducted from the safe outcome to obtain the same utility as the risky outcome is the risk premium
- $E\left[U\left(V\right)\right] \quad \leq = U\left(E\left[V\right] - \Pi\right)$

Risk premium

- → We will now look at how much utility an individual loses if the outcome is risky, compared to an outcome that is safe.
- ▶ We will compare the utility of two choices with the same expected outcomes, one which is safe and the other which is risky.
- ▶ With risk aversion, the utility of the choice with the risky outcome should be lower than that of the risk-free outcome. As the outcome determines the utility an individual obtains, the utility cannot be known when making the choice; therefore the individual can only determine the expected utility and we use this expected utility as the decision criterion. The safe outcome is the expected value of the risky outcome.
- ▶ We can now determine how much of the safe outcome we need to deduct such that the expected utility of the risky outcome is identical to the utility of the safe outcome. This deduction we all the risk aversion.
- ▶ *Formula*
- → We will now seel to determine this risk premium. Intuitively we would expect the risk premium to be larger the larger the risk is; as risk is disliked, the larger the risk the lower the utility and the more needs to be deducted from the safe outcome to achieve the same utility. The more individuals dislike risk, the lower the utility and hence the large the risk premium; thus higher risk aversion, measuring this dislike of risk, implies a higher risk premium.

Approximating the risky outcome

- We use a quadratic approximation of the expected utility of the individual facing risky outcomes
- We use the expected value as a starting point, then make a linear adjustment, and a quadratic adjustment
- $$\mathsf{E}\left[U\left(V\right)\right] = \mathsf{E}\left[U\left(\mathsf{E}\left[V\right]\right) + \frac{\partial U(\mathsf{E}[V])}{\partial V}(V - \mathsf{E}[V]) + \frac{1}{2}\frac{\partial^2 U(\mathsf{E}[V])}{\partial V^2}\left(V - \mathsf{E}[V]\right)^2\right]$$
$$= U(\mathsf{E}[V]) + \frac{1}{2}\frac{\partial^2 U(\mathsf{E}[V])}{\partial V^2}\mathsf{Var}\left[V\right]$$

Approximating the risky outcome

→ In order to determine the risk aversion, we will first approximate the expected utility of the risky outcome. We di such an approximation to not having to relay on a specific utility function, but obtain general results.

- To capture the risk of the risky outcome, we use a quadratic approximation. The quadratic term will allow us to recover the variance of outcomes.
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  - The expected outcome is the anchor point of our approximation of the utility function as this will coincide with the safe outcome.
  - We first introduce a linear term were we adjust for the difference between the actual outcome and the expected outcome. The slope of the utility function at the expected value determines how much this adjustments needs to be.
  - We than repeat the same for a quadratic adjustment and the second derivative will gives the degree of adjustment needed. The term $\frac{1}{2}$ emerges from mathematical considerations in this Taylor series approximation.
- *Formula*
- We can now use the expected value and see that the first term is risk-free as it is the utility of the expected outcome and hence its expected value is the same. The second term will vanish as its becomes $\mathsf{E}[V] - \mathsf{E}[\mathsf{E}[V]] = \mathsf{E}[V] - \mathsf{E}[V] = 0$. In the final term we see that the definition of the variance emerges.

→ We can now retain this approximation and compare it with that of the safe outcome.

Approximating the safe outcome

- We use a linear approximation of the expected utility of the individual facing a safe outcome
- We use the expected value as a starting point and then make a linear adjustment
- $U(\mathsf{E}[V]-\Pi)=U(\mathsf{E}[V])+\frac{\partial U(\mathsf{E}[V])}{\partial V}\Pi$
- Setting this equal to the approximation of the risky outcome gives $\Pi=\frac{1}{2}\left(-\frac{\frac{\partial^2 U(\mathsf{E}[V])}{\partial V^2}}{\frac{\partial U(\mathsf{E}[V])}{\partial V}}\right)\mathsf{Var}\left[V\right]$
- The risk premium is increasing in the risk and the risk aversion

Approximating the safe outcome

→ After having approximated the utility of the risky outcome, we can now repeat the same for the safe outcome.

- For the safe outcome we use a linear approximation as the quadratic term which led to the emergence of the variance; but as this will be zero without any risk, we do not need to consider this quadratic term.
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  - Again, we use the expected value as our anchor point.
  - The linear adjustment from this anchor point will be the risk premium, and the degree of adjustment will be given by the first derivative of the utility at the expected outcome.
- *Formula*
- We can now set these approximations for the risky and safe outcomes equal and solve for the risk premium.
- As discussed above, the risk premium is increasing in the risk, the variance and in this expression using the derivatives of the utility function. We will interpret this as the risk aversion. This interpretation is based on the observation that the risk premium should be increasing in the risk aversion.

→ We can now define risk aversion more formally.

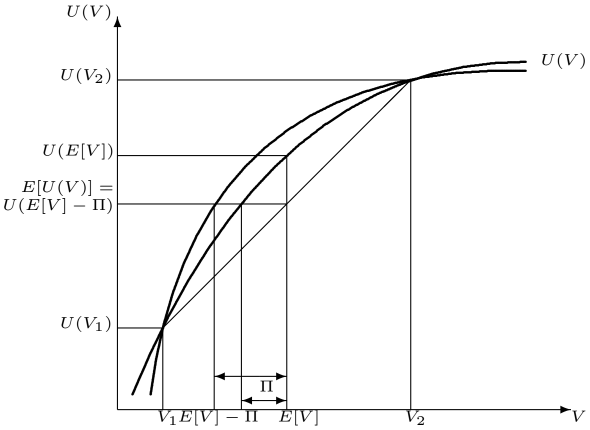
Arrow-Pratt measure of risk aversion

- We define $z = -\frac{\frac{\partial^2 U(\mathsf{E}[V])}{\partial V^2}}{\frac{\partial U(\mathsf{E}[V])}{\partial V}}$ as the Arrow-Pratt measure of absolute risk aversion
- The more risk averse an individual is, the higher the risk premium: $\Pi = \frac{1}{2} z \text{Var}[V]$
- The second derivative of the utility function reflects the curvature of the utility function

Arrow-Pratt measure of risk aversion

- → The risk aversion as defined here is also known as the Arrow-Pratt measure as these two economists developed these independently of each other.
- ▶ We define the term in brackets as a measure of risk aversion. This is sometimes called the measure of absolute risk aversion as other measures of risk aversion exist.
- ▶ We can rewrite the risk premium using this risk aversion and easily see that a higher risk aversion and higher risk increase the risk premium.
- ▶ We know from utility theory that the second derivative of the utility function is negative as the marginal utility (first derivative) is decreasing. The more strongly curved the utility function is, the larger, in absolute terms, the second derivative and the larger the risk aversion. Thus the second derivative drives the level of risk aversion.
- → We have thus established a connection between the utility function and risk aversion.

Risk premium with two outcomes



Risk premium with two outcomes

→ We can now illustrate these results graphically.

- We will look at how the outcome affects the utility an individual derives from this outcome.
- We have a generic utility function that is increasing (higher outcomes is better), but this increase is reducing (reducing marginal utility).
- We consider that for the risky outcome, there are two possible outcomes, V_1 as indicated here
- and V_2 as indicated here. For simplicity we assume in this illustration that both possible outcomes are equally likely.
- The safe outcome gives the expected value of the risky outcome as indicated here.
- We now derive the expected utility of the risky outcome. The expected utility will be a linear combination between the utilities of the two possible outcomes.
- With the outcomes being equally likely this will be exactly in the middle point; this will always be where the expected outcome is also located. The utility will be on the connecting lines between the utilities of the possible outcomes and not the utility function; the reason is that either of the possible outcomes are achieved, and not the expected outcome itself.
- From the definition of the risk premium we know that if we deduct it from the expected outcome of the safe choice, the utility is the same.
- From this we can easily obtain the risk premium.
- We now consider a more curved utility function, which has the same utility at the two possible outcomes. We can always make transformations to the utility function to achieve such a situation.
- Following the same steps, we easily see that the risk premium is larger.

→ This graphical illustration confirms that a more curved utility function, which ahs a larger second derivative, increases the risk premium and thus the individual has a higher risk aversion.

Approximating expected utility

- We have the expected utility given as $E\left[U\left(V\right)\right]=U\left(E\left[V\right]-\Pi\right)$
- The risk premium was $\Pi=\frac{1}{2}z\text{Var}\left[V\right]$

$\Rightarrow$ $E\left[U\left(V\right)\right]=U\left(E\left[V\right]-\frac{1}{2}z\text{Var}\left[V\right]\right)$

- As marginal utility is positive, we can often maximize expected utility by maximizing only the argument $E\left[V\right]-\frac{1}{2}z\text{Var}\left[V\right]$
- We can maximize expected utility without knowing the utility function, we only require the risk aversion

Approximating expected utility

→ We can now use out results and derive and approximation of the utility function that is often used in finance.

- We had set the expected utility of the risky outcome equal the utility of the safe outcome, less the risk premium.
- The risk premium as given from above as in the *formula*.
- [⇒] Inserting the risk premium, we obtain the expected utility as the utility of the expected value, less the product of risk aversion and the risk.
- When maximizing the expected utility, we can maximize the argument only as the utility function itself is increasing and hence maximizing the argument maximizes the utility itself.
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    - Using this approximation we do not need to know the specific details of the utility function, we use the expected outcome and the risk as the relevant information/
    - The only property we need to know in addition to the expected outcome and risk is the risk aversion of the decision-maker.

→ We have thus established that with risky outcomes we can optimize decisions using the expected outcome, the risk, and risk aversion alone, without detailed knowldge of the utility function.

Summary

- Risk aversion measures the attitude towards risk
- Risk aversion will be dependent on the preferences of individuals
- Risk itself can be measured objectively through the variance

Summary

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- Risk aversion is a measure of the attitude towards risk, the more risk averse an individual is, the more he dislikes risk.
- We only need to know the first and second derivative of the utility function, but not the shape of the utility function itself to approximate the optimal decisions. This is where the personal preferences of the decision-maker are important; different decision-makers will have different risk aversions.
- The measurement of risk by the variance of outcomes is not dependent on any preferences, but will be identical for all individuals (as long as they agree on the information that leads to the risk measurement).

→ It is often, however, that individuals differ in their risk assessment due to different information. This is a common problem in many models of financial markets seeking to explain trading behaviour.

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